Annales de la Fondation Louis de Broglie - Vol. 4, n° 4, 1979

ZERO-POINT ENERGY, PLANCK'S LAW AND THE PREHISTORY OF STOCHASTIC ELECTRODYNAMICS PART 1 : EINSTEIN AND HOPF'S PAPER OF 1910

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(manuscrit reçu le 30 Novembre 1978)

Abstract : The development of quantum physics is far less direct than as it is described in popular reconstructions. It is not generally known, for instance, that in 1913 Einstein and Stern had derived Planck's law from the hypothesis of a zero-point energy, without the need of a specific quantization postulate. We give here a commented translation of an article by Einstein and Hopf, which is the logical premise of the Einstein-Stern paper ; the translation of the latter is postponed to a forthcoming paper. We think that putting these texts at the disposal of a wider reading public might be particularly interesting on the occasion of the centenary of Einstein's birth, and for promoting investigations into a side aspect of his research work. Furthermore, these early studies may be considered as the first stages of the development of stochastic electrodynamics, a theory which has been recently put forward as a possible alternative to quantum mechanics and has received considerable attention in this journal.

Résumé : Le développement de la physique quantique est beaucoup moins linéaire qu'on le représente communément. On ne connaît généralement pas, par exemple, qu'en 1913 Einstein et Stern avaient

dérivé la loi de Planck en partant de l'hypothèse de l'énergie au zéro absolu, sans nécessité d'introduire un postulat indépendant de quantification. Nous donnons ici une traduction commentée, en anglais, d'un article par Einstein et Hopf, qui constitue la prémisse logique à l'ouvrage de Einstein et Stern, tout en renvoyant celle de cette dernière à un article à paraître. Il nous semble que rendre possible la lecture de ces ouvrages à un public plus étendu pourrait être particulièrement intéressant à l'occasion du centenaire de la naissance d'Einstein et stimuler, dans cette circonstance, l'investigation d'un aspect collatéral de son activité de recherche. Comme autre motif d'intérêt, nous soulignons que ces études peuvent être considérées comme une préhistoire de l'électrodynamique stochastique, une théorie qui a été proposée comme une alternative possible à la mécanique quantique et qui a requis l'attention de cette revue auparavant.

1. Introduction

It is a widespread conviction that the advent of quantum physics was determined by the utter failure which classical physics met in trying to account for the results concerning the black-body spectrum, the specific heats of solids and the corpuscular aspects of radiation. Planck's law, in particular, appears thus as an absolute necessity imposed by experimental facts. From a *historical* point of view the rise of quantum mechanics has been extensively analysed and shown to be far less direct[1] than as accounted for in popular reconstructions. The above conviction has, however, to be modified also from a *logical* point of view. On the one hand, the failure of classical physics concerning, for instance, the specific heat problem, is closely linked to the use of the equipartition theorem ; but this theorem, as has been widely discussed[2], was never rigorously justified for the cases in question. On the other hand, it has been shown that Planck's law may be derived from the assumption of a zero-point energy with no need for an independent quantum hypothesis[3]. As Boyer himself acknowledges[4], this latter result had already been obtained by Einstein and Stern in 1913[5]. It seems relevant to recall this fact for two orders of reasons. In the first place, to stress once more the complex nature of the growth of scientific knowledge, in which ideas and results are sometimes put aside for no apparent, strictly logical, motivation, often in the presence of the advent of a positive practice determined by the success of different formulations[6]. In the second place, to trace the historical origins of what has now become a field of research, with its own methodology and traditions, that is, stochastic electrodynamics, which came to life in the early papers by Welton[7a], Braffort and Tzara[7b] and

Marshall[7c] and was recently developed by several authors[8]. We may recall that stochastic electrodynamics is a classical electron theory implemented with the hypothesis of a random classical zero-point radiation. Lorentz-invariance fixes the energy per degree of freedom proportional to the frequency ; Planck's constant enters into the theory as the proportionality constant.

It therefore seemed appropriate to us to document these facts by providing an accurate translation of the papers by Einstein and co-workers on this topic. We say papers since the above-mentioned article by Einstein and Stern does in fact have its formal and logical premises in a preceding work by Einstein and Hopf[9]. This phase of Einstein's activity has not received particular attention by the scholars (see, however, reference 1). Our contribution aims also at promoting investigations into a side aspect of Einstein's research work. This may acquire further relevance in the occurrence of the centenary of his birth, which is an invitation to an extended reflection on his scientific personality as a whole.

Einstein and Hopf's paper derives its motivation from the following consideration : that the current derivations of the Rayleigh-Jeans law, in which classical physics meets with failure, are faulty, owing to an acritical use of the equipartition theorem. Einstein and Hopf aim at freeing the derivation of this flare. They are actually able to re-derive the Rayleigh-Jeans formula through a procedure which, to their eyes, no longer lays itself open to criticism : hence classical physics does indeed lead to the result which is the touchstone of its failure. One may say that, at this stage, Einstein's (and Hopf's) attitude is that of clinching the necessity of the quantum hypothesis. At a later stage, taking note of Planck's so-called second theory[10] and taking over his hypothesis of the zero-point energy, Einstein and Stern, in a paper of wider scope, repeat the calculation obtaining, instead of Rayleigh's, Planck's law, which thus appears, in a certain sense, to be "classically" based. Should one then conclude that Einstein had in the meanwhile changed his attitude towards the quantum hypothesis ? This conclusion does not seem documentable on the basis of the published papers. It should be added that, apart from a paper by Nernst[11] of 1916 (which we intend to discuss in a subsequent paper), Einstein and Stern's arguments were dropped by the authors themselves and did not undergo further developments until the recent revival we mentioned above.

In this first part we provide an English version of Einstein and Hopf's paper, postponing that of Einstein and Stern's work to a forthcoming paper. While perusing these articles it soon became apparent that for a real understanding it was necessary to re-do all the calculations. There are two reasons for this : in the first place, throughout the authors give only the results of calculations wich are often cumbersome, though not difficult ; in the second place, the notation is often irksome when it does not appear to be altogether incorrect for today's standards. We may recall that Boyer, in his paper where Einstein and Stern's results are re-derived[3], writes explicitly : "Notations involving Lorentz transformations have changed so much in the half-century since Einstein and Hopf's work that the present author found their manipulations on the surface incomprehensible. Repeating the full calculations, the author arrived at values identical with their results". Throughout this process of reconstruction we found several mistakes in the formulae ; in the majority of cases it was a matter of misprints, which, since there is no record of the intermediate steps, nonetheless make reading difficult ; there are also, however, some minor formal errors which have no bearing on the final results, and which we have pointed out in the comments. We thought reading would be made easier by the insertion of some further comments on the essential steps of the calculations, updated in the notation. Words or phrases in square brackets are either additions required in the English version or original German words recalled to stress the impossibility of an adequate English rendering.

The translation itself is the work of one of the present authors (S.B.). The English rendering was improved with the help of Dr. G. Venturi and Mrs. G. Beale, to whom we express our sincere thanks, and further checked against the German text to preserve the flavour of the original language as much as possible.

Statistical investigation of the motion of a resonator in a radiation field, by A. Einstein and L. Hopf

1. Line of thought   It has already been shown in several ways and is today generally accepted that our present views on the distribution and emission of electromagnetic energy on the one hand, [and] on the statistical distribution of energy on the other hand, cannot, through correct use in the radiation theory, lead to anything else but the so-called Rayleigh (Jeans) radiation law. Since this stands in total contradiction with the experiment, it is necessary to carry out an alteration in the foundations of the theories applied to the derivation and people have repeatedly presumed[12] that the application of the law of the statistical distribution of energy to radiation or to rapidly oscillating motions (resonators) may not be devoid of objections. The following investigation shall now show that there is no need for a dubious use of this sort and that it is sufficient to use the principle for the *translatory*

motion of molecules and oscillators to arrive at Rayleigh's radiation law. The applicability of the principle to the translatory motion is sufficiently proved through the achievements of the kinetic theory of gases ; we shall therefore be able to conclude that only a more fundamental and deep-reaching change in the current views can lead to a radiation law which corresponds better to the experiment.

We treat an electromagnetic oscillator[13] in motion which on the one hand undergoes the actions [Wirkungen] of a radiation field, and on the other hand is loaded [behaftet] with a mass m and enters in interaction with the molecules existing in the space [occupied by] the radiation. If only this latter interaction existed, the mean square value of the momentum of the translatory motion of the oscillator would be completely determined through statistical mechanics. In our case there also exists the interaction of the oscillator with the radiation field. In order that statistical equilibrium be possible this latter interaction cannot alter that mean value. In other words : the mean square value of the momentum of the translatory motion, which the oscillator assumes under the influence *of the radiation alone*, must be the same as that which it would assume, on the basis of statistical mechanics, under the mechanical influence of the molecules alone[14]. The problem is thus reduced to that of ascertaining the mean square value $\overline{(mv)^2}$ of the momentum that the oscillator assumes under the influence of the radiation field alone.

This mean value must be the same at the time t = 0 as at the time t = τ, so that one has :

$$\overline{(mv)^2_{t=0}} = \overline{(mv)^2_{t=\tau}}$$

For the following it is convenient to distinguish [between] actions of two sorts[15] through which the radiation field influences the oscillator, namely

1.) The reaction K that the radiation pressure opposes to a rectilinear motion of the oscillator. This, if terms of the order of $\left(\frac{v}{c}\right)^2$ are neglected, is proportional to the velocity v ; we may thus write K = -Pv. If we further assume that during the time τ the velocity v cannot change observably then the impulse deriving from this force becomes = -P v τ.

2.) The oscillations Δ of the electromagnetic impulse which intervene as a consequence of the motion of electric masses in the disordered radiation field. These may be positive as well as negative and are, in first approximation, independent of the circumstance that the oscillator is in motion.

These impulses superpose themselves during the time τ on the impulse $(mv)_{t=0}$ and our equation becomes :

$$(1) \qquad \overline{(mv)^2_{t=0}} = \overline{(mv_{t=0} + \Delta - P\, v\, \tau)^2}$$

By increasing the mass m we can generally achieve [the result] that the term multiplied by τ^2 which appears in the right-hand side of Equation (1) may be neglected. Furthermore the term multiplied by $\overline{v\Delta}$ vanishes, as v and Δ can, in complete mutual independence, become positive as well as negative. If we further replace $\overline{mv^2}$ through the temperature Θ by means of the equation known from the theory of gases :

$$\overline{mv^2} = \frac{R}{N}\,\Theta$$

(R = absolute gas constant, N = Loschmidt's number), then Equation (1) takes on the form :

$$(2) \qquad \overline{\Delta^2} = 2\,\frac{R}{N}\,P\,\Theta\,\tau .$$

We have thus only to ascertain $\overline{\Delta^2}$ and P (i.e. K) through electromagnetic considerations ; then Equation (2) yields the radiation law.

2. Computation of the force $\overline{K}$[16] In order to compute the force which the radiation opposes to an oscillator in motion, let us first compute the force on an oscillator at rest and then transform this, with the aid of the formulae which follow from the theory of relativity.

Let the oscillator of proper frequency ν_0 freely oscillate in the z-direction of an orthogonal reference system x, y, z. Furthermore, if $\vec{E}$ and $\vec{H}$ respectively denote the electric and magnetic forces [Kraft] of the external field, then the [electric dipole] moment f of the oscillator obeys, according to Planck, the differential equation[17]

$$(3) \qquad 16\,\pi^4\,\nu_0^3\,f + 4\,\pi^2\,\nu_0\,\ddot{f} - 2\,\sigma\,\dddot{f} = 3\,\sigma\,c^3\,E_z$$

In addition, here σ is a constant characteristic for the damping of an oscillator through radiation.

Now let a plane wave fall[18] onto the oscillator ; let the

ray form an angle ω with the x-axis. If we split this wave into two waves polarized at right angles to each other, such that the electric force of the former lies in the plane of the oscillator and the ray, [and] that of the latter at right angles to it, then it is clear that only the first imparts a certain momentum to the oscillator. If we write the electric force of this first wave as a Fourier series[19]

$$(4) \qquad E = \sum_n A_n \cos\left[\frac{2\,\pi\,n}{T}\left(t - \frac{\alpha x + \beta y + \gamma z}{c}\right) - \theta_n\right],$$

whereby T means a very large time, then the direction cosines α, β, γ, of the ray are expressed through ϕ and ω in the following way :

$$\alpha = \sin\phi\,\cos\omega, \quad \beta = \sin\phi\,\sin\omega, \quad \gamma = \cos\phi$$

and the components of the electric and magnetic forces coming into consideration for our later computation are[20] :

$$(5) \qquad \begin{cases} E_x = E\,\cos\phi\,\cos\omega, \\ E_z = -E\,\sin\phi, \\ H_y = E\,\cos\omega . \end{cases}$$

The ponderomotive force which is exerted on the oscillator is[21]

$$\vec{R} = f\,\frac{\partial \vec{E}}{\partial z} + \frac{1}{c}\left(\frac{d\vec{f}}{dt} \wedge \vec{H}\right)$$

In order that this equation, as well as Equation (3), be exact, it must be assumed that the dimensions of the oscillator are always small with respect to the wave-length of the radiation coming into consideration. The x-component K_x of the ponderomotive force is

$$(6) \qquad R_x = \frac{\partial E_x}{\partial z}\,f - \frac{1}{c}\,H_y\,\frac{df}{dt}$$

By solving (3)[22], we obtain[23], by taking (4) and (5) into account :

$$f = -\frac{3c^3}{16\pi^3}\,T^3\,\sin\phi\,\sum_n A_n\,\frac{\sin\gamma_n}{n^3}\,\cos\left(\tau_n - \gamma_n\right)$$

$$\dot{f} = \frac{3c^3}{8\pi^2}\,T^2\,\sin\phi\,\sum_n A_n\,\frac{\sin\gamma_n}{n^2}\,\sin\left(\tau_n - \gamma_n\right)$$

whereby, for abbreviation,

$$\tau_n = 2\pi n \frac{t}{T} - \theta_n$$

is set and γ_n is given through the equation

$$\cot g\, \gamma_n = \frac{\pi}{\sigma} \frac{\nu_0(\nu_0^2 - n^2/T^2)}{n^3/T^3}$$

Since, furthermore[24] :

$$\frac{\partial E_x}{\partial z} = \frac{2\pi}{cT} \cos^2\phi \cos\omega \sum_n A_n n \sin\tau_n \qquad [25]$$

R_x appears as the double sum :

$$R_x = -\frac{3c^2}{8\pi^2} T^2 \cos^2\phi \sin\phi \cos\omega \sum_n \sum_m A_n \frac{\sin\gamma_n}{n^3} A_m\, m \cos(\tau_n - \gamma_n) \sin\tau_m -$$

$$-\frac{3c^2}{8\pi^2} T^2 \sin\phi \cos\omega \sum_n \sum_m A_n \frac{\sin\gamma_n}{n^2} A_m \sin(\tau_n - \gamma_n) \cos\tau_m$$

In the formation of the mean value, owing to the mutual independence of the phase angles, only the terms with n = m come into consideration[26] and

$$\overline{R_x} = \frac{3c^2}{16\pi^2} T^2 \sin^3\phi \cos\omega \sum_n A_n^2 \frac{\sin^2\gamma_n}{n^2} =$$

(7)

$$= \frac{3c^2}{16\pi^2} \overline{A^2_{\nu_0 T}}\, T \frac{\sigma}{2\nu_0} \sin^3\phi \cos\omega \text{ [27]}$$

is obtained[28]. This is the mean value of the x-component of the force which a wave incident in direction ϕ, ω exerts on the oscillator at rest.

If the oscillator is moving in the x-direction with a velocity v, then we replace the angles ϕ, ω, as [they are] more convenient, through the angle ϕ_1 between the ray and the x-axis and the angle ω_1 between the projection of the ray on the yz plane and y-axis. Then the [following] relations hold :

$$\cos\phi_1 = \sin\phi \cos\omega$$

$$\sin\phi_1 \cos\omega_1 = \sin\phi \sin\omega$$

$$\sin\phi_1 \sin\omega_1 = \cos\phi$$

For [computing] the value of the force $\overline{R_{x'}}$ which operates on the moving oscillator, we carry out the transformation formulae[29] of the theory of relativity[30]

$$A' = A\left\{1 - \frac{v}{c}\cos\phi_1\right\},$$

$$T' = T\left\{1 + \frac{v}{c}\cos\phi_1\right\},$$

$$\nu' = \nu\left\{1 - \frac{v}{c}\cos\phi_1\right\},$$

$$\cos\phi_1' = \frac{\cos\phi_1 - \frac{v}{c}}{1 - \frac{v}{c}\cos\phi_1}, \qquad \omega_1' = \omega_1.$$

[Then]

$$\overline{R_{x'}} = \frac{3c^2}{16\pi^2} \overline{A'^2_{\nu'_0 T'}}\, T' \frac{\sigma}{2\nu'_0} (1 - \sin^2\phi_1' \sin^2\omega_1')\cos\phi_1'$$

is obtained. Now, when terms in $(v/c)^2$ are neglected, it is :

$$\overline{A'^2_{\nu'_0 T'}} = \overline{A^2_{\nu_0 T}} \left(1 - 2\frac{v}{c}\cos\phi_1\right)$$

or, since we have to refer everything to the proper frequency ν_0' of the moving oscillator[31] :

$$\overline{A'^2_{\nu'_0 T'}} = \overline{A^2_{\nu'_0}}\left\{1 + \frac{v}{c}\cos\phi_1\right\}T \left(1 - 2\frac{v}{c}\cos\phi_1\right) =$$

$$= \left\{\overline{A^2_{\nu'_0 T}} + \nu'_0 \frac{v}{c}\cos\phi_1 \left(\frac{d\overline{A^2}}{d\nu}\right)_{\nu'_0 T}\right\}\left(1 - 2\frac{v}{c}\cos\phi_1\right)$$

Further on we express the quantity $\overline{A^2 T}$ through the mean radiation density ρ. The mean energy of a plane wave, which comes from a definite direction, we set equal to the energy density in a cone of solid angle $d\Omega$. If, in addition, we take heed of the equality of the electric and magnetic forces and of both polarization planes, then we arrive at the relation[32] :

$$\rho \frac{d\Omega}{4\pi} = \frac{1}{8\pi} \frac{\overline{A^2 T}}{2} .2.2$$

Our expression for the force becomes :

8) $\overline{R'_x} = \frac{3c^2}{16\pi^2}\frac{\sigma}{2\nu'_0}\left\{\rho_{\nu'_0}+\nu'_0\frac{v}{c}\cos\phi_1\left(\frac{d\rho}{d\nu}\right)_{\nu'_0}\right\}\left(\cos\phi_1 - \frac{v}{c}\right)$

$$\left(1 - \frac{\sin^2\phi_1}{1-2\frac{v}{c}\cos\phi_1}\sin^2\omega_1\right)d\Omega$$

If we finally integrate over all solid angles[33], then we obtain the total force sought :

$$(9)\quad \overline{K} = -\frac{3c\sigma}{10\pi\nu'_0}\,v\left\{\rho_{\nu'_0} - \frac{\nu'_0}{3}\left(\frac{d\rho}{d\nu}\right)_{\nu'_0}\right\}$$

3. Computation of the impulse-oscillations $\overline{\Delta^2}$ The computation of the impulse-oscillations may be considerably simplified with respect to the computation of the force since a transformation according to the theory of relativity is unnecessary[34]. It is sufficient to develop the electric and magnetic forces in the origin, as if [they were] solely dependent on time, in a Fourier series, when it is only possible to carry out the proof that the single components of the force which intervene in this expression are mutually independent.

The impulse that the oscillator experiences in the time τ in the x-direction is :

$$J = \int_0^\tau K_x\,dt = \int_0^\tau\left(\frac{\partial E_x}{\partial z}f - \frac{1}{c}H_y\frac{df}{dt}\right)dt$$

Partial integration produces :

$$\int_0^\tau H_y\frac{df}{dt}\,dt = \left(H_y f\right)_0^\tau - \int_0^\tau\frac{\partial H_y}{\partial t}f\,dt$$

The first integrand vanishes, when τ is suitably chosen, i.e. when τ is sufficiently large[35]. If, in addition, one sets -according to Maxwell's equation-

$$\frac{1}{c}\frac{\partial H_y}{\partial t} = \frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}$$

then one arrives at the simple expression :

$$10)\quad J = \int_0^\tau\frac{\partial E_z}{\partial x}f\,dt$$

Now only the component E_z and its derivative $\partial E_z/\partial x$ enter into our expression. Their independence cannot, however, be easily proved. Since we treat only two wave-trains travelling in opposite directions (of equal solid angle), then we can write :

$$E_z = \sum\left\{a_n\sin\frac{2\pi n}{T}\left(t - \frac{\alpha x+\beta y+\gamma z}{c}\right) + b_n\cos\frac{2\pi n}{T}\left(t - \frac{\alpha x+\beta y+\gamma z}{c}\right)\right.$$
$$\left. + a'_n\sin\frac{2\pi n}{T}\left(t + \frac{\alpha x+\beta y+\gamma z}{c}\right) + b'_n\cos\frac{2\pi n}{T}\left(t + \frac{\alpha x+\beta y+\gamma z}{c}\right)\right\}$$

and

$$\frac{\partial E_z}{\partial x} = \sum\left\{\frac{2\pi n\alpha}{Tc}\left(-a_n\cos\frac{2\pi n}{T}(\ldots) + b_n\sin\frac{2\pi n}{T}(\ldots)\right.\right.$$
$$\left.\left. + a'_n\cos\frac{2\pi n}{T}(\ldots) - b'_n\sin\frac{2\pi n}{T}(\ldots)\right)\right\}$$

The quantities $a_n + a_{n'}$, $a_n - a_{n'}$, ... are mutually independent and of the same character as those denoted by S in the preceding essay[36] ; there, for such [quantities], it is proved that the probability law of a combination is represented as a product of Gauss's error functions of the single quantities. From the aforesaid it is easily concluded that between the coefficients of the developments E_z and $\partial E_z/\partial x$ no probability relation whatsoever can hold.

We now express E_z and $\partial E_z/\partial x$ as a Fourier series :

$$E_z = \sum_n B_n\cos\left(2\pi n\frac{t}{T} - \theta_n\right)$$
$$\frac{\partial E_z}{\partial x} = \sum_m C_m\cos\left(2\pi m\frac{t}{T} - \xi_m\right)$$

Then

$$f = \frac{3c^3}{16\pi^3}T^3\sum_n B_n\frac{\sin\gamma_n}{n^3}\cos\left(2\pi n\frac{t}{T} - \theta_n - \gamma_n\right)$$

and

$$J = \frac{3c^3}{16\pi^3} T^3 \int_0^\tau dt \sum_m \sum_n C_m B_n \frac{\sin\gamma_n}{n^3} \frac{1}{2}\left(\cos\left\{2\pi(n+m)\frac{t}{T} - \xi_m - \theta_n - \gamma_n\right\} + \right.$$

$$\left. + \cos\left\{2\pi(n-m)\frac{t}{T} + \xi_m - \theta_n - \gamma_n\right\}\right)$$

are obtained. In the integration over t two integrands with the factors $1/(n + m)$ and $1/(n - m)$ are obtained ; as n and m are very large numbers, the former is very small and can therefore be neglected. On thus arrives at the expression[37] :

$$(11) \quad J = \frac{3c^3}{32\pi^4} T^4 \sum_m \sum_n C_m B_n \frac{\sin\gamma_n}{n^3} \frac{1}{n-m} \cos\delta_{mn} \sin\pi(n-m)\frac{\tau}{T}$$

with the abbreviation

$$\delta_{mn} = \pi(n - m)\frac{\tau}{T} + \xi_m - \theta_n - \gamma_n .$$

J^2 appears then as a multiple sum over n, m and two more variables n' and m'. If we construct the mean value $\overline{J^2}$, then we have to pay attention [to the fact] that the angles δ_{mn} and $\delta_{m'n'}$ are completely independent of one another ; therefore, in constructing the mean value only the terms for which this independence is nullified come into consideration. Visibly this is only the case when

$$m = m' \quad \text{and} \quad n = n' ;$$

we arrive at the mean value sought :

$$\overline{J^2} = \left(\frac{3c^3\ T^4}{32\pi^4}\right)^2 \sum_m \sum_n \frac{1}{2} C_m^2 B_n^2 \left(\frac{\sin\gamma_n}{n^3}\right)^2 \frac{1}{(n-m)^2} \sin^2\pi(n-m)\frac{\tau}{T} ;$$

since

$$\sum_m \frac{1}{(n-m)^2} \sin^2\pi(n-m)\frac{\tau}{T} = \frac{1}{T}\int_0^\infty \frac{1}{(\nu-\mu)^2}\sin^2\{(\nu-\mu)\pi\tau\}d\mu = \frac{\pi^2\tau}{T}$$

and

$$\sum_n \frac{\sin^2\gamma_n}{n^6} = \frac{1}{T^5}\int_0^\infty \frac{\sin^2\gamma_n}{\nu^6} d\nu = \frac{1}{T^5}\frac{\sigma}{2\nu_0^5} ,$$

$$12) \quad \overline{J^2} = \left(\frac{3c^3}{32\pi^3}\right)^2 \frac{\sigma\tau}{4\nu_0^5} \overline{B^2_{\nu_0 T}}\ \overline{C^2_{\nu_0 T}}\ T^2$$

is obtained. Now it is :

$$\overline{J^2} = \overline{(\bar{J} + \Delta)^2} = \overline{J}^2 + 2\bar{J}\bar{\Delta} + \overline{\Delta^2}$$

and since the mean values $\bar{J}$ and $\bar{\Delta}$ vanish, expression (12) gives the value of the impulse-oscillations Δ^2 themselves. The mean values of the quantities $B^2_{\nu_0 T}$ and $C^2_{\nu_0 T}$ remain to be expressed through the radiation density ρ_{ν_0}.

To this end we must again treat the radiation coming from different directions and, as above, set the amplitude of the radiation coming from a definite direction in relation to the energy through the equation :

$$\overline{A^2_{\nu_0 T}}\ T = \rho_{\nu_0}\ d\Omega$$

The amplitude

$$B_{\nu_0 T} = \sum A_{\nu_0 T} \sin\phi$$

[must be integrated] over all the angles of incidence[38] ; thus

$$13) \quad \overline{B^2_{\nu_0 T}}\ T = \overline{A^2_{\nu_0 T}}\ T \sum \sin^2\phi = \frac{8}{3}\pi\rho_{\nu_0}$$

Similarly[39]

$$14) \quad \overline{C^2_{\nu_0 T}}\ T = \left(\frac{2\pi\nu}{c}\right)^2 \overline{A^2_{\nu_0 T}}\ T \sum \sin^4\phi\cos^2\omega = \frac{64}{15}\frac{\pi^3\nu_0^2}{c^2}\rho_{\nu_0}$$

is obtained. So, by inserting (13) and (14) in (12), we finally obtain :

$$15) \quad \overline{\Delta^2} = \frac{c^4\ \sigma\ \tau}{40\pi^2\nu_0^3}\rho^2_{\nu_0}$$

4. The radiation law Now, in addition, we only have to insert values (9) and (15) found in our equation (2) ; so we arrive at the differential equation containing the radiation law :

$$\frac{c^3 N}{24\pi R\Theta\nu^2}\rho^2 = \rho - \frac{\nu}{3}\frac{d\rho}{d\nu}$$

which, integrated, gives :

$$(16) \quad \rho = \frac{8\pi R\Theta\nu^2}{c^3 N}$$

This is the well-known Rayleigh's radiation law, which stands in striking contrast with the experience. In the foundations of our derivation must thus hide an assertion which does not find itself in harmony with the real phenomena concerning temperature radiation.

Let us therefore deal more closely and critically with these foundations :

People have wanted to find grounds for the fact that all the exact statistical treatments in the field of the radiation doctrine [Lehre] lead to Rayleigh's law in the very application of these procedures to radiation. Planck[40], with good reason, opposes this argument to Jean's derivation. In the above derivation, however, it was certainly not the case of a somewhat arbitrary transposition of statistical considerations to radiation ; the principle of equipartition of the energy was only applied to the translatory motion of the oscillator. The achievements of the kinetical theory of gases show, however, that for the translatory motion this principle can be considered as thoroughly demonstrated.

The basic theoretical argument used in our derivation, which must contain an unfounded assumption, is thus nothing else but that based on the light dispersion theory for completely transparent bodies. The real phenomena are distinguished from the results to be inferred from this basic argument in that in the former impulse-oscillations of another sort are also made perceptible, which for short-wave radiation of low density vastly prevail over those yielded by the theory[41].

REFERENCES AND NOTES

1) We think, in particular, of the writings by M.J. Klein : M.J. Klein, Science, 148, 173 (1965) ; and : lectures given at the International School "E. Fermi", Course LVII, Academic Press, New York (1977) ; see also : M. Jammer, *The Conceptual Development of Quantum Mechanics*, McGraw Hill, New York (1966).

2) For a review of these matters see, for instance : L. Galgani, A. Scotti, Rivista del Nuovo Cimento 2, 189 (1972).

3) This point has been particularly stressed by Boyer ; see : T.H. Boyer, Phys. Rev. 182, 1374 (1969).

4) In : T.H. Boyer, Phys. Rev. D, 1, 2257 (1970). Boyer, however, stresses the existing differences in the derivation and points out what he judges to be an inconsistency in Einstein and Stern's derivation. We shall come back to this matter in the following.

5) A. Einstein and O. Stern, Annalen der Physik, 40, 551 (1913).

6) We make wide reference to the debate reported in *Criticism and the Growth of Knowledge*, edited by A. Musgrave and I. Lakatos, Cambridge University Press (1970).

7a) T.A. Welton, Phys. Rev. 74, 1157 (1948).

b) P. Braffort and C. Tzara, C. R. Acad. Sc. (Paris) 239, 1179 (1954).

c) T.W. Marshall, Proc. Roy. Soc. 276A, 475 (1963).

8) See, in particular : T.H. Boyer (op. cit.) ; M. Surdin, Ann. Inst. H. Poincaré , 15A, 203 (1971) ; E. Santos, N.C. 19B, 57 (1974) ; L. De la Peña-Auerbach and A.M. Cetto, J. Math. Phys. 18, 1612 (1977). We refer to the following review papers for further references : T.H. Boyer, Phys. Rev. D11, 790 (1975) ; P. Claverie and S. Diner, Ann. Fond. L. de Broglie 1, 73 (1976) ; *Stochastic Electrodynamics and Quantum Theory*, Proceedings of the Symposium on : "Quantum Chemistry -a scientific melting pot", to be published in a special issue of the Int. Journ. of Quantum Chemistry ; A.M. Cetto, P. Claverie, S. Diner,

L. De la Peña-Auerbach and E. Santos, in preparation.

9) A. Einstein and L. Hopf, Annalen der Physik, 33, 1105 (1910).

10) This theory is exposed in Planck's *Quantum Theory of Radiation*, Dover, New York (1959).

11) W. Nernst, Ver. d. Deut. Phys. Ges. 18, 83 (1913).

12) The English theorists, in particular Rayleigh and Jeans, had come back to this matter over and over since 1900. Already in his historical paper, where he had first applied the equipartition theorem to ether and ponderable matter [Lord Rayleigh, *Remarks upon the Law of Complete Radiation*, Phil. Mag., 5th series, XLIX, 539 (1900)] , Rayleigh, after recalling that "according to this doctrine every mode of vibration should be alike favoured" and stressing that "for some reason not yet explained the doctrine fails in general", conjectures that "it may apply to the graver modes". The problem came into focus after the appearance of the first edition of Jeans's book on *The dynamical theory of gases*, where, according to Rayleigh, "he attacks the celebrated difficulty of reconciling the "law of equipartition of energy with what is known respecting the specific heat of gases" and "shows that in an approximately steady state the energy of vibrational modes may bear a negligible ratio to that of the translational and rotational modes" [Lord Rayleigh, Nature, vol. LXXI, 559 (1905)] . The difficulty seems to revive for Rayleigh (ibidem) "when we consider a gas, not radiating into empty space, but bounded by a perfectly reflecting enclosure". Jeans's answer [J.M. Jeans, Nature, vol. LXXI, 607 (1905)] is resumed in a subsequent paper of Rayleigh's [Lord Rayleigh, Nature, vol. LXXII, 54 (1905)] as follows : "The various modes of molecular motion are divided into two sharply separated groups. Within one group, including the translatory modes, equipartition of energy is supposed to establish itself within a small fraction of a second ; but between the modes of this group, equipartition may require, Mr. Jeans thinks, millions of years". After reexamining the whole matter and discussing Planck's law, Rayleigh concludes by expressing his "dissatisfaction with the way in which great potential energy is dealt with in the general theory leading to the law of equipartition". The discussion on the applicability of the equipartition theorem remained alive for the whole period preceding the appearance of Einstein and Hopf's paper [see, in particular, the classical papers by Jeans in : Phil. Mag., 6th series, vol. 10, 91 (1905) and : Phil. Mag., 6th series, vol. 17, 229 (1909)] and went on even after, in particular at the first Solvay Conference, where it was discussed in detail by Lorentz and Jeans [see : H.A. Lorentz, *Sur l'application au rayonnement du théorème de l'équipartition de l'énergie* ; J.M. Jeans, *La théorie cinétique de la chaleur spécifique d'après Maxwell et Boltzmann* ; in : *La théorie du rayonnement et les quanta* ; publiés par MM. P. Langevin et M. de Broglie, Gauthier-Villars, Paris (1912)] .

13) For the sake of simplicity we shall assume that the oscillator swings only in the z-direction and moves only in the x-direction (Note of the authors).

14) We can further persuade ourselves of the correctness of the authors' conclusion on the basis of the following consideration : we can imagine an enclosure divided by an ideal partition into two zones, the former containing thermal radiation, the latter a gas of particles, both at a temperature T and separately at equilibrium with an oscillator like the one described in the paper. It is evident that removing the partition does not alter the equilibrium.

15) For a discussion on these points, see, for instance, R. Becker, Theorie der Elektrizität, Band II, Verlag und Druck Von B.G. Teubner, Leipzig und Berlin (1933), p. 388 ff.

16) See also M. Abraham, Ann. d. Phys. 14, p. 273 ff. (1904). (Note of the authors).

17) See, for instance : W.K.H. Panofsky and M. Phillips, *Classical Electricity and Magnetism*, Addison-Wesley Publishing Company, Cambridge (1955), p. 322 ff., in particular formula 21-16, with the identification :

$$\sigma = \frac{4\pi\nu_0 e^2}{3m\ c^3}$$

σ is directly related to the parameter $\tau_{st.}$ of stochastic electrodynamics :

$$\tau_{st.} = \frac{2e^2}{3m\ c^3}$$

18) Actually the wave must be "outgoing", if α, β, γ are to be its direction cosines. It is by this choice that H_y has a positive sign and the terms in $\sin\phi$ in $\partial E_x/\partial z$ are cancelled out (see

below).

19) As the authors will specify below, the phases θ_n are "at random", i.e. they are, as we would say, random variables. The field E is therefore a stochastic process. This is an essential physical hypothesis that is introduced in the discussion. It is rooted in the assumption typical of the first phase of the Planckian studies, of "natural radiation", necessary in his discussion of the achievement of equilibrium in the enclosure filled with thermal radiation, just as the "Stosszahlansatz" was necessary to Boltzmann for the derivation of the H-theorem. Eqs. (3) and (4) appear nowadays as the basic equations of stochastic electrodynamics, which, in some enunciations, starts from a Fourier analysis of the stochastic process. It should be noted that the amplitudes A_n are fixed, that is, they are not themselves random variables. We recall that the assumption that E is a stochastic process results in the prescription that any physical result should follow from taking an ensemble average ; in this case this then implies averaging over the ensembles of the θ_n only. This is one of the possible alternatives ; the other one, which is equivalent in the limit of infinite degrees of freedom, is in terms of random amplitudes (see, for instance : S.O. Rice, in : *Selected Papers on noise and stochastic processes*, edited by Nelson Wax, Dover, New York (1954)).

20) The third of Eqs. (5) is wrong in the original text, where, however, in the following use is made of the correct version reported here.

21) We have introduced a notation for vectors since the original used in the text is unreadable. The first term arises from applying the expression $\nabla(\vec{f}.\vec{E})$ for the force due to the electric field ; clearly

$$\nabla(\vec{f}.\vec{E}) = (\vec{f}.\nabla)\vec{E} = f_z \frac{\partial\vec{E}}{\partial z} = f\frac{\partial\vec{E}}{\partial z}$$

The second term arises from the Lorentz force.

22) M. Planck, l.c. p. 114 (Note of the authors).

23) The field is computed in the origin of the coordinates, where one can always choose to localize the dipole. The discussion in terms of Fourier transforms of Eq. (3) gives us as the

response spectrum F(ν)

$$F(\nu) = 3\sigma c^3 \frac{\mathcal{F}_\nu\left(E_z\right)}{2i\sigma\nu^3 - 4\pi^2\nu_0\nu^2 + 16\pi^4\nu_0^3}$$

where

$$\mathcal{F}_\nu\left(E_z\right) = -\sin\phi \sum_n A_n\sqrt{\frac{\pi}{2}}\left(e^{-i\theta_n}\delta\left(\nu - \frac{2\pi n}{T}\right) + e^{i\theta_n}\delta\left(\nu + \frac{2\pi n}{T}\right)\right)$$

The results follow by integrating term by term.

24) After correcting two obvious misprints in the formulae for $\partial E_x/\partial z$ and K_x.

25) Actually this expression for $\partial E_x/\partial z$ as well as the one for H_y should be completed through the components of the wave which is polarized at right angles to the one which excites the oscillator ; however, it is clear that these expressions, because of the independence of their phases from those of the oscillator, do not contribute to the mean value (Note of the authors).

26) This independence follows from the final result of the preceding essay (Note of the authors). For a comment, see note (36).

27) M. Planck, l.c. p. 122 (Note of the authors).

28) An expression like that given for K_x has a definite value for a given assignment of the random phases θ_n. A physically meaningful value is obtained by carrying out an ensemble average, over the θ_n, that is $\overline{K}_x$. This implies taking the expressions $\overline{\cos(\tau_n-\gamma_n)\sin\tau_m}$ and $\overline{\sin(\tau_n-\gamma_n)\cos\tau_m}$, that is, in turn, $\overline{\cos\tau_n\ \sin\tau_m}$, $\overline{\sin\tau_n\ \sin\tau_m}$, $\overline{\cos\tau_m\ \cos\tau_n}$. By integrating between $-\pi$ and π over θ_n and θ_m, the only terms that survive are

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos^2\left(\frac{2\pi n}{T} - \theta_n\right)d\theta_n = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sin^2\left(\frac{2\pi n}{T} - \theta_n\right)d\theta_n = \frac{1}{2}$$

We observe that the factor $\dfrac{T^2\sin^2\gamma_n}{n^2}$ can be put in the more familiar form

$$\frac{T^2\sin^2\gamma_n}{n^2} = \frac{(2\pi)^2\gamma^2\omega^4}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^6}$$

where we have introduced

$$\omega = \frac{2\pi n}{T}, \quad \omega_0 = 2\pi\nu_0 = \frac{2\pi}{T}, \quad \gamma \equiv \tau_{st.} = \frac{2e^2}{3mc^3}$$

The above expression is the typical Lorentzian for the case of a damping proportional to $\dddot{r}$. The final expression in Eq. (7) arises from taking T very large, so that one may substitute $\frac{T}{2\pi}\sum_n \to \int d\omega$; the integration is formal once $\tau_{st.} << T$ is consistently assumed. The characteristic damping is thus assumed to be very small with respect to the coherence time of the wave packet. While this is not explicitly the typical quantum-mechanical limit of stochastic electrodynamics, in a certain sense it prefigures it. The amplitudes are assumed to vary much more slowly than the Lorentzian and are taken at their mean value, corresponding to $\bar{n} = \nu_0 T$.

29) A. Einstein, Ann. d. Phys. 17, 914 (1905) (Note of the authors).

30) The formulae that follow hold actually at the first order in v/c.

31) A slight misprint has been corrected.

32) As it is written, the formula is incorrect, inasmuch as it sets an infinitesimal quantity equal to a finite quantity. We arrive at

$$\rho\frac{1}{4\pi} = \frac{1}{8\pi}\frac{\overline{A^2}\,T}{2}.2.2$$

on the basis of the following considerations : the energy density carried by a packet of plane waves propagated within a solid angle $\Delta\Omega$ in the direction ω, ϕ in the frequency interval $\Delta\nu$ stays in the ratio $\Delta\Omega/4\pi$ to the total density ρ. Such a density may be written as

$$\int_{\Delta\Omega} \frac{2.2.\overline{A^2}\,T}{2.8\pi} d\Omega = \frac{2.2.\overline{A^2}\,T}{2.8\pi}\Delta\Omega$$

where $\Delta n = T\Delta\nu$ has been exploited and it is assumed that, to a good approximation, $\overline{A^2}$ does not depend on the direction within $\Delta\Omega$ (the factor 1/2 is implied by the averaging operation). It is also evident that the subsequent expression of the ponderomotive force is formally incorrect for the same reason : it actually measures the ponderomotice action of the wave packet discussed above.

33) The result follows straighforwardly by separating the various terms if the limit $v/c << 1$ is taken.

34) Namely, the impulses of alternating signs arising from the irregularities of the radiation process may be determined for a resonator *at rest* (Note of the authors).

35) In practice $\tau > T$ so as to make the field components vanish.

36) The authors refer to a previous paper of theirs : A. Einstein, L. Hopf, Ann. d. Phys. 33, 1096 (1910). The expression given above for E_z is the obvious development of a superposition of a progressive and a regressive wave along a definite direction. In the origin the expression of E_z and $\frac{\partial E_z}{\partial x}$ becomes

[1]

$$E_z = \sum_n \left[(a_n + a_n')\sin\frac{2\pi n}{T}t + (b_n + b_n')\cos\frac{2\pi n}{T}t\right]$$

$$\frac{\partial E_z}{x} = -\sum_n \frac{2\pi n\alpha}{Tc}\left[(a_n - a_n')\cos\frac{2\pi n}{T}t - (b_n - b_n')\sin\frac{2\pi n}{T}t\right]$$

They fall therefore into the class of general expressions

[2]

$$\sum_n \left[A_n \sin\frac{2\pi n}{T}t + B_n\cos\frac{2\pi n}{T}t\right]$$

examined by Einstein and Hopf in the above-mentioned paper. There the authors face the problem of whether it may correctly be assumed that, in the case where the A_n and B_n are random variables, their distributions are actually independent, as, they say, is normally assumed. The authors imagine a plane subdivided into several elements, each one radiating independent, as, they say, is normally assumed. The authors imagine a plane subdivided into several elements, each one radiating independently of the others at a time t_s ; the plane lies at a very large distance from the point of observation, so that all the single waves emitted travel approximately in the same direction. The wave emitted by the s-element is represented by a development of the type :

$$\sum_n a_n \sin 2\pi n \frac{t - t_s}{T}$$

the wave of Eq.[2] is thus given by

$$[3] \quad \sum_s \sum_n a_n \left(\sin 2\pi n \frac{t}{T} \cos 2\pi n \frac{t_s}{T} - \cos 2\pi n \frac{t}{T} \sin 2\pi n \frac{t_s}{T} \right) ;$$

By comparing [3] with [2], one has the identification

$$A_n = a_n \sum_s \cos 2\pi n \frac{t_s}{T}$$

[4]

$$B_n = a_n \sum_s \sin 2\pi n \frac{t_s}{T}$$

The stochastic character of the wave given by [3] is determined by the various terms t_s which are characteristic of every element of the plane and completely uncorrelated with one another. From the independence of the various terms of the sums over s in Eqs.[4], follows the statistical independence of the coefficients A_n and B_n. This shows that, in Eq.[1], the coefficients $(a_n + a_{n'})$, $(b_n + b_{n'})$ and respectively $(a_n - a_{n'})$, $(b_n - b_{n'})$ are independent of each other. As $(a_n + a_{n'})$, $(a_n - a_{n'})$ and $(b_n + b_{n'})$, $(b_n - b_{n'})$ are, for obvious reasons, mutually independent, the mutual independence of all the coefficients appearing in the developments follows.

37) Two misprints corrected.

38) Actually it is the quantity $A_{\nu_0 T} \sin\phi$ that must be squared, averaged over the ensemble and then integrated over all the angles of incidence. One then obtains the quantity

$$\overline{B^2_{\nu_0 T}} = \overline{A^2_{\nu_0 T}} \sum_{\text{over all angles}} \sin^2\phi$$

or

$$\overline{B^2_{\nu_0 T}}\, T = \overline{A^2_{\nu_0 T}}\, T \sum_{\text{over all angles}} \sin^2\phi =$$

$$= \rho_{\nu_0} \int \sin^2\phi \, d\Omega = \rho_{\nu_0} \int_0^{2\pi} d\omega \int_{-1}^{1} d\cos\phi \, \sin^2\phi = \frac{8}{3} \pi \rho_{\nu_0}$$

39) From

$$E_z = \sum_n A_n \cos\left[\frac{2\pi n}{T}\left(t - \frac{\alpha x + \beta y + \gamma z}{c}\right) - \theta_n\right] \sin\phi$$

one derives, for the coefficients C_n of the development of $\frac{\partial E_z}{\partial x}$:

$$C_n = \frac{2\pi n}{T} \frac{\alpha}{c} \sin\phi \, A_n$$

with

$$\alpha = \sin\phi \cos\omega$$

For the term $\nu_0 = \frac{1}{T}$ (n=1) we thus obtain

$$\frac{2\pi\nu_0}{c} \sin^2\phi \cos\omega \, A_{\nu_0 T} .$$

This is the quantity that must be squared, averaged over the ensemble and then integrated over all the angles of incidence. One then obtains the quantity

$$\overline{T C^2_{\nu_0 T}} = \left(\frac{2\pi\nu_0}{c}\right)^2 \overline{A^2_{\nu_0 T}}\, T \sum_{\text{over all angles}} \sin^4\phi \cos^2\omega =$$

$$= \left(\frac{2\pi\nu_0}{c}\right)^2 \rho_{\nu_0} \int d\Omega \, \sin^4\phi \cos^2\omega = \left(\frac{2\pi\nu_0}{c}\right)^2 \rho_{\nu_0} \int_0^{2\pi} d\omega \cos^2\omega \int_{-1}^{1} d\cos\phi \sin^4\phi =$$

$$= \frac{64}{15} \frac{\pi^3 \nu_0^2}{c^2} \rho_{\nu_0}$$

[40]) M. Planck, l.c. p. 178 (Note of the authors).

[41]) See A. Einstein, Phys. Zeitschr. 10, p. 185 ff. What is essentially new in the work in question lies in the fact that the impulse oscillations were for the first time computed exactly (Note of the authors).