ONE SOME FUNDAMENTAL ASPECTS OF STOCHASTIC ELECTRODYNAMICS

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Abstract : *We discuss some fundamental questions related to the Langevin-Braffort equation of stochastic electrodynamics, and derive from it, by a simple and direct procedure, the corresponding Fokker-Planck-type equation to order* e^2.

Résumé : *On discute certaines questions fondamentales relatives à l'équation de Langevin-Braffort de l'électrodynamique stochastique et on en dérive, suivant une méthode simple et directe, une équation du type Fokker-Planck à l'approximation* e^2.

I. INTRODUCTION

In view of the accumulating evidence in favour of stochast[ic] electrodynamics (SED) as a foundation to quantum mechanics (QM), w[e] consider that an analysis of some of its basic hypotheses is necessary and can be of help for a further development of the theory. W[e] have in mind, in particular, certain questions related to the equation of motion postulated in SED, namely, the so-called Langevin-Braffort equation

$$m\ddot{\vec{x}} = \vec{F}(\vec{x}) + m\tau\dddot{\vec{x}} + e\vec{E}. \quad (\tau = 2e^2/3mc^3). \quad (1)$$

One of the main lines of research of SED is to use this equation, with adequately selected statistical properties for the zero-point radiation field $\vec{E}$. (which SED takes for granted), as a basis fo[r] a statistical description of the electron, and try to show that it coincides at least in its most relevant aspects with that given by QM. This point of view-which is shared also by Boyer(1), Braffort et al(2), Claverie and Diner(3), Marshall(4), Santos(5) and others -is producing what we may hopefully call a stochastic alternative to orthodox QM: a physical explanation of the quantum mechanical behaviour of matter from a very reduced set of postulates. We feel that the results obtained to date are interesting and promising enough as to call for a thorough discussion of these postulates; the present note is intended as a partial answer to this demand.

The equation $m\ddot{\vec{x}} = \vec{F}(\vec{x}) + m\tau\dddot{\vec{x}}$ is a well-known classic equation of motion; however, the derivation of its stochastic counpart - the Langevin-Braffort equation - from first principles seem to have been overlooked. Moreover, it implies some - also well-

known - formal problems which usually are dealt with in a formal way, without entering into a deeper discussion about their origin and meaning. It seems useful, therefore, to derive the Langevin-Braffort equation from first principles and use the opportunity to discuss the difficulties associated with it.

Another fundamental problem of SED is the transition from a stochastic description in terms of the Langevin-Braffort equation, to a statistical description in terms of a Fokker-Planck-type equation. In this note (Section III) we present a simple and direct method of performing this transition, which is carried out only to order e^2, for simplicity.

We wish to emphasize that this note is limited to the discussion of certain methological questions of SED; the relevance of this theory to QM can be grasped by resorting to results published elsewhere(3,6,7). In this sense, the present note is one in a series of papers intended to demonstrate that SED can in fact be used as a fundamental theory of QM, or even of QED, as concerns the (non-relativistic, spinless) electron.

II. THE LANGEVIN-BRAFFORT EQUATION

The description of the motion of an individual (non-relativistic) particle in SED is usually made in terms of the Langevin-Braffort eq. (1), where $\vec{E}_0$ is the random zero-point electric field with spectral energy density given by

$$\rho(\omega) = \frac{\hbar\omega^3}{2\pi^2 c^3} , \qquad (2)$$

corresponding to an energy $\frac{1}{2}\hbar\omega$ per normal mode. Let us take a close look at the origin of this postulate.

The non-relativistic hamiltonian for a charged particle embedded in the electromagnetic radiation field is

$$H = \frac{1}{2m}(\vec{p} - \tfrac{e}{c}\vec{A})^2 + V(\vec{x}) + H_r \qquad (3)$$

where $\vec{p}$ is the canonical momentum conjugate to the position coordinate $\vec{x}$ of the particle and H_r is the hamiltonian of the radiation field.

The field may be decomposed into plane waves; in terms of the canonical variables $q_{n\sigma}$, $p_{n\sigma}$ associated with each mode n and polarization σ, we write, as usual,

$$\left.\begin{aligned}
&\phi = 0 \\
&\vec{A} = \sqrt{\frac{4\pi c^2}{L^3}} \sum_{n,\sigma} \hat{\epsilon}_{n\sigma}\left(q_{n\sigma}\cos\vec{k}_n\cdot\vec{x} - \frac{p_{n\sigma}}{\omega_n}\sin\vec{k}_n\cdot\vec{x} \right) \\
&\vec{k}_n\cdot\hat{\epsilon}_{n\sigma} = 0 , \quad \hat{\epsilon}_{n\sigma}\cdot\hat{\epsilon}_{n\sigma'} = \delta_{\sigma\sigma'} \quad (\sigma = 1,2) , \\
&k_n = \omega_n/c , \quad \vec{k}_n = \frac{2\pi}{L}(n_1\hat{\imath} + n_2\hat{\jmath} + n_3\hat{k}) .
\end{aligned}\right\} \qquad (4)$$

Eqs. (4) are written in the Coulomb gauge and under the assumption that the field is contained in a cavity of volume L^3 with perfectly conducting walls; we may let $L \to \infty$ and replace the sums

over n_1, n_2, n_3 by integrals, so that

$$\frac{1}{L^3}\sum_n \xrightarrow[L\to\infty]{} \frac{1}{(2\pi c)^3}\int d\omega\cdot\omega^2\cdot d\Omega_k . \qquad (5)$$

Eqs. (4) give for the hamiltonian of the field within the cavity:

$$H_r = \frac{1}{8\pi}\int d^3x\,(\vec{E}^2 + \vec{H}^2) = \frac{1}{2}\sum_{n,\sigma}(p_{n\sigma}^2 + \omega_n^2 q_{n\sigma}^2) . \qquad (6)$$

The hamiltonian equations for particle and field are therefore:

$$\dot{\vec{x}} = \nabla_p H = \frac{1}{m}(\vec{p} - \tfrac{e}{c}\vec{A}) \qquad (7)$$

$$\dot{\vec{p}} = -\nabla H = \vec{F} - e'\sum_{n,\sigma}(\dot{\vec{x}}\cdot\hat{\epsilon}_{n\sigma})\,\vec{k}_n\left[q_{n\sigma}\sin\vec{k}_n\cdot\vec{x} + \frac{p_{n\sigma}}{\omega_n}\cos\vec{k}_n\cdot\vec{x}\right]$$

$$\begin{aligned}
\dot{q}_{n\sigma} &= \frac{\partial H}{\partial p_{n\sigma}} = p_{n\sigma} + e'\,\dot{\vec{x}}\cdot\hat{\epsilon}_{n\sigma}\,\frac{\sin\vec{k}_n\cdot\vec{x}}{\omega_n} \\
\dot{p}_{n\sigma} &= -\frac{\partial H}{\partial q_{n\sigma}} = -\omega_n^2 q_{n\sigma} + e'\,\dot{\vec{x}}\cdot\hat{\epsilon}_{n\sigma}\cos\vec{k}_n\cdot\vec{x}
\end{aligned} \qquad (8)$$

where

$$e' = e\sqrt{4\pi/L^3} .$$

In the nonrelativistic, long-wavelength or dipole approximation (which we can take without missing essential information) the radiation field becomes homogeneous in space. The equation of motion for the particle reduces then, according to eqs. (7), to

$$m\ddot{\vec{x}} = \vec{F} - e'\sum_{n,\sigma}\hat{\epsilon}_{n\sigma}\,\dot{p}_{n\sigma} = \vec{F} + e\vec{E} \qquad (9)$$

where $\vec{E} = -\frac{1}{c}\frac{\partial\vec{A}}{\partial t}$.

The equations of motion for the field variables, eqs. (8), can be readily solved, yielding

$$q_{n\sigma}(t) = q^{0}_{n\sigma}(t) + \frac{e'}{\omega_n}\int_0^t dt'\ \hat{\epsilon}_{n\sigma}\cdot\dot{\vec{x}}(t')\sin\omega_n(t-t') \tag{10a}$$

$$p_{n\sigma}(t) = p^{0}_{n\sigma}(t) + e'\int_0^t dt'\ \hat{\epsilon}_{n\sigma}\cdot\dot{\vec{x}}(t')\cos\omega_n(t-t') \tag{10b}$$

where

$$q^{0}_{n\sigma}(t) = q_{n\sigma}(0)\cos\omega_n t + \frac{p_{n\sigma}(0)}{\omega_n}\sin\omega_n t \tag{11a}$$

$$p^{0}_{n\sigma}(t) = p_{n\sigma}(0)\cos\omega_n t - \omega_n q_{n\sigma}(0)\cos\omega_n t \tag{11b}$$

are the source-free contributions to the radiation field, and we have assumed that the particle-field interaction is suddenly connected at $t=0$. Hence, the electric force acting on the particle consists of two terms:

$$e\vec{E} = e\vec{E}_0 + e\vec{E}_s \tag{12}$$

$\vec{E}_0$ being the vacuum field:

$$e\vec{E}_0 = -e'\sum_{n,\sigma}\hat{\epsilon}_{n\sigma}\,p^{0}_{n\sigma} \tag{13a}$$

and $\vec{E}_s$ being the electric field radiated by the particle itself:

$$e\vec{E}_s = -e'^2\sum_{n,\sigma}\hat{\epsilon}_{n\sigma}\int_0^t dt'\ \hat{\epsilon}_{n\sigma}\cdot\dot{\vec{x}}(t')\cos\omega_n(t-t')\ . \tag{13b}$$

Eqs. (9), (12) and (13) combined yield an integro-differential equation of motion for the particle, which is usually transformed into a differential equation of third order by means of two integrations by parts. In fact, according to eq. (5), as $L\to\infty$ eq. (13b) goes over to

$$e\vec{E}_s = -\frac{2m\tau}{\pi}\int_0^\infty d\omega\cdot\omega^2\int_0^t dt'\ \dot{\vec{x}}(t')\cos\omega(t-t') = \frac{2m\tau}{\pi}\int_0^\infty d\omega\int_0^t dt'\ \dot{x}(t')\frac{d^2}{dt'^2}\cos\omega(t-t'$$

$$= m\tau\left[-\frac{2}{\pi}\ddot{\vec{x}}(t)\int_0^\infty d\omega + \dddot{\vec{x}}(t)\right]. \tag{14}$$

This result reveals that the non-locality in time of the self-force given by eq. (13b) may be expressed as the combined action of an electromagnetic contribution to the inertia of the point particle $m_{em} = \frac{2}{\pi}m\tau\int_0^\infty d\omega$ and the Lorentz radiation reaction force $m\tau\dddot{\vec{x}}$. Therefore, the physical mass m of the particle is, in this approximation, the sum of the bare mass m_{bare} and the electromagnetic mass m_{em}; but since m in the hamiltonian (being the observable mass, already contains m_{em}, this mass m_{em} mus be subtracted. This mass renormalization may be formally done, although, as is usually the case with the classical point electron, m_{em} comes out infinite.

We small return below to an analysis of the preceding results. Here we simply adopt the renormalized mass as the physica

one, and write the equation of motion in the form

$$m\ddot{\vec{x}} = \vec{F} + m\tau\dddot{\vec{x}} + e\vec{E}_0 \tag{15}$$

with the vacuum field $\vec{E}_0$ given by (13a).

We now proceed to define the statistical properties of the vacuum field $\vec{E}_0$. The random character of the field is introduced through the canonical variables $p_{n\sigma}, q_{n\sigma}$; it is clear that in the absence of particles, the vacuum field must average to zero, i.e.,*

$$\langle p^{\circ}_{n\sigma} \rangle = \langle q^{\circ}_{n\sigma} \rangle = 0 \tag{16}$$

and the average energy per mode must be $\frac{1}{2}\hbar\omega_n$, i.e.,

$$\langle (p^{\circ}_{n\sigma})^2 \rangle = \omega_n^2 \langle (q^{\circ}_{n\sigma})^2 \rangle = \frac{1}{2}\hbar\omega_n . \tag{17}$$

In addition, we will assume that the vacuum field had reached equilibrium before the interaction is connected, so that the initial distribution of the field variables is stationary. These conditions are not sufficient to define uniquely the distribution of the vacuum field amplitudes, but they are sufficient for our present purposes, as we shall see below.

From eqs. (13a), (16) and (17) we obtain

$$\langle \vec{E}_0 \rangle = 0 \tag{18}$$

and

$$\langle \vec{E}_0(t)\cdot\vec{E}_0(t') \rangle = \frac{4\pi\hbar}{L^3} \sum_n \omega_n \cos\omega_n (t-t')$$

* The symbol $\langle\,\rangle$ denotes average over the distribution S of amplitudes, i.e., $\langle q \rangle = \int d^N q_{n\sigma}\, d^N p_{n\sigma}\, q(\{q_{n\sigma}\},\{p_{n\sigma}\},\vec{x},\vec{p},t)\, S(\{q_{n\sigma}\},\{p_{n\sigma}\},$

or if we let $L \to \infty$,

$$\langle \vec{E}_0(t)\cdot\vec{E}_0(t') \rangle = \frac{2\hbar}{\pi c^3}\int_0^\infty d\omega\cdot\omega^3 \cos\omega(t-t') = 4\pi\int_0^\infty d\omega\, \rho(\omega)\cos\omega(t-t'), \tag{19}$$

where the spectral energy density of the vacuum field is given b eq. (2). We have thus recovered, in the non-relativistic long-wa length limit, the basic postulates of SED, namely, eqs. (1) and

The radiation reaction force $m\tau\dddot{\vec{x}}$, being of third orde allows for spurious solutions to eq. (15); since such unphysical solutions are characterized by being self-accelerating (coloquia runaway) we can eliminate them by the usual expedient of imposin the asymptotic condition $\ddot{\vec{x}} \xrightarrow[t\to\infty]{} 0$, i.e., by writing eq. (15) in form

$$m\ddot{\vec{x}} = \frac{1}{\tau}e^{t/\tau}\int_t^\infty dt'\, e^{-t'/\tau}\left[\vec{F}(\vec{x}(t')) + e\vec{E}_0(t')\right] . \tag{20}$$

In view of the smallness of τ ($= 2e^2/3mc^3 \sim 10^{-23}$ sec.), we may e $\vec{F}(\vec{x}(t'))$ in Taylor's series around $\vec{F}(\vec{x}(t))$; retaining the firs two terms only, we obtain an equation which is correct only up t terms containing e^2 (or τ):

$$m\ddot{\vec{x}} = \vec{F}(\vec{x}) + \tau(\dot{\vec{x}}\cdot\nabla)\vec{F} + e\vec{E}_m(t) \tag{21}$$

where $\vec{E}_m$ is the vacuum electric field with a modified spectral energy density:

$$\rho_m(\omega) = \frac{\rho(\omega)}{1+\tau^2\omega^2} . \tag{22}$$

We have met as yet with four important difficulties in our treatment, which are clearly related among each other, namel i) the divergent electromagnetic mass: ii) the appearance of a third-order term in the equation of motion, allowing the introduc of an additional initial condition, iii) the appearance of runaw solutions; iv) the unphysical spectral energy density assigned to the vacuum field.

The need for a mass renormalization can be avoided from the beginning, by an adecuate modification of the basic postulates of the theory: for instance, by redefining the action for the field in such a form as to cancel the troublesome terms without introduci undesirable effects, or by using an appropriate combination of retarded and advanced potentials à la Dirac. A discussion of this point, including references to the most relevant work, may be found in Rohrlich's book(11).

Recognizing that both difficulties (ii) and (iii) above are a product of our procedure, since they do not appear in the original hamiltonian formulation, we have proposed to get rid of them by using one and the same mathematical trick, namely, by impos a condition on the acceleration which guarantees that it remain bounded. The need for this sort of asymptotic condition was alread recognized by Dirac in his early work on the subject (1938). Thoug the method is not entirely satisfactory, we know of no other proced to write down in closed form the equation of motion for the radiati particle in the electromagnetic field, without resorting to integro differential equations - or to eq. (9) in its primitive form. This problem (in its relativistic version) is discussed in the same book by Rohrlich(11).

The above considerations point to eq. (21) as a reasonable equation of motion for the radiating electron, devoid of the classical difficulties (i)-(iii). As to the fourth difficulty, namely the unphysical spectral density, we will not have opportunit

in this paper to discuss its consequences; we therefore content ourselves with stressing that it is apparently due to our neglect of several relativistic effects. For instance, by taking account of pair production the modes with energies greater than $\omega_c = 2mc^2/\hbar$ would be considerably affected; we must expect that all such considerations taken together carry us to a real, integrable spect density for the vaccum field. This points to the need of developi a wholly consistent relativistic stochastic theory, able to accou for the quantal properties of the field.

III. GENERALIZED FOKKER-PLANCK EQUATION

In order to construct a statistical description for the electron, it is necessary to eliminate any explicit dependence on the random field variables. Different procedures way be followed for this purpose. For instance, one may try to solve eq. (21) for $\vec{x}(t)$ and use the solution to compute the correlations of the various dynamical variables for the particle. This technique has been successfully applied in the linear case, showing that the harmonic oscillator of SED possesses indeed quantum-mechanical properties(4,8,9). However, it is virtually impossible to solve eq. (21) for a non-linear force; one must resort to other procedur

In the following we present a procedure applicable to any conservative-force problem, but we shall, for simplicity, work it out in the non-relativistic, dipole approximation, and through second order in the electric charge e.

We start by observing that the density of states of the whole system (particle + field) obeys a Liouville equation

$$\frac{\partial R}{\partial t} + \hat{L} R = 0 \tag{23}$$

where the liouvillian operator $\hat{L}$ is determined by the hamiltonian equations (10) and (11). Let us redefine $\vec{p} \equiv m\dot{\vec{x}}$, then we can write $\hat{L}$ in the form

$$\hat{L} = \hat{L}_p + \hat{L}_r + \hat{L}_i \equiv \hat{L}_o + \hat{L}_i \tag{24}$$

where the particle-, field- and interaction liouvillians are, respectively,

$$\hat{L}_p = \frac{1}{m}\vec{p}\cdot\nabla + \vec{F}\cdot\nabla_p \tag{25a}$$

$$\hat{L}_r = \sum_{n,\sigma}\left(p_{n\sigma}\frac{\partial}{\partial q_{n\sigma}} - \omega_n^2 q_{n\sigma}\frac{\partial}{\partial p_{n\sigma}}\right) \tag{25b}$$

$$\hat{L}_i = e\vec{E}\cdot\nabla_p + \frac{e'}{m}\sum_{n,\sigma}\vec{p}\cdot\hat{\epsilon}_{n\sigma}\frac{\partial}{\partial p_{n\sigma}}. \tag{25c}$$

A reduced Liouville equation for the particle is obtained by avera ing eq. (23) over the field variables:

$$\frac{\partial Q}{\partial t} + \hat{L}_p Q + \nabla_p\cdot\langle e\vec{E}\rangle Q = 0 \tag{26}$$

where

$$\langle A\rangle Q \equiv \int A(\vec{x},\vec{p},\{q_{n\sigma}\},\{p_{n\sigma}\},t)\, R\, d^N q_{n\sigma}\, d^N p_{n\sigma} \tag{27a}$$

for any function A, and in particular,

$$Q(\vec{x},\vec{p},t) = \int R\, d^N q_{n\sigma}\, d^N p_{n\sigma} \tag{27b}$$

is the phase-space distribution function for the particle.

Incidentally, an entirely analogous procedure leads to a reduced Liouville equation for the field, by integration over the variables $\vec{x}$, $\vec{p}$, namely:

$$\frac{\partial S}{\partial t} + \hat{L}_i S + \frac{e'}{m}\sum_{n,\sigma}\frac{\partial}{\partial p_{n\sigma}}\hat{\epsilon}_{n\sigma}\cdot\langle\vec{p}(\{q_{n\sigma}\},\{p_{n\sigma}\})\rangle S = 0,$$

which can be used for a statistical analysis of the radiation field in the presence of matter.

Returning to eq. (26), we see that it contains an unknown function, namely, the (phase-space) local average electric force $e\langle\vec{E}\rangle$. In order to compute this term we resort once more to the complete Liouville equation (23), and write its solution formally as

$$R(t) = e^{-\hat{L}t}R(o)$$

where $R(0)$ is the distribution function at some initial time $(t=0)$ at which we assume that the particle-field interaction is suddenly connected.

Since we want to compute the electric force only through second order in e, an expression for $R(t)$ correct fo first order in e will suffice. This is obtained by applying the formula

$$e^{-(A+B)t} = e^{-At} - \int_0^t dt'\, e^{-(A+B)(t-t')}\, B\, e^{-At'}$$

with A= $\hat{L}_o$ and B= $\hat{L}_i$, and making the substitution $e^{-(\hat{L}_o+\hat{L}_i)t} \rightarrow e^{-\hat{L}_o t}$ in the integrand:

$$R(t) = e^{-\hat{L}t}R(o) \simeq e^{-\hat{L}_o t}R(o) - \int_0^t dt'\, e^{-\hat{L}_o(t-t')}\hat{L}_i\, e^{-\hat{L}_o t'}R(o). \tag{29}$$

Hence the average electric force becomes

$$e\langle\vec{E}\rangle Q = \int d^N q_{n\sigma}\, d^N p_{n\sigma}\, e\vec{E}\left[e^{-\hat{L}_0 t}R(0) - \int_0^t dt'\, e^{-\hat{L}_0(t-t')}\hat{L}_i e^{-\hat{L}_0 t'}R(0)\right]. \tag{30}$$

For times $t \leq 0$, when particle and field are mutually independent, one can write the distribution function as a product of two functions:

$$R(t) = Q_0(\vec{x},\vec{p},t)\, S_0(\{q_{n\sigma}\},\{p_{n\sigma}\},t) \qquad (t \leq 0)$$

satisfying the uncoupled Liouville equations:

$$\frac{\partial Q_0}{\partial t} + \hat{L}_p Q_0 = 0 \tag{31a}$$

$$\frac{\partial S_0}{\partial t} + \hat{L}_r S_0 = 0\,. \tag{31b}$$

If we assume that the vacuum field reached equilibrium before $t=0$, then S_0 is stationary:

$$\frac{\partial S_0}{\partial t} = 0\ ; \qquad \hat{L}_r S_0 = 0$$

and hence, is a function of the vacuum-field energy. Actually, sin the vacuum normal modes are mutually independent, we may write S_0 as a product of functions of the normal-mode energies:

$$S_0 = \prod_{n,\sigma} S_{n\sigma}(E_{n\sigma})\ ; \qquad E_{n\sigma} = \tfrac{1}{2}\left[(p^0_{n\sigma})^2 + \omega_n^2 (q^0_{n\sigma})^2\right].$$

According to eqs. (10), the source-free variables $q^0_{n\sigma}$, $p^0_{n\sigma}$ coincide at $t=0$ with $q_{n\sigma}$, $p_{n\sigma}$; we may therefore write

$$S(0) = \prod_{n,\sigma} S_{n\sigma}(E_{n\sigma})\ ; \qquad E_{n\sigma} = \tfrac{1}{2}\left(p_{n\sigma}^2 + \omega_n^2 q_{n\sigma}^2\right), \tag{32}$$

whereby

$$e^{-\hat{L}_0 t}R(0) = e^{-(\hat{L}_p+\hat{L}_r)t}Q(0)S(0) = S(0)\,e^{-\hat{L}_p t}Q(0) \tag{33}$$

since $\hat{L}_r S(0) = 0$.

On the other hand, we recall from eq. (9) that the approximate expression for the electric force in the long-wavelength limit is

$$e\vec{E} = -e' \sum_{n,\sigma} \hat{\epsilon}_{n\sigma} p_{n\sigma} \tag{34}$$

and this gives for the interaction liouvillian, eq.(26),

$$\hat{L}_i = -e' \sum_{n,\sigma} p_{n\sigma}\hat{\epsilon}_{n\sigma}\cdot\nabla_p + \frac{e'}{m}\sum_{n,\sigma}\vec{p}\cdot\hat{\epsilon}_{n\sigma}\frac{\partial}{\partial p_{n\sigma}}\,. \tag{35}$$

By using eqs. (32)-(35) we see that the first term on the r.h.s. of eq. (30) integrates to zero and the remaining term yields

$$\begin{aligned} e\langle\vec{E}\rangle Q = {} & -e'^2\int d^N q_{n\sigma}\, d^N p_{n\sigma}\, S(0)\sum_{n,\sigma}\hat{\epsilon}_{n\sigma}p_{n\sigma}\int_0^t dt'\left(e^{-\hat{L}_r(t-t')}p_{n\sigma}\right)e^{-\hat{L}_p(t-t')}\hat{\epsilon}_{n\sigma}\cdot\nabla_p e^{-\hat{L}_p t'}Q(0) \\ & + \frac{e'^2}{m}\int d^N q_{n\sigma}\, d^N p_{n\sigma}\sum_{n,\sigma}\hat{\epsilon}_{n\sigma}p_{n\sigma}\int_0^t dt'\left(e^{-\hat{L}_r(t-t')}\frac{\partial S(0)}{\partial p_{n\sigma}}\right)\hat{\epsilon}_{n\sigma}\cdot e^{-\hat{L}_p(t-t')}\vec{p}\, e^{-\hat{L}_p t'}Q(0) \\ \equiv {} & e\langle\vec{E}_0\rangle Q + e\langle\vec{E}_s\rangle Q\,. \end{aligned} \tag{36}$$

From $\hat{L}_r p_{n\sigma} = -\omega_n^2 q_{n\sigma}$ and $\hat{L}_r q_{n\sigma} = p_{n\sigma}$ we obtain

$$e^{-\hat{L}_r(t-t')}p_{n\sigma} = p_{n\sigma}\cos\omega_n(t-t') - \omega_n q_{n\sigma}\sin\omega_n(t-t')$$

whereby the first term in eq. (36) takes on the form

$$\begin{aligned} e\langle\vec{E}_0\rangle Q = {} & -e'^2\int d^N q_{n\sigma}\, d^N p_{n\sigma}\, S(0)\sum_{n,\sigma}\hat{\epsilon}_{n\sigma}p_{n\sigma}^2\int_0^t dt'\cos\omega_n(t-t')\,e^{-\hat{L}_p(t-t')}\hat{\epsilon}_{n\sigma}\cdot\nabla_p e^{-\hat{L}_p t'}Q(0) \\ & + e'^2\int d^N q_{n\sigma}\, d^N p_{n\sigma}\, S(0)\sum_{n,\sigma}\hat{\epsilon}_{n\sigma}\omega_n p_{n\sigma}q_{n\sigma}\int_0^t dt'\sin\omega_n(t-t')\,e^{-\hat{L}_p(t-t')}\hat{\epsilon}_{n\sigma}\cdot\nabla_p e^{-\hat{L}_p t'}Q(0) \end{aligned}$$

This expression involves the initial distribution $S(\{q_{n\sigma}\},\{p_{n\sigma}\},0)$ which coincides with the source-free distribution $S_0(\{q^0_{n\sigma}\},\{p^0_{n\sigma}\},0)$; we can therefore use eqs. (16) and (17), whereby we obtain:

$$e\langle\vec{E}_0\rangle Q = -\frac{e'^2\hbar}{2}\sum_{n,\sigma}\hat{\epsilon}_{n\sigma}\omega_n\int_0^t dt'\cos\omega_n(t-t')e^{-\hat{L}_p(t-t')}\hat{\epsilon}_{n\sigma}\cdot\nabla_p e^{-\hat{L}_p t'}Q(0)$$

or letting $L\to\infty$,

$$e\langle\vec{E}_0\rangle Q = -\frac{m\tau\hbar}{\pi}\int_0^\infty d\omega\cdot\omega^3\int_0^t dt'\cos\omega(t-t')\,e^{-\hat{L}_p(t-t')}\nabla_p Q(\vec{x},\vec{p},t')$$

$$= -e^2\int_0^t dt'\langle\vec{E}_0(t)\vec{E}_0(t')\rangle\cdot e^{-\hat{L}_p(t-t')}\nabla_p Q(\vec{x},\vec{p},t') \qquad (37)$$

where we have used eq. (19) and have written $e^{-\hat{L}_p t'}Q(0) = Q(t')$, which is correct to zero order in e, according to eq. (26).

The second term in eq. (36) can be integrated by parts over $p_{n\sigma}$, yielding

$$e\langle\vec{E}_s\rangle Q = -\frac{e'^2}{m}\sum_{n,\sigma}\hat{\epsilon}_{n\sigma}\int_0^t dt'\cos\omega_n(t-t')\,e^{-\hat{L}_p(t-t')}\hat{\epsilon}_{n\sigma}\cdot\vec{p}\,Q(t')$$

$$\xrightarrow[L\to\infty]{} -\frac{2\tau}{\pi}\int_0^\infty d\omega\cdot\omega^2\int_0^t dt'\cos\omega(t-t')\,e^{-\hat{L}_p(t-t')}\vec{p}\,Q(t').$$

A procedure similar to that used in the calculation of eq. (14) yie

$$e\langle\vec{E}_s\rangle Q = -\frac{2\tau}{\pi}\vec{F}(\vec{x})Q(t)\int_0^\infty d\omega + \frac{\tau}{m}\left[(\vec{p}\cdot\nabla)\vec{F}\right]Q(t)$$

which is precisely the (phase-space) local average of eq.(14), excep that the radiative reaction force is written already to order τ; cf. eq. (21).

Therefore, after mass renormalization we are left with

$$e\langle\vec{E}_s\rangle Q = \frac{\tau}{m}\left[(\vec{p}\cdot\nabla)\vec{F}\right]Q(t)\,. \qquad (38)$$

We recall that in deriving eq. (21), the vacuum field spectrum was modified according to eq. (22), due to the presence of the radiation reaction. Close inspection shows that the same modification would have appeared in eq. (37), had we not prematurely approximated to second order in e. We can still repair this fault, by writing instead of eq. (37):

$$e\langle\vec{E}_0\rangle Q = -e^2\int_0^t dt'\langle\vec{E}_m(t)\vec{E}_m(t')\rangle\cdot e^{-\hat{L}_p(t-t')}\nabla_p Q(t') \qquad (39)$$

where $\vec{E}_m(t)$ is the vacuum electric field with spectral density modified according to eq. (22).

Introducing into eq. (27) the two contributions to the average electric force, given by (38) and (39), we obtain a generalized Fokker-Planck equation for the electron in phase-space, correct to order τ (or e^2):

$$\frac{\partial Q}{\partial t} + \frac{\vec{p}}{m}\cdot\nabla Q + \nabla_p\cdot\left(\vec{F} + \frac{\tau}{m}\vec{p}\cdot\nabla\vec{F}\right)Q -$$

$$-e^2\nabla_p\cdot\int_0^t dt'\langle\vec{E}_m(t)\vec{E}_m(t')\rangle e^{-\hat{L}_p(t-t')}\cdot\nabla_p Q(t') = 0\,. \qquad (40)$$

The exact version of eq. (40) would contain an infinite number of integral terms with higher powers of e, representing, of course, higher-order contributions of the electromagnetic interaction between particle and field.

It is instructive to compare this result with the one obtained previously[6] using the (approximate) method of the stochastic Liouville equation. In following this method, we use eq. (21) to write down a stochastic Liouville equation for the particle only:

$$\frac{\partial Q}{\partial t} + \frac{\vec{p}}{m}\cdot\nabla Q + \nabla_p\cdot\left(\vec{F} + \frac{\tau}{m}\vec{p}\cdot\nabla\vec{F} + e\vec{E}_m\right)Q = 0 \qquad (41)$$

and use it to derive an equation for the averaged phase-space distribution function

$$Q = \langle \mathcal{Q} \rangle .$$

This is accomplished by introducing a smoothing operator $\hat{P}$ the action of which is equivalent to averaging over the distribution of the random field variables:

$$\hat{P} A = \langle A \rangle . \qquad (42)$$

In particular, $\hat{P}\mathcal{Q} = \langle \mathcal{Q} \rangle = Q$.

By applying this smoothing procedure to eq. (41) we obtain two coupled equations for $\hat{P}\mathcal{Q} = Q$ and $(1-\hat{P})\mathcal{Q} = \delta Q$, which can be combined to eliminate the irrelevant contribution δQ . The ensuing equation for Q reduces in the e^2 - approximation to:

$$\frac{\partial Q}{\partial t} + \frac{\vec{p}}{m}\cdot\nabla Q + \nabla_p\cdot\left(\vec{F} + \frac{\tau}{m}\vec{p}\cdot\nabla\vec{F}\right)Q -$$
$$- e^2\,\nabla_p\cdot\int_0^t dt'\;\hat{P}\,\vec{E}_m(t)\vec{E}_m(t')\,e^{-\hat{L}_p(t-t')}\cdot\nabla_p Q(t') = 0 \qquad (43)$$

which coincides with eq. (40), since $\hat{P}\,\vec{E}_m(t)\vec{E}_m(t') = \langle \vec{E}_m(t)\vec{E}_m(t')\rangle$.

It should be noted that the smallness of the non-classical terms appearing in eq. (40) - or (43) - does not at all imply that we can dismiss them; it is precisely through the interaction with the radiation field that the material system may reach a state of equilibrium. Only in the equilibrium régime, when the interaction has already accomplished the task of determining the essentially non-classical structure of the distribution function, do the non-classical terms represent corrections. In fact, as is shown in refs. (6,7), one can derive the Schrödinger equation from eq. (40) when

system is close to equilibrium; the remanent effect of the radiation field gives then rise to the (nonrelativistic) radiative corrections of QED, namely, the Lamb shift and the decay of otherwise stable excited atomic states. It is precisely in connection with results of this sort that the present note acquires its full meaning.

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