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A NEW SOLUTION OF THE BORN-INFELD EQUATION

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Abstract : *A solution of the Born-Infeld equation has been found which is time-dependent and spherically symmetric. With the change of variable* $\phi = (\log f)^{1/2}$, *the equation for f becomes separable into a product of a Gaussian function of r times an elliptic function of time.*

Résumé : *On donne une solution de l'équation de Born-Infeld qui dépend du temps et possède la symétrie sphérique. Grâce au changement de variable* $\phi = (\log f)^{1/2}$, *l'équation en f devient séparable sous la forme du produit d'une fonction gaussienne de r par une fonction elliptique du temps.*

INTRODUCTION

One of the most tantalizing problems of mathematical physics is that of obtaining solutions of non-linear wave equations which represent disturbances having particle-like and periodic-wave attributes. The particle aspect is manifest when two or more localized disturbances pass through each other and asymptotically,

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at large distances of separation, maintain the integrity, or wholeness, of a single isolated disturbance. Such entities are called 'solitons'. Some non-linear equations possessing N-soliton solutions given by explicit formulae haven been tabulated by Scott et al (Scott 1973) in a significant review paper. A general method for generating special equations that have the N-soliton property -i.e.- the Korteweg-de-Vries, the cubic Schrödinger, and the sine-Gordon equations, has been recently derived (Lamb 1976) from the non-linear equations of classical differential geometry (the Seret-Frenet formulas) that describe the dynamical behavior of twisted filaments. Sixteen years earlier, it was shown that the equations representing minimal surfaces in differential geometry can be used to represent solutions of the well-known Born-Infeld equation (Lochak, 1960).

This feature of possessing multiple-soliton solutions is a special property of a few, select, non-linear equations. Moreover, of such, only the sine-Gordon and Born-Infeld equations are relativistically invariant. It has been pointed out (Klein 1976) that the separability of the sine-Gordon equation, after the change of dependent variable $\phi = 4 \tan^{-1} f$, into a function of space times a function of time, leads to the dualistic interpretation of a localized particle accompanied by a plane wave. This gives a natural interpretation to the dualism postulated on other grounds, by Louis de Broglie in 1923-24.

SOLVING THE BORN-INFELD EQUATION

A variety of solutions of the two-dimensional (one space and one time dimension) Born-Infeld equation can be obtained using the same mathematical techniques as have been developed for solving Plateau's Problem. In particular, a method found by Weingarten a century ago (Weingarten 1887), and rediscovered recently by Frêchet (Frêchet 1956), can be used to find minimal surfaces imbedded in Euclidean 3-space, having the form

(1) $$f(x) + g(y) + h(z) = 0.$$

If z stands for ϕ, the dependent variable in the Born-Infeld equation, one immediately has a separable solution in the form :

(2) $$\exp h(\phi) = \exp \{-f(x)\} \, . \, \exp \{-g(y)\}.$$

These solutions are analogous to the simple 'breathers' in sine-Gordon equation theory. The method, of surprising simplicity and generality, has been elaborated in some detail, in the beautiful treatise 'Vorlesungenüber Minimal Flächen' (Nitsche, 1975, esp. sections 82-87). Of special interest is the fact that the variables x, y, z, are interchangeable so that any of them may play the role of the dependent variable ϕ. An embarrassing richness of solutions follow from Weingarten's method ; for example, the minimal surface of Scherk :

(3) $$e^{\phi} = \cos x \,/\, \cos y = \cos t \,/\, \cosh y \quad (\text{putting } y \to iy)$$

can be readily derived using the formulas given in Nitsche's text.

The method can readily be applied to the four-dimensional 'surface' imbedded in a five-dimensional Euclidean space. One seeks a solution ; $f(\phi) = R(r) \, . \, T(t)$, where R is a localized function of r, and T is a periodic function of time. After a Lorentz transformation :

(4) $$r^2 \to x^2 + y^2 + \gamma^2 (z - \beta t)^2$$

$$t \to \gamma \,(t - \beta x),$$

it is evident that such a separable solution represents a localized particle associated with a plane wave whose dispersion relation is $\omega^2 = k^2 + 1$, in accordance with de Broglie's theory of wave-particle dualism. The separable solutions of the Born-Infeld equation having spherical symmetry, together with a time-dependent term, can only have a special form. Thus the requirement of separability in the radial coordinate r, and the time t, can be shown to give a unique result for the solution.

Using Weingarten's method (see Appendix of this paper) for a 5- dimensional space readily gives the solution :

(5) $$\phi^2 = \mathrm{Cos}^2 t - r^2$$

where Cos t = e(t) is the function derived by inversion of :

(6) $$t = \int_0^e \frac{dx}{\sqrt{1 - x^6}}$$

and we have put $x_1 = \phi$, $x_2^2 + x_3^2 + x_4^2 = r^2$, and $x_5 = it$.

Evidently, one may exponentiate this formula, giving :

$$(7) \qquad \exp(\phi^2) = e^{-r^2} \exp(\mathrm{Cos}^2 t).$$

The integral in (6) can be expressed in terms of standard elliptic integrals. An explicit representation can be then obtained :

$$(8) \qquad \exp(\phi^2) = \exp\{a(\mathrm{cn}\,\tau - 1)/(\mathrm{cn}\,\tau + b)\}^2 \exp(-r^2)$$

where

$$(9) \qquad a = 1/2(4k^2 - 3), \quad b = 4k'^2, \quad k = 1/2\sqrt{2+\sqrt{3}}.$$

The symbol τ stands for $(2\,\sqrt[4]{3}\,t)$. cn is the elliptic cosine with period $4K(k)$.

If one transforms the dependent variable in the Born-Infeld equation via the substitution

$$(10) \qquad \phi = (\log f)^{1/2}$$

one obtains a complicated non-linear equation. Being directly separable in (r,t) this equation in f would be analogous to that obtained in the sine-Gordon equation theory after the change of variable $\phi = 4\tan^{-1} f$. The sine-Gordon equation, in four dimensional form, has recently been found to possess solutions representing N solitons (Kobayashi and Izutsu, 1976). However interesting these solutions are, they do not represent entities having the spherical symmetry and dualistic nature wich an elementary particle should have. 'Breathers' have not been found in four-dimensional space time, for the sine-Gordon equation.

CONCLUDING REMARKS

In 1934, Born an Infeld proposed an equation which possessed a spherically symmetric, static solution which gave a natural explanation in terms of non-linear electrodynamics, for the existence of (spin-less) particles that possess charge. These solutions were static, in the rest frame of the entity. In the present paper, we have shown that an isolated 'breather' solution can exist also, with the proper time-dependence to represent an entity having the

dualism of de Broglie 's theory.

It is not known, even in one space-dimension, whether solutions representing two or more breathers can exist. However, the work of Lochak (1960) and the later Russian papers of Barbashov and Chernikov (1966-67) are certainly suggestive in this regard. Obviously this needs to be looked into.

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APPENDIX

The implicit representation of a 4-dimensional surface imbedded in a 5-dimensional Euclidean space is $\Phi\ (s,t,x,y,z) = 0$. One must extremize $\int [\Phi_s^2 + \Phi_t^2 + \Phi_x^2 + \Phi_y^2 + \Phi_z^2]^{1/2}\,ds\,dt\,dx\,dy\,dz$ subject to the constraint that Φ be zero. One obtains :

$$\text{(A1)}\quad \frac{\partial}{\partial s}\left(\Phi_s/\Delta\right) + \frac{\partial}{\partial t}\left(\Phi_t/\Delta\right) + \frac{\partial}{\partial x}\left(\Phi_x/\Delta\right) + \frac{\partial}{\partial y}\left(\Phi_y/\Delta\right) + \frac{\partial}{\partial z}\left(\Phi_z/\Delta\right) = 0$$

Now assume that $\Phi = c(s) + e(t) + f(x) + g(y) + h(z) = 0$. One obtains :

(A2) $[e'^2(t) + f'^2(x) + g'^2(y) + h'^2(z)]c''(s)$ ... plus four more

similar terms = 0.

Next introduce new variables $m = c(s)$, $p = e(t)$, $u = f(x)$, $v = g(y)$, $w = h(z)$. Write the abbreviations $V(m) = c'^2(s)$, $W(p) = e'^2(t)$, $X(u) = f'^2(x)$, $Y(v) = g'^2(y)$, $Z(w) = h'^2(z)$. Then (A2) becomes :

$$\text{(A3)}\quad (W + X + Y + Z)V' + (V + X + Y + Z)W' + (V + W + Y + Z)X' + (V + W + X + Z)Y' + (V + W + X + Y)Z' = 0.$$

Moreover, $m + p + u + v + w = 0$.

The capitalized variables V,W,X,Y,Z are functions of m,p,u,v,w. Spherical symmetry demands that :

$$e(t) = t^2 \qquad f(x) = x^2 \qquad g(y) = y^2.$$

Hence :

$$\text{(A4)}\quad \begin{aligned} W(p) &= 4p \\ X(u) &= 4u \\ Y(v) &= 4v \end{aligned}$$

One is left with an equation involving V(s) and Z(w) which cannot have any solution unless :

(A5) $V(m) = 4m$ or $Z(w) = 4w$.

Choosing $V(m) = 4m$, we find that (A3) reduces to an ordinary differential equation in $Z(w)$:

(A6) $-Z'w + 4Z - 12w = 0$

with solution :

(A7) $Z(w) = c\,w^4 + 4w = -4\,w^4 + 4w$ (putting $c = -4$).

Restoring the original variables gives :

(A8) $\left(\frac{dh}{dz}\right)^2 = -4z^4 + 4z.$

Therefore

(A9) $z = \int^{\sqrt{h}} \frac{dx}{\sqrt{1 - x^6}}$

so that the inverse of this elliptic integral gives h as a function of z.

The quantity $t^2 + x^2 + y^2$ may evidently be written as r^2. However, either of the two remaining variables s, z may be selected to represent ϕ, the solution of Plateau's problem. The other variable can be replaced by (it), where the symbol t stands for time, and so the Plateau solution is then a solution of the Born-Infeld equation.

The identification of z with ϕ in equation (A9) is no special interest, since one simply obtains the equation of a hypercatenoid imbedded in 5-space. On the other hand, identifying variable s with ϕ, gives

(A10) $m(s) + r^2 + w(z) = 0$ or $\phi^2 + r^2 + h(z) = 0.$

Using the inverse of the elliptic integral (A9) gives :

(A11) $\phi^2 + r^2 + \mathrm{Cos}^2 z = 0$

Writing $z = it$ (t being the time variable), and exponentiating

the above equation yields :

(A12) $\exp(\phi^2) = \exp(-r^2)\exp(\mathrm{Cos}^2 t),$

which we may denote by the new symbol f.

Since f(r,t) describes a localized disturbance in the form of a Gaussian function of radial distance r, accompanied by a function varying periodically in time, the entity has the double nature of wave and particle. The partial differential equation in f is cumbersome to write out explicitly but it is special in being separable in r and t, and the solution has no discontinuities or singularities.