Annales de la Fondation Louis de Broglie - Vol. 3, n° 3, 1978

SUR LA CERTITUDE DES VALEURS DU MOMENT CINÉTIQUE EN MÉCANIQUE QUANTIQUE JOUISSANT D'UNE RÈGLE DE CORRESPONDANCE NON-NEUMANIENNE

par M. I.A. LYABIS
Peoples' Friendship University,
Chair of Theoretical Physics,
3, Ordjonikidze Street,
V-302, Moscow, U.S.S.R.

(manuscrit reçu le 4 Juillet 1977)

Abstract : In the present paper an unique correspondence rule from the class of the Non-Neumann rules of constructing quantum operators is fixed and problems of certainty of angular momentum of a particle are considered. Equation, which solutions are the states of maximum certainty of a physical quantity, is solved for angular momentum and one of its components.

Résumé : Dans ce travail on fixe une règle de correspondance appartenant à la classe des règles non-neumaniennes de construction des opérateurs quantiques. On considère des questions relatives à la certitude du moment cinétique de la particule. On trouve des solutions de l'équation du moment cinétique et de ses composantes ; ces solutions sont les états de certitude maximale de la grandeur physique.

On Certainty of the Angular Momentum Values in Quantum Mechanics with the Non-Neumann Correspondence Rule

I.A. Lyabis

INTRODUCTION

In the previous paper [1] the class of the Non-Neumann rules of constructing quantum operators (correspondence rules) has been determined. The main consequences to which the application of such rules to quantum theory leeds are obtained. So, in the quantum theory with the Non-Neumann correspondence rule the concept of certainty becomes more general and fundamental than it is in the conventional quantum theory. The accuracy of determining even a single physical quantity is limited. The states, in which the averege value $\langle A \rangle$ of the physical quantity A is given with maximum accuracy (i.e. the dispersion $\langle (\Delta A)^2 \rangle$ reaches its minimum in these states), satisfy some nonlinear equation. (In the generally accepted quantum theory such states are the eigenstates of the operator $O(A)$ representing the physical quantity A.)

So, the question of certainty of physical quantities characterising concrete quantum-mechanical systems is of natural interest. In order to solve a quantum-mechanical problem one should know a concrete form of operators (i.e. it is necessary to use some correspondence rule). In the present paper for this purpose

Kuryshkin's rule of constructing quantum operators [2], which satisfies the Non-Neumann definition, has been taken.

Quantum operators $O(A)$ corresponding to physical quantities A according to this rule depend on some set of functions of coordinates and time $\{\varphi_K(q,t)\}$. The rule [2] is unique when the set $\{\varphi_K\}$ is fixed, but choice of another set changes operators and, hence, the theory itself. That is why the rule of constructing quantum operators [2] determines indeed the whole class of unique correspondence rules and all of them are the Non-Neumann.

A set of functions $\varphi_K(q,t)$ determines according to the accepted terminology [3] some "subquantum situation" in which a mechanical system is placed. It is supposed that "subquantum situation", being some physical reality, interacts with a mechanical system and determines its quantum behaviour. In the present work on base of the equation obtained in paper [1] question of certainty of the angular momentum $\vec{L} = [\vec{q} \times \vec{p}]$ of a particle, placed in different "subquantum situations", is considered.

1. Operators

Kuryshkin's correspondence rule [2] is defined by the action of the operator $O(A)$ associated to the physical quantity A on the arbitrary state $\Psi(q,t)$

$$(1) \quad O(A)\Psi(q,t) = (2\pi\hbar)^{-N} \int \varphi(\xi,\eta,t) A(q+\xi, p+\eta, t) e^{\frac{i}{\hbar}((q-q')p)} \Psi(q',t)\, d\xi\, d\eta\, dq'\, dp$$

where $A(q,p,t)$ is a function of coordinates $q(q_1,\dots,q_N)$, momentum $p(p_1,\dots,p_N)$ and time t, characterising the considered system in the classical theory, (ξp) is the scalar product of the vectors ξ and p, $\varphi(\xi,\eta,t)$ is the auxiliary function of phase-space and time

$$(2) \qquad \varphi(\xi,p,t) = (2\pi\hbar)^{-\frac{N}{2}} e^{-\frac{i}{\hbar}(\xi p)} \sum \varphi_k(\xi,t)\,\tilde{\varphi}_k^*(p,t),$$

$$\tilde{\varphi}_k(p,t) = (2\pi\hbar)^{-\frac{N}{2}} \int \varphi_k(\xi,t)\, e^{-\frac{i}{\hbar}(\xi p)}\, d\xi .$$

The set of functions $\varphi_k(\xi,t)$ is normalised as follows

$$(3) \qquad \sum_k \int \varphi_k(\xi,t)\,\varphi_k^*(\xi,t)\, d\xi = 1 .$$

In order to make culculations simpler we shall consider only stationary homogeneous isotropic and real "subquantum situation" [3]. It means that functions $\varphi_k(\vec{\xi},t)$ satisfy the following conditions

$$(4) \qquad \varphi_k(\vec{\xi},t) = \varphi_k(|\vec{\xi}|) = \varphi_k^*(|\vec{\xi}|) .$$

The correspondence rule (1), the requirements (3), (4) make possible to write down main operators connected with the angular momentum.

$$(5) \qquad \begin{cases} O(L_l) = \sum\limits_{m,n} \varepsilon_{lmn}\, \xi_m \hat{p}_n ; \quad O(\vec{L}) = [\vec{\xi} \times \hat{\vec{p}}] ; \\ O(L_l^2) = O^2(L_l) + D(L_l) ; \quad O(\vec{L}^2) = O^2(\vec{L}) + D(\vec{L}) ; \\ D(L_l) = (\delta p)^2(\vec{\xi}^2 - \xi_l^2) + (\delta \xi)^2(\hat{\vec{p}}^2 - \hat{p}_l^2) + \mathcal{L}^2 ; \\ D(\vec{L}) = 2(\delta p)^2\, \vec{\xi}^2 + 2(\delta\xi)^2\, \hat{\vec{p}}^2 + 3\mathcal{L}^2 . \end{cases}$$

Here $\hat{p}_l = -i\hbar \frac{\partial}{\partial \xi_l}$, l, m, $n = 1,2,3$, ε_{lmn} is the complctly antisymmetrical pseudotensor, $(\delta\xi)^2$, $(\delta p)^2$, $\mathcal{L}^2$ are the functionals of the auxiliary function (2)

$$(6) \qquad \begin{cases} (\delta\xi)^2 = \frac{1}{3}\int \vec{\xi}^2\, \varphi(\vec{\xi},\vec{p})\, d\vec{\xi}\, d\vec{p} , \\ (\delta p)^2 = \frac{1}{3}\int \vec{p}^2\, \varphi(\vec{\xi},\vec{p})\, d\vec{\xi}\, d\vec{p} , \\ \mathcal{L}^2 = \frac{1}{3}\int [\vec{\xi}\times\vec{p}]^2\, \varphi(\vec{\xi},\vec{p})\, d\vec{\xi}\, d\vec{p} . \end{cases}$$

All the permutational correlations of the operators (5) are determined by the commutators

$$(7) \qquad \begin{cases} [O(L_l), O(L_m)] = i\hbar \sum\limits_n \varepsilon_{lmn}\, O(L_n) ; \\ [D(L_l), O(L_m)] = 0 . \end{cases}$$

The operators $O(\vec{L}^2)$ and $O(L_l^2)$ do not commute, that is why the equation determining the state's of maximum certainty of angular momentum differs from that for its component.

2. The states of the angular momentum maximum certainty

In accordance with the result of the paper [1] the states, in which the average value of angular momentum is determined with maximum accuracy, are the solutions of the equation

$$(8) \qquad O((\vec{L} - \langle \vec{L} \rangle)^2)\, \Psi(\vec{\xi}) = d_L^2\, \Psi(\vec{\xi}) .$$

The commutators (7) and operators (5) make possible to search for the solutions of the equation (8) in the form

$$(9)\qquad \Psi(\vec{\xi}) = \Psi(\xi,\theta,\varphi) = f(\xi)\, Y_{\ell m}(\theta,\varphi),$$

where $Y_{\ell m}$ are the spherical harmonics, $\xi = |\vec{\xi}|$, $\theta = \arccos(\frac{z}{\xi})$, $\varphi = \operatorname{arctg}(y/x)$. With the help of the expression (9) and the following substitutions

$$(10)\qquad \begin{cases} u(\xi) = \xi f(\xi), \quad \xi = R\zeta = \sqrt{\dfrac{\hbar(\delta\xi)}{(\delta p)}}\,\zeta, \\ d^2_{\vec{L}} - 3\mathcal{L}^2 = 4\hbar(\delta\xi)(\delta p)B + \hbar^2(\ell^2+\ell-m^2), \end{cases}$$

from the correlation (8) we obtain the equation for the function $u(\zeta)$

$$(11)\qquad \frac{d^2}{d\zeta^2}u(\zeta) + \left(2B - \zeta^2 - \ell(\ell+1)\zeta^{-2}\right)u(\zeta) = 0.$$

Taking into account the behaviour of u at zero and in infinity we shall seek the solutions of (11) in the form

$$u(\zeta) = \exp\left\{-\frac{\zeta^2}{2}\right\}\zeta^{\ell+1}\, v(\zeta).$$

Substitution of this expression in the equation (11) leads to

$$(12)\qquad \left[\frac{d^2}{d\zeta^2} + 2\left(\frac{\ell+1}{\zeta} - 2\zeta\right)\frac{d}{d\zeta} + 2\left(B-\ell-\frac{3}{2}\right)\right]v(\zeta) = 0.$$

The solution of the equation (12) is the degenerate hipergeometrical function $F\{\frac{1}{2}(\ell+\frac{3}{2}-B), \ell+\frac{3}{2}, \zeta^2\}$, its asymptotical behaviour together with the requirement of decrement of $f(\xi)$ in infinity gives the restriction

$$(13)\qquad \ell + \frac{3}{2} - B = -2K,$$

where $K = 0,1,\ldots$. In this case the degenerate hipergeo-

metrical function $F(-K, c, x)$ is proportional to the generated Laguerre's polynomial $L^c_K(x)$. So, the states of the angular momentum maximum certainty are found

$$(14)\qquad \Psi_{\ell m K} = \frac{2^{\ell+K+1}}{(\pi R^6)^{1/4}}\sqrt{\frac{(\ell+K)!}{K!(2\ell+2K+1)!}}\left(\frac{\xi}{R}\right)^{\ell} e^{-\frac{\xi^2}{2R^2}} L_K^{\ell+1/2}\left(\frac{\xi^2}{R^2}\right) Y_{\ell m}(\theta,\varphi).$$

The correlations (10) and (13) determine the spectrum of the angular momentum and its despersion values

$$(15)\qquad \langle\vec{L}\rangle_{\ell m K} = \langle\vec{L}\rangle_m = (0, 0, m\hbar),$$

$$(16)\qquad (d^2_{\vec{L}})_{\ell m K} = \hbar^2(\ell^2+\ell-m^2) + 4\hbar(\delta\xi)(\delta p)(2K+\ell+3/2) + 3\mathcal{L}^2;$$

where $\ell = 0,1,\ldots$; $m = -\ell,\ldots,0,\ldots,\ell$; $K = 0,1,\ldots$.
The average values of the main quantities characterising the motion of a particle in the states of the angular momentum maximum certainty are the following

$$(17)\qquad \begin{cases} \langle\vec{\xi}\rangle_{\ell m K} = (0,0,0), \qquad \langle\vec{p}\rangle_{\ell m K} = (0,0,0), \\ \langle\vec{\xi}^2\rangle_{\ell m K} = R^2\left(2K+\ell+\frac{3}{2}\right) + 3(\delta\xi)^2, \\ \langle\vec{p}^2\rangle_{\ell m K} = \dfrac{\hbar^2}{R^2}\left(2K+\ell+\frac{3}{2}\right) + 3(\delta p)^2. \end{cases}$$

3. The states of maximum certainty of the angular momentum component

In this section the states, in which the angular momentum component is determined with maximum accuracy, will be obtained. Let us choose z-component for consideration, then the sought for states are the solutions of the equation (see paper [1])

$$(18)\qquad O\left((L_z-\langle L_z\rangle)^2\right)\Psi(\vec{\xi}) = d^2_{L_z}\,\Psi(\vec{\xi}).$$

Equalities (5), (7) make possible to find the solutions of equation (18) in the form

$$\Psi(\vec{\xi}) = \Psi(z,\rho,\varphi) = Z(z)\,u(\rho)\,e^{im\varphi},$$

where $\rho=\sqrt{x^2+y^2}$, $\varphi = \operatorname{arctg}(y/x)$.

The procedure of obtaining solutions of equation (18) is the same as it is for equation (8). After substitutions

$$(19)\qquad \begin{cases} \rho = \sqrt{\dfrac{\hbar(\delta\xi)}{(\delta p)}}\;\xi = R\xi, \\ d^2_{L_z} - \mathcal{L}^2 = 2\hbar(\delta\xi)(\delta p)C, \end{cases}$$

it is not so difficult to get the eigenstates and eigenvalues of equation (18).

$$(20)\qquad \langle L_z\rangle_{mk} = \langle L_z\rangle_m = m\hbar,$$

$$(21)\qquad (d^2_{L_z})_{mk} = 2\hbar(\delta\xi)(\delta p)(2k+|m|+1) + \mathcal{L}^2;$$

$$(22)\qquad \Psi_{mk} = \frac{Z(z)}{\sqrt{2\pi}\,R}\sqrt{\frac{2}{k!(k+|m|)!}}\left(\frac{\rho}{R}\right)^{|m|} e^{-\frac{\rho^2}{2R^2}} L_k^{|m|}\left(\frac{\rho^2}{R^2}\right) e^{im\varphi},$$

where $m=0,\pm1,\ldots$; $k=0,1,\ldots$.

The average values of the main dynamic quantities in states (22) are the following

$$(23)\qquad \begin{cases} \langle y\rangle_{mk} = \langle x\rangle_{mk} = 0, \qquad \langle p_x\rangle_{mk} = \langle p_y\rangle_{mk} = 0, \\ \langle x^2+y^2\rangle_{mk} = R^2(2k+|m|+1) + 2(\delta\xi)^2, \\ \langle p_x^2+p_y^2\rangle_{mk} = \dfrac{\hbar^2}{R^2}(2k+|m|+1) + 2(\delta p)^2. \end{cases}$$

The z-component values of coordinate and momentum remain arbitrary because of the function $Z(z)$ arbitrariness.

Formulae (16) and (21) determine the minimum possible dispersion ("subquantum uncertainty") of angular momentum

$$(24)\qquad (\delta L)^2 = 2\hbar(\delta\xi)(\delta p) + \mathcal{L}^2,$$

i.e. the dispersion values in any state are subject to the limitations

$$\langle(\Delta\vec{L})^2\rangle \geq 3(\delta L)^2, \qquad \langle(\Delta L_z)^2\rangle \geq (\delta L)^2.$$

Expression (24) establishes the physical meaning of the functional $\mathcal{L}^2$ (6).

Note that the average value of the square of angular momentum differs from zero if $\ell = 0$. (In the generally accepted quantum mechanics this value equals zero.) The interpretation of this result and appearence of a new quantum number k may be found in the further investigation.

Acknowledgements

The author expresses his sincere gratitude to Drs. Yu. I. Zaparovanny and V. V. Kuryshkin for guidance, help and encouragement.

References

[1] V.V. Kuryshkin, I.A. Lyabis, Y.I. Zaparovanny, Ann. Fond. Louis de Broglie, 3, 45, 1978.

[2] V.V. Kuryshkin, Ann. Inst. H. Poincaré, v. XVII, 81, 1972.

[3] V.V. Kuryshkin, Int. J. Theor. Phys., 7, 451, 1973.