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A REAL PLANE REPRESENTATION OF THE ELECTRODYNAMICS OF THE COMPLEX SCHRÖDINGER CONFIGURATION PLANE

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Abstract : *The occurence, in the Schrödinger equation type of description of electronic process, of the apparently extra unclassical force additional to that due to the given external potential is analysed and explained using only classical electrodynamics and thermodynamics.*

Résumé : *L'énergie potentielle non-classique, supplémentaire à l'énergie potentielle extérieure, qui se présente dans la mécanique ondulatoire de Schrödinger est analusée et expliquée en employant seulement l'électrodynamique classique et la thermodynamique.*

1. Introduction(*)

In recent work ([1,2,3,4]), the present author has shown that there are a variety of ways in which the mathematical and physical structure of the one dimensional Schrödinger equation can be deduced from classical fluid and thermodynamical principles. It is thus possible to construct a classical basis for quantum mechanics. Possibly, the most direct route ([2]) into this area of analysis is by analytic continuation of the configuration variable x of the one dimensional Schrödinger equation into the complex $z = x + iy$ plane. This route is direct but the various new roles of familiar mathematical quantities all require physical identification, interpretation and explanation. The less direct route which was followed through in reference ([1]) involves starting from a two dimensional mixed fluid basis and showing how assumed properties for the fluid structure lead inevitably to the Schrödinger equation for the fluid process taking place on the real x coordinate axis. By this second route fluid and thermodynamical identifications naturally attach

* Le soulignage d'une lettre signifie que celle-ci représente un vecteur.

themselves to the older quantum mechanical mathematical quantities in the course of the derivation. Having seen the light, as it were, yet a third route into this alternative theory becomes recognisable. An appropriate classical variational principle can be constructed and from it the new structure can be derived ([3]). Thus it seems that much evidence can be gathered to show that the orthodox Schrödinger description for the quantum process is a powerful streamlined partial mathematical complex plane representation of a real two dimensional physical system. A similar situation occurs in classical two dimensional fluid dynamics where sometimes the complex plane representation supplies a strong analytic tool for handling the mathematics of some types of real two dimensional problem ([5]). The word '*partial*' is used above in the sense that, in the orthodox Schrödinger theory, although complex functions are employed the full potential of the structure is not utilised in that the formalism is kept, usually in theory and in practice, firmly planted on the real x axis.

The Schrödinger wave function method confined to the real x axis has great computational facility but there is in it an inherent loss of '*explanatory*' physical back up for the quantities used. It supplies an outstanding successful techniques, but at the same time involves a loss of deeper understanding of underlying physical processes and also carries a tendency for paradox to seem to arise. On the other hand, it seems that if we look at a real plane representation for the analytically continued Schrödinger equation a whole new world of explanatory classical physical images emerges. Once one has identified the well known mathematical quantities within the new two dimensional context, physical pictures of possible underlying fluid and thermal processes begin to appear in profusion. Whereas before, one came to a dead stop with the philosophical interpretation of quantities such as the wave function which according to many theoreticians should not be analysed further but rather regarded as a given basic unanalysable concept ; in the alternative view to be presented here, analysis is natural and the wave function, for example, is the exponential of the complex potential of classical fluid dynamics. With regard to the wave function, of course, this view is only new in the present theory in the sense that we are here talking of the complex plane fluid process whereas earlier authors (6,7,8) used such an interpretation for the wave function in the context of an actual two dimensional subspace of the Euclidean three space of common experience. The particular aspect of this new point of view with its power to invoke and support classical images to be studied in this paper is the electromagnetic thermal interaction taking place in the complex plane. Here we find that a real plane representation for this aspect conforms to a recognisable classical Maxwell equation type of description and that in such terms we can give a clear account of how the extra *'quantum potential'* (7), which appears in earlier theory without explanation, arises in an electromagnetic thermal arena away from the real x axis of the Schrödinger equation while being generated by the passage of the primary *'particle'*. Although we can detect many classical configurations and structures within this extended area of analysis, there is an important sense in which the *'material'* involved in the structure is not familiar in more usual classical studies. It seems that the $\sqrt{-1}\,y$ direction movement should be interpreted as the movement of negative mass. However, this is a matter of constitution

rather than one of a theoretical label of *'classical'* or *'non-classical'*. In fact, this feature of the theory is acceptable in the sense that in regions between singularities it is a process of mass polarisation which carries the distributed quantum information and the familiar quantum density $\rho\ (x,\ t)$ represents the degree of this polarisation on the real x axis. In work with the Schrödinger theory, it is usually assumed that there is no mass present between singularities other than in a probabilistic sense. In the present theory, there is still no mass present between singularities. Instead, equal quantities of positive and negative mass coexist in such regions and make their presence felt by having different states of average motion while keeping their total energy of *'thermal'* motion also zero. However, we shall see that in studying the electromagnetics of this problem in the extended planar region, the negative mass aspect hardly surfaces. This is because the basic quantity associated with the moving fluid particle is an electric current density,

$$\mathbf{j} = \bar{e}\ \rho\ (v_1\underline{i} - v_2\underline{j}), \qquad (1.1)$$

and the negative mass movement variables $(-\rho m v_2\underline{j})$ for the y direction become subsumed in the associated current density $(-\bar{e}\rho v_2\underline{j})$ for the y direction. $(-\rho m v_2\underline{j})$ is, of course, the momentum density associated with the negative mass density $m\rho_2 = -\ \rho m$ for the second fluid. Thus from this point of view, we see that it is the vector,

$$m\bar{\mathbf{v}} = m(v_1\underline{i} - v_2\underline{j}), \qquad (1.2)$$

that plays the role of *'classical'* momentum and it is in such a capacity that it appears in the

Newtonian like equation of motion for the system. However, as with Newtonian dynamics, in differentiating a moving system with respect to time we shall follow the '*particle*' movement which is determined by the '*number*' density velocity,

$$\mathbf{v} = v_1\underline{i} + v_2\underline{j} \qquad (1.3)$$

and consequently the substantive time derivative operator will be denoted by $\frac{d}{dt}\Big|_v$ and by which be meant differentiation following the v velocity field, and so, in the case of rate of change of momentum following the particles, we have

$$\frac{d}{dt}(m\bar{\mathbf{v}})\Big|_{\mathbf{v}} = m\left(\frac{\partial \bar{\mathbf{v}}}{\partial t} + \mathbf{v}\cdot\nabla\bar{\mathbf{v}}\right) \qquad (1.4)$$

This formula will be used a little later. We shall now briefly outline how the various extended two dimensional quantities of this alternative theory are defined and indicate from what sort of physical argument their existence can be inferred.

The most important information containing function occurring in the theory is the two dimensional density function ρ (x, y, t) . ρ is a real function and in terms of the wave function Ψ (x, t) it has the form,

$$\rho\,(x, y, t) = \Psi^{*}(x - iy, t)\,\Psi\,(x + iy, t) \qquad (1.5)$$

The meaning of the star $*$ symbol used in relation (1.5) is that of complex conjugation of the functional form '*only*'.

That is if

$$\Psi\,(Z) = \Psi\,(x + iy)$$

$$= \Psi_1\,(x, y) + \sqrt{-1}\,\Psi_2\,(x, y)$$

and Ψ_1 and ψ_2 satisfy the Cauchy Riemann relations,

$$\frac{\partial \Psi_1}{\partial x} = \frac{\partial \Psi_2}{\partial y}, \qquad \frac{\partial \Psi_1}{\partial y} = -\frac{\partial \Psi_2}{\partial x} \qquad (1.6)$$

and further we use the bar - symbol to donote the usual complex conjugation operation then,

$$\bar{\Psi}\,(Z) = \Psi^{*}\,(\bar{Z}) = \Psi_1\,(x, y) - i\Psi_2\,(x, y)$$

and $\Psi^{*}(Z) = \bar{\Psi}\,(\bar{Z}) = \Psi_1\,(x,-y) - i\Psi_2\,(x,-y)$.

It follows that

$$\rho\,(x, y, t) = \Psi^{*}(\bar{Z})\,\Psi\,(Z)$$

$$= \bar{\Psi}\,(Z)\,\Psi\,(Z)$$

$$= |\Psi\,(Z)|^2$$

$$= \text{A Real function.} \qquad (1.7)$$

When y = o, we recover the usual quantum density function

$$\rho\,(x, o, t) = \Psi^{*}(x, t)\,\Psi\,(x, t), \qquad (1.8)$$

though possibly with an unusually dimensioned numerical factor. Expression (1.5) is not just an ad hoc definition for ρ (x, y, t) in terms of the usual wave function analytically continued. That ρ (x, y, t) should have a form such as (1.5) is a consequence of the fact deducible from thermodynamics that the total thermal energy of the two fluids is zero or equivalently that they have numerically equal but opposite signs for their temperatures,

$$T_1 + T_2 = 0. \qquad (1.9)$$

Also from thermodynamics, we can infer that

$$kT_1 = -\frac{m\nu^2 \partial^2 \ell n\rho}{\partial x^2}$$

$$(1.10)$$

and

$$kT_2 = -\frac{m\nu^2 \partial^2 \ell n\rho}{\partial x^2}$$

where $\nu = \hbar/2m$.

Therefore

$$\nabla^2 \ell n\rho = 0 \qquad (1.11)$$

and this last equation ensures that ρ (x, y, t) is of the form (1.5). Defining the temperature T_2 of the negative mass fluid by (1.10) so that it is of opposite sign to that of the positive mass fluid enhances the symmetrical form of the relationship between the two fluids and the mathematical expressions used to describe them. The two temperatures T_1 and T_2 can be derived by the usual thermodynamical steps,

$$kT_1 = \frac{\partial \varepsilon_1}{\partial \ell n\rho}\bigg|_y \qquad (1.12)$$

and

$$kT_2 = \frac{\partial \varepsilon_2}{\partial \ell n\rho}\bigg|_x \qquad (1.13)$$

where ε_1 and ε_2 are internal energies of the two fluids and $k\ell n\rho$ is an entropy associated with the density ρ. Later in the paper, we shall use only T_1 and then denote it by plain T. The internal energies ε_1 and ε_2 are given by

$$\varepsilon_1 = -\frac{1}{2} m v_2^2 \qquad (1.14)$$

and

$$\varepsilon_2 = -\frac{1}{2} m v_1^2 \qquad (1.15)$$

The subscript 'y' or 'x' in the formulas (1.12) and (1.13) is meant to indicate that y is kept constant while differentiating ε_1 with respect to $\ell n\rho$ or x is kept constant while differentiating ε_2. That the internal energy of one fluid is the negative of kinetic energy for the other is an inevitable consequence of the two fluid properties that $\left(\frac{\partial P}{\partial x}\right)\Big/\rho$, inspite of its appearence, is a one dimensional gradient and that the movement of the mixed positive and negative mass fluids on the real plane is non-isotropic.

The product of the fundamental constant $\nu = \hbar/2m$ with $\ell n\rho$ turns out to be the stream function for the '*momentum*' velocity field, $v_1\underline{i} - v_2\underline{i}$. That is to say,

$$v_1 = -\nu \frac{\partial \ell n\rho}{\partial y} \tag{1.16}$$

and

$$v_2 = -\nu \frac{\partial \ell n\rho}{\partial x} \tag{1.17}$$

Pressure is conveniently defined for both fluids by the ideal gas law,

$$\frac{P}{\rho} = kT_1, \tag{1.18}$$

which is itself a derivable consequence of the equilibrium fluid thermodynamical structure. With this definition for ρ, we have

$$\frac{P}{\rho} = -\frac{m\nu^2 \partial^2 \ell n\rho}{\partial x^2} = \frac{m\nu \partial v_2}{\partial x} \tag{1.19}$$

by (1.17). The expressions (1.16) and (1.17) for v_1 and v_2 are also derivable consequences of the structure and can be obtained in a number of ways by invoking classical images ([1], [3], [4]). Whatever route is taken in obtaining these various relations and definitions, we eventually arrive at the two equations,

$$\frac{\partial v_1}{\partial t} = -\frac{\partial}{\partial x}(E_1/m), \tag{1.20}$$

$$\frac{\partial v_2}{\partial t} = +\frac{\partial}{\partial y}(E_1/m), \tag{1.21}$$

where E_1 is given by the Bernouli equation,

$$E_1 = \frac{m}{2}(v_1^{\ 2} - v_1^{\ 2}) + \frac{P}{\rho} + V_1 \tag{1.22}$$

with P satisfying (1.18),

$$E_1 = \mathrm{Re.}\left(i\hbar \frac{\partial}{\partial t} \ell n\, \Psi\right) \tag{1.23}$$

and ψ (Z) an analytically continued solution of Schrödinger's one dimensional equation in x - with also the relation,

$$m\,(v_1 + iv_2)\,\Psi(Z) = -i\hbar \frac{\partial \Psi}{\partial z}(Z) \tag{1.24}$$

arising from the formalism. Ψ (Z) is also found to be of the form,

$$\Psi(Z) = \exp\left(-\frac{i\hbar}{m}\,\omega(Z)\right), \tag{1.25}$$

where ω (Z) is the complex potential of classical fluid dynamics for the $v_1\underline{i} - v_2\underline{j}$ velocity field. There are a number of different ways in which the equations (1.20), (1.21), and (1.22) can be expressed which are worth recording. In terms of the vorticity $\zeta_3\underline{k} = \nabla_\wedge \underline{v}$ of the two dimensional **v** velocity field, we have

$$\frac{\partial v_1}{\partial t} - v_2\zeta_3 = -\frac{\partial}{\partial x}\left(\frac{P}{m\rho} + \frac{v^2}{2} + \frac{V_1}{m}\right), \tag{1.26}$$

$$\frac{\partial v_2}{\partial t} + v_1\zeta_3 = - \frac{\partial}{\partial y}\left(\frac{P}{m(-\rho)} + \frac{v^2}{2} - \frac{V_1}{m}\right) \quad (1.27)$$

where $v = |\mathbf{v}|$.

Another familiar Eulerian form into which these equations may be shaped is as fallows,

$$\frac{\partial v_1}{\partial t} = - \frac{\partial}{\partial x}\left(\frac{v_1^2}{2}\right) - \frac{1}{m\rho}\frac{\partial P}{\partial x} - \frac{1}{m}\frac{\partial V_1}{\partial x} \quad (1.28)$$

$$\frac{\partial v_2}{\partial t} = - \frac{\partial}{\partial y}\left(\frac{v_2^2}{2}\right) + \frac{1}{m\rho}\frac{\partial P}{\partial y} + \frac{1}{m}\frac{\partial V_1}{\partial y} \quad (1.29)$$

A vector equation of motion on the real $\underline{i}$, $\underline{j}$ plane can be obtained by using (1.18) (1.16) and (1.17) to convert (1.26) and (1.27) into,

$$- m\left(\frac{\partial v_1}{\partial t} + v_1\frac{\partial v_1}{\partial x} - v_2\frac{\partial v_2}{\partial x}\right) = \frac{\partial}{\partial x}(kT+V_1), \quad (1.30)$$

$$- m\left(\frac{\partial v_2}{\partial t} - v_1\frac{\partial v_1}{\partial y} + v_2\frac{\partial v_2}{\partial y}\right) = - \frac{\partial}{\partial y}(kT+V_1), \quad (1.31)$$

and then using the Cauchy Riemann equations for v_1 and v_2 these last two equations then become

$$m\left(\frac{\partial v_1}{\partial t} + v_1\frac{\partial v_1}{\partial x} + v_2\frac{\partial v_1}{\partial y}\right) = - \frac{\partial}{\partial x}(kT+V_1) \quad (1.32)$$

$$m\left(\frac{\partial v_2}{\partial t} + v_1\frac{\partial v_2}{\partial x} + v_2\frac{\partial v_2}{\partial y}\right) = \frac{\partial}{\partial y}(kT+V_1). \quad (1.33)$$

Combining these together in terms of the components of the real vector $\bar{\mathbf{v}} = v_1\underline{i} - v_2\underline{j}$ we get,

$$m\frac{d\bar{\mathbf{v}}}{dt} = - \nabla(kT) - \nabla V_1 \quad (1.34)$$

as the vector equation of motion for the system. Equation (1.34) is an equation between real quantities described on a real two dimensional surface. It is also of the classical Newtonian form. However, besides the *'given'* external potential,

$$V_1(x, y) = \text{Re.}\{V(z)\} = \frac{\bar{V}(z) + V(z)}{2},$$

there occurs in it an extra potential kT, somehow generated *'internally'* within the system. It is worth noting another expression for the energy E_1 which is easily derived from the equations (1.22), (1.16), (1.17) and (1.18),

$$\{E_1(x, y, t) - V_1(x, y)\}\rho = \nu^2\left(\frac{\partial^2\rho}{\partial y^2} - \frac{\partial^2\rho}{\partial x^2}\right) \quad (1.35)$$

$$= \frac{1}{2}\rho_+\left(v_1^2 + \frac{kT_1}{m}\right) + \frac{1}{2}\rho_-\left(v_2^2 + \frac{kT_2}{m}\right),$$

where $\rho_+ = \rho = -\rho_-$.

This expression shows how the energy E_1 is a combination of kinetic and thermal energies in the usual classical pattern. Finally, we note that in the real plane representation the fluid mass current density, $m\rho(v_1\underline{i} + v_2\underline{j})$ satisfies an equation of continuity of the form,

$$\frac{\partial \rho}{\partial t} = - \frac{1}{2} \nabla \cdot (\rho \mathbf{v}) + \Gamma, \tag{1.36}$$

where Γ is a source function. This reduces to the usual quantum equation of continuity,

$$\frac{\partial \rho}{\partial t} = - \frac{\partial (\rho v_1)}{\partial x} \tag{1.37}$$

on the x axis, where Γ = 0, and generally off the x axis (1.36) can be used to convert it into

$$\frac{\partial \rho}{\partial t} = \frac{\nu \partial^2 \rho}{\partial x \partial y} + \Gamma, \tag{1.38}$$

where $\Gamma = \dfrac{\rho V_2}{m\nu}$,

with V_2 the imaginary part of the analytically continued given external potential, V(x).

Equation (1.38) looks a little like the equation of diffusion. In fact it can be derived from what might be called '*centrifugal diffusion*'. That is a '*banking up*' of the density ρ locally with a tendency to level out by random decay, becomes balanced by centrifugal forces arising from the local rotation of the fluid ([4]). This paper will now be devoted to showing how the extra potential kT arises from the two dimensional fluid, thermodynamic and electromagnetic interactions taking place off the real axis. To carry through the next steps we shall make quite general classical assumptions about the form one would expect would be taken by a two dimensional equation of motion for a system

such as we are studying. We shall then show that the general form reduces to (1.34) when various internal equilibrium conditions are satisfied and we shall find, as might be expected, that magnetism arising from various currents plays an important role. The density function ρ (x, y, t) arises from the wave function by (1.5) and so with the usual probabilistic density interpretation it would, for the one dimensional wave function ψ (x, t), have the dimension of L^{-1}. However, we shall use a density function ρ (x, y, t) with the dimensions of L^{-3} in the following work in order to keep our electromagnetic equations directly comparable with the usual three dimensional Maxwell forms. The connection between the ρ to be used here and the usual quantum ρ for the one dimensional case is simply that the former is obtained form the latter by the formula (1.5) followed by multiplication by a constant of dimension L^{-2} or that the analytically continued wave function is modified by multiplication by a constant of dimension L^{-1}. However, although we shall employ a z direction given by a unit vector $\underline{k}$ in order to be able to use the vector form for vector products and to give vector expression to vorticity and other quantities on the $\underline{i}$, $\underline{j}$ plane, our problem is basically two dimensional, so that for all function f to be used in this paper $\partial f/\partial z = 0$.

The $- \nabla(kT)$ term in (1.34) is the term which renders the version of quantum mechanics given by (1.1) apparently non classical. It is not quite the same term as the extra force used by Bohm and Vigier ([7,8]) in discussing the deviations of quantum mechanics from classical forms. The term they used and which is sometimes called the Bohm potential has the form $\rho^{-1}\partial^2\rho/\partial x^2$. An extensive and comprehensive account of attempts to reformulate and explain quantum mechanics together with many references can be found in the book by Max Jammer [9]. Discussions of fluid and electromagnetic aspects of quantum mechanics can also be found in the works of Janossy, Ziegler-Naray [10,11,12,13] and Takabayasi [14,15].

Working with analytically continued functions has the consequence that frequently the Cauchy Riemann equations are applicable to various quantities. In particular, in the case of the velocity field $\mathbf{v} = v_1\underline{i} + v_2\underline{j}$, we have,

$$\frac{\partial v_1}{\partial x} = \frac{\partial v_2}{\partial y} \tag{1.39}$$

and

$$\frac{\partial v_1}{\partial y} = -\frac{\partial v_2}{\partial x}, \tag{1.40}$$

and these two equations play a very important part in the following work. The reason for the applicability of these equations to the components of the velocity field $\mathbf{v}$ is that $\bar{\mathbf{v}}$ simultaneously satisfies the two conditions, $\nabla\cdot\bar{\mathbf{v}} = \nabla_\wedge\bar{\mathbf{v}} = 0$. The reader will recognise that in terms of $\bar{\mathbf{v}}$ these are the same conditions that are also used in superfluid theory for the momentum p_s. For example see page 73 of London's book [16]. However, when applied to the three dimensional case, unlike here where $\bar{\mathbf{v}}$ is two dimensional, they do not have equations (1.39) and (1.40) as a consequence. Physically the fluid structure to be studied here is thus related to superfluid theory except that here only planar movement occurs.

However, it should be emphasised that all the following analysis is done, not in the complex plane, but rather in a real $\underline{i}$, $\underline{j}$ plane where often vector components are fortuitously also the real and imaginary parts of a complex function of a complex variable. Thus the notation $\bar{\mathbf{v}} = v_1\underline{i} - v_2\underline{j}$ will be used to distinguish $\bar{\mathbf{v}}$ from $\mathbf{v} = v_1\underline{i} + v_2\underline{j}$ as it parallels the notation $\bar{v} = v_1 - \sqrt{-1}\, v_2$ used to represent the complex conjugate of the function $v = v_1 + \sqrt{-1}\, v_2$.

2. The Physical Connections

The system to be studied here is essentially an electronic fluid and so it will have those physical properties normally possessed by either an electron or possibly a positron but it will be regarded as being distributed over some space region and under the influence of an external potential. We are concerned with the internal electromagnetic thermal structure of this system thought of as having two parts. The two parts being, the particle proper, and the rest of the total system which exists in the sense that it is a background or support against which the particle's behaviour can be analysed. The background becomes a physical frame of reference but intimately connected with the particle. In the terminology of elementary particle physics, the particle here is the '*bare*' particle and the induced background which coexists with it, is that assembly of interactions which gives the particle substance.

It will be assumed that the very existence of the particle and its state of motion represented by such parameters and functions as mass m, charge e', magnetic moment $\underline{\beta}$, density ρ and velocity field v causes the induction of various internal forces and reactions within '*the whole system*' which can be regarded as the particle and its immediate local environment. Mostly these forces and reactions, being internal, mutually cancel. However, it appears that in quantum theory, somehow, some remnant of the possible internal interactions survives and consequently influences the changes of motion of the particle along with the influence of the external field. This residual force is the $-\nabla(kT)$ term in equation (1.34). It will be shown, in this paper, how such an unbalanced internal force can arise from electromagnetic and thermal considerations in a classical context in a real two dimensional fluid system. The physical sequence of interactions arising out of the existence and state of motion of the particle will now be described and afterwards all the mathematical connections will be derived and discussed.

Firstly, the existence of the particle as part of an electromagnetic fluid, causes an internal electric field $\underline{E}$ which acts on the particle and its surroundings. It can be anticipated that this total internal electric field will have its origin in charges and currents present in the medium. That part of $\underline{E}$ which originates directly from the primary '*particle*' whose motion through the medium we are studying will be denoted by $\underline{E}^{(r)}$ and this part in its direct action will exert forces on any induced charges which have appeared in the medium as a result of the disturbance due to the primary particle and its motion. Charges induced in the medium will themselves contribute to the total electric field and this part of the electric field we shall denote by $\underline{E}^{(t)}$. $\underline{E}^{(t)}$ then will directly act back on the primary particle. It will be assumed that the electronic fluid has a conductivity σ.

Consequently, given the induced field $\underline{E}$ together with induced charges created by a local $\pm$ charge polarisation of the background, a conduction current $\underline{J} = \sigma\underline{E}$ will flow. We can divide the conduction current $\underline{J}$ into two parts $\underline{J}^{(r)}$ and $\underline{J}^{(t)}$.

The part $\underline{J}^{(r)}$ being the component of $\underline{J}$ in the direction of $\underline{E}^{(r)}$ with $\underline{J}^{(t)}$ representing the rest of the induced conduction current. The induced current $\underline{J}$ will be regarded as being essentially that part of the total system which is not the particle, but rather as being a feature of the medium in which the particle moves. We shall attribute a total dipole moment $\beta\underline{k}$ to the particle and so if ρ (x, y, t) is the density of particle distribution (either a probability distribution or a mass distribution divided by total mass m) over the $\underline{i}$, $\underline{j}$ plane, then the particle will carry a distributed magnetic field $\underline{M}_o = \rho\beta\underline{k}$ and a related magnetic induction field $\underline{B}_o = \mu_o\underline{M}_o$. It follows that the induced currents $\underline{J}^{(t)}$ and $\underline{J}^{(r)}$ experience Lorentz forces per unit volume,

$$\underline{L}_1 = \underline{J}^{(t)} \wedge \underline{B}_o \text{ and } \underline{L}_2 = \underline{J}^{(r)} \wedge \underline{B}_o,$$

which have their origin in the magnetic induction $\underline{B}_o$ of the particle. Il follows that the primary particle which carries the $\underline{B}_o$ field and which, in fact, is distributed over the $\underline{i}$, $\underline{j}$ plane will experience locally the reactions $-\underline{L}_1$ and $-\underline{L}_2$ from these forces by Newtons third law. Suppose that $\mathbf{f}^{(r)}$ is the force per unit volume acting on the induced charges which have arisen locally from the inducing primary charged particle. We shall then '*define*' the electric field $\underline{E}^{(r)}$ arising from rhe primary particle by

$$\acute{e}\,\rho\underline{E}^{(r)} = \mathbf{f}^{(r)},$$

where ρ is given by (1.5) and is the density function for the primary particle and é is the change on it. The advantage of '*defining*' $\underline{E}^{(r)}$ in this way is that,

with such a definition, $\underline{E}^{(r)}$ and $\underline{E}^{(t)}$ are then on the same footing in relation to the primary density function ρ. Thus with this definition for $\underline{E}^{(r)}$, $-\acute{e}\,\rho\underline{E}^{(r)}$ is the Newtonian third law reaction force per unit volume experienced by the primary partical locally, while $-\underline{E}^{(r)}$ can be regarded as the local electric field reaction. $\acute{e}\,\rho\underline{E}^{(t)}$, on the other hand, is the force per unit volume exerted on the primary particle by charges induced in the background. Given internal stability of the system, the tendency is for these forces to cancel in pairs. That is to say, one might expect that $\mathfrak{F}_1 = -\rho e'\underline{E}^{(r)} - \underline{L}_1 \to 0$ and that $\mathfrak{F}_2 = \rho e'\underline{E}^{(t)} - \underline{L}_2 \to 0$. However, we shall reserve judgement on these two possible équilibria until we have considered another chain of interactions. The complicating feature is that $\underline{J}^{(t)}$ is that part of the induced current which occurs naturally in the form of the curl of a vector. Thus $\underline{J}^{(t)}$ can sustain a magnetic induction field $\underline{\tilde{B}}$, say, which is entirely additional to the magnetic induction field $\underline{B}_o$ of the particle via the Maxwell equation $\mu_o\underline{J}^{(t)} = \nabla \wedge \underline{\tilde{B}}$. Thus $\underline{\tilde{B}}$ can disturb the simple equilibria $\tilde{\mathfrak{F}}_1 = 0$ and $\mathfrak{F}_2 = 0$ suggested earlier by causing further interactions. Given the the possible existence of $\underline{\tilde{B}}$ we can then expect that a more complicated equilibrium situation is involved in which $\underline{\tilde{B}}$ interacts with $\underline{M}_o$. At this point we shall define a magnetic field $\underline{H}$ by the relation $\underline{B} = -\mu_o\underline{H}$ and another related magnetisation field $\underline{M}$ by $\underline{M} = -\underline{H}$. Thus if $\underline{\tilde{B}}$ is the external magnetic induction vector, we have $\underline{B} = \mu_o(\underline{H} + \underline{M})$. This $\underline{B}$ is identically zero so that we can regard $\underline{\tilde{B}}$ as being an internal magnetic induction field in a medium which is perfectly diamagnetic. This is again a condition which is thought to hold in a superconductor under some conditions. See page 14 of London's book ([16]). Equilibria involving magnetic fields, magnetisation, density and which insolve the temperature T also are well-known classically and the connection between these various physical quantities is then under some conditions described by Curie's law, equation (3.31). If such a law holds in this context, temperature naturally becomes involved in the overall situation and then thermal effects must be taken into account in addition to the electromagnetic effects. The existence of $\tilde{\underline{B}}$ also reacts back on to the particle in a more direct way than just through a relationship such as (3.31). Given $\tilde{\underline{B}}$, the particle will experience a direct force per unit volume in virtue of its own charge and motion. We shall denote this third Lorentz force by $\underline{L}_o$ where $\underline{L}_o = e'\rho\bar{\mathbf{v}}_\wedge\tilde{\underline{B}}$. The essential physical content of a relationship such as Curie's law is that an increase of thermal disorder caused by a temperature rise causes a decrease in magnetisation for a fixed $\bar{\underline{H}}$. However, for the situation that $\underline{M}_o$ becomes fixed, the consequence is that the magnetic field $\underline{H}$ which is an internal field in this work becomes proportional to T.

Thus the collection of interactions analysed so far in a classical way leads to the consideration of the possible involvement of a thermal field T (x, y, t) and so implies that further forces will have to be taken into account which have so far not been brought into the scheme. As we have density gradients and we are now considering the involvement of temperature it is clear that we must bring in the force that Einstein ([17]) called the osmotic force per unit volume $\underline{f}^{(o)} = -kT\nabla\rho$ and also a thermal

force $\underline{f}^{(th)}$, say, which would be responsible for the existence of a thermal current, $\underline{J}^{(th)} = -\chi\nabla T$, where χ is the thermal conductivity of the fluid. It turns out that the osmotic force when intruduced naturally balances the additional Lorentz force $\underline{L}_o$ mentioned earlier so producing a third equilibrium, $\mathfrak{F}_3 = 0$, while the thermal force per unit volume $f^{(th)} = -\nabla(kT)$ on the particle actually only cancels the electric force $-e'\underline{E}^{(r)}$ which we provisionally paired with $\underline{L}_1$ by $\mathfrak{F}_1 = \bar{0}$. Thus $\tilde{\mathfrak{F}}_1 = 0$ does not in fact hold upon the more comprehensive analysis and consequently $\tilde{\mathfrak{F}}_1 = 0$ has to be replaced by $\mathfrak{F}_1 = -\rho e'\underline{E}^{(r)} - \rho\nabla(kT) = 0$ and $-\underline{L}_1$ is left free and uncancelled and acting on the particle $\mathfrak{F}_2 = 0$ is not affected by the more detailed analysis and so can still be assumed to hold. This completes the physical description of how influences originating at the particle are spread and reflected back on to the particle in the form of the remnant Lorentz force $-\underline{L}_1$ which then assumes the equilibrium value $-\underline{L}_1 = -\rho\nabla(kT)$ and so accounts for the form of equation (1.34).

3. Equations of motion and equilibria

The function $m\mathbf{v} = m(v_1\underline{i} - v_2\underline{j})$ represents the local momentum field assciated with the particle. $m\mathbf{v} = m\mathbf{v}(x, y, t)$ is a function of space and time and so the operator d/dt needed to calculate the rate of change of momentum will need to be rate of change following the velocity field $\mathbf{v} = v_1\underline{i} + v_2\underline{j}$. Thus

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \tag{3.1}$$

We shall assume that the equation of motion of the particle is '*classical*' and by that term is meant that only forces understandable in terms of classical thermodynamics and Maxwell's electrodynamical equations will be involved in our analysis. Thus we take the equation of motion expressed in terms of forces per unit volume to be,

$$\rho m\frac{d\mathbf{v}}{dt} = \rho e'(\underline{E}^{(t)} + \mathbf{v}_\wedge\tilde{\underline{B}}) - \nabla P + \underline{R} - \rho\nabla V_1 \tag{3.2}$$

V_1 is the external field. $\underline{E}^{(t)}$ and $\tilde{\underline{B}}$ are internal fields induced in the vicinity of the particle by its existence and state of motion. P is the pressure contribution and $\underline{R}$ is a reaction term given by

$$\underline{R} = - e'\underline{E}^{(r)} - \underline{J}_\wedge\underline{B}_o \; , \tag{3.3}$$

where $\underline{J}$ is the induced current related to the induced $\underline{E}$ by

$$\underline{J} = \sigma\underline{E}. \tag{3.4}$$

σ is the conductivity of the fluid assembly and $\underline{B}_o$ is the magnetic induction field associated with the particle's magnetic dipole moment $\beta\underline{k}$, say. $e' = - e$, where $e > 0$ is the magnitude of the charge of one electron. Thus

$$\underline{B}_o = \mu_o\rho\beta\underline{k}, \tag{3.5}$$

and the reaction force $\underline{R}$ is just the Lorentz forces due to motion experienced by the induced conduction current $\underline{J}$, per unit volume, from the magnetic induction field $\underline{B}_o$ of the particle, together with the electric field $\underline{E}^{(r)}$ impressed on the charges induced in the medium by the particle.

We shall make the further assumption that the electric fluid satisfies the simple gas law,

$$\frac{P}{\rho} = kT, \tag{3.6}$$

connecting number density, pressure and temperature. Evidence for (3.6) can be found in references ([1],[2]). In (3.6), P, ρ and T are all functions of x, y and t. The induction field $\underline{B}_o$ carried by the particle arises from the current $e'\rho\mathbf{v}$. This can be seen by considering the curl of $\underline{B}_o$ in order to find the current responsible for $\underline{B}_o$. Thus by (3.5)

$$\nabla_\wedge\underline{B}_o = \beta\mu_o\rho\left(\frac{\partial \ell n\rho}{\partial y}\,\underline{i} - \frac{\partial \ell n\rho}{\partial x}\,\underline{j}\right). \tag{3.7}$$

however, $\nu\ell n\rho$ is the stream function ([1]) for the $v_1\underline{i} - v_2\underline{j}$ flow field and so

$$\underline{k}_\wedge \overline{\mathbf{v}} = -\nu\nabla \ell n\rho , \tag{3.8}$$

or

$$v_1 = -\nu\frac{\partial \ell n\rho}{\partial y} , \tag{3.9}$$

and

$$v_2 = -\nu\frac{\partial \ell n\rho}{\partial x} , \tag{3.10}$$

Using (3.9) and (3.10) in (3.7), we have,

$$\nabla_\wedge \underline{B}_o = \mu_o e'\rho\overline{\mathbf{v}} , \tag{3.11}$$

and this establishes that the current $e'\rho\overline{v}$ is the source of $\underline{B}_o$. Using (3.6) we can expand the pressure term, $-\nabla P$, into the two contributions,

$$-\nabla P = -\rho\nabla(kT) - kT\nabla\rho . \tag{3.12}$$

Returning to the induced electric field $\underline{E}$, we shall assume that this can be decomposed into two parts. The first part we shall denote by $\underline{E}^{(r)}$ and also assume that this part is expressible in the form $\underline{E}^{(r)} = -\nabla\phi$ and that it represents the electric field experienced by the medium but arising from the particle. The second part we shall denote by $\underline{E}^{(t)}$ and consider that this represents the field arising from induced charges in the medium and that its acts on the particle directly. We shall further assume that both of these fields can drive conduc-

tion currents which we shall denote by,

$$\underline{J}^{(r)} = \sigma\underline{E}^{(r)} , \tag{3.13}$$

and

$$\underline{J}^{(t)} = \sigma\underline{E}^{(t)} , \tag{3.14}$$

$\underline{J}$ is the total induced current,

$$\underline{J} = \sigma\underline{E}^{(r)} + \sigma\underline{E}^{(t)} \tag{3.15}$$

$$= -\nabla(\sigma\phi) + \sigma\underline{E}^{(t)} \tag{3.16}$$

by the assumed form for $\underline{E}^{(r)}$. Thus (3.16) would represent a natural decomposition of $\underline{J}$ by the Helmots's theorem if $\underline{J}^{(t)} = \sigma\underline{E}^{(t)}$ were expressible as the curl of some quantity. In other words, given the assumed form for $\underline{E}^{(r)}$ it is natural to assume that $\underline{J}^{(t)}$ can support a magnetic field $\underline{\tilde{B}}$, say. As $\underline{J}$ can be decomposed into two parts by (3.4), (3.13) and (3.14), the reaction term $\underline{R}$ in (3.3) can be put into the form,

$$\underline{R} = -\underline{L}_1 - \underline{L}_2 - e'\rho\underline{E}^{(r)} , \tag{3.17}$$

where

$$\underline{L}_1 = \underline{J}^{(t)}{}_\wedge \underline{B}_o , \tag{3.18}$$

and

$$L_2 = \underline{J}^{(r)}{}_\wedge \underline{B}_o , \tag{3.19}$$

$\underline{L}_1$, $\underline{L}_2$ and $e'\rho\underline{E}^{(r)}$ are forces per unit volume impressed on the induced currents and charges by the particles magnetic induction field $\underline{B}_o$ and its electric field $\underline{E}^{(r)}$. Using (3.12),(3.17) and (3.19), the equation of motion (3.2) can be expressed in the form,

$$m\rho\frac{d\mathbf{V}}{dt} = \mathcal{F}_1 + \mathcal{F}_2 + \mathcal{F}_3 - (\underline{L}_1 + \rho\nabla V_1), \quad (3.20)$$

where

$$\mathcal{F}_1 = -\rho e'\underline{E}^{(r)} - \rho\nabla(kT), \quad (3.21)$$

$$\mathcal{F}_2 = \rho e'\underline{E}^{(t)} - \underline{L}_2, \quad (3.22)$$

and

$$\mathcal{F}_3 = \rho e'\mathbf{V}_\Lambda\underline{\tilde{B}} - kT\nabla\rho. \quad (3.23)$$

The terms have now been paired in anticipation of the various equilibria discussed in the previous section. The problem now reduces to showing that the set of equations,

$$\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3 = 0, \quad (3.24)$$

represent an internally consistent set under the laws of classical physics and at the same time that they have a solution consistent with supporting a thermal field of the form (1.10) which we know to be a necessary condition if equation (3.2) is to reduce exactly to (1.34).

Let us start this analysis to prove consistency by considering the term L_1 which occurs in $\mathcal{F}_2$ given by (3.22). By (3.5) and (3.19)

$$\underline{L}_2 = \underline{J}^{(r)}\,{}_\Lambda\underline{B}_o = \underline{J}^{(r)}{}_\Lambda\mu_o\rho\beta\underline{k} \quad (3.25)$$

Thus, if

$$\underline{J}^{(r)} = \underline{J}^{(t)}{}_\Lambda\,\underline{k}\,, \quad (3.26)$$

a condition to be discussed further in a moment, then

$$\underline{L}_2 = -\underline{J}^{(t)}\mu_o\rho\beta$$
$$= -\sigma\underline{E}^{(t)}\mu_o\rho\beta, \quad (3.27)$$

by (3.4), and if we take the dipole moment β to be such that

$$\sigma\beta\mu_o = -e' \quad (3.28)$$

then the equation,

$$\mathcal{F}_2 = 0, \quad (3.29)$$

will certainly hold as can be see from (3.22). (3.26) still remains to be explained. The induced electric currents $\underline{J}^{(t)}$ and $\underline{J}^{(r)}$ will involve no local charge occuring provided that for every negative charge induced one positive charge is also induced into the conduction current streams. Thus if two equal and opposite charges always emerge from their creative polarisation of a local state of zero charge into perpendicular directions on the $\underline{i}$, $\underline{j}$ plane a condition such as (3.26) is bound to hold. In fact, the mathematical reason for condition (3.26) is related to the fact that the components of $\underline{J}^{(t)}$ and $\underline{J}^{(r)}$ are solutions of Laplace's equation in two dimensions which in turn is a consequence of the Cauchy Riemann equations (1.39) and (1.40) but this is not obvious at this stage of the analysis. We shall now take

(3.28) to be the actual physical relationship between the parameters σ, β, μ_o and e' because it is a necessary condition if (3.29) is to hold. Let us now look more closely at $\underline{J}^{(t)}$, the part of the current that can be the source of the magnetic induction field $\underline{B}$ through the Maxwell equation,

$$\mu_o \underline{J}^{(t)} = \nabla_\wedge \tilde{\underline{B}}. \qquad (3.30)$$

We can relate $\tilde{\underline{B}}$ to a field of magnetisation $\underline{M}$ say by the relation

$$\tilde{\underline{B}} = \mu_o \underline{M}, \qquad (3.31)$$

and if $\underline{M}$ is related to a magnetic field $\underline{H}$ as in a perfect diamagnetic substance we have,

$$\underline{M} = - \underline{H}, \qquad (3.32)$$

$$\tilde{\underline{B}} = - \mu_o \underline{H}. \qquad (3.33)$$

As all the equation $\underline{B}$, $\underline{M}$ and $\underline{H}$ are internally induced quantities the question as to which of them is primary is somewhat obscure unlike in the more usual situation when the symbol $\underline{H}$ is used for the external driving magnetic field and then $\underline{H}$ is clearly the primary quantity in that it causes the magnetic induction $\underline{B}$. We now wish to bring thermal processes into the magnetisation relationships. Curie's law, which we shall express in the form,

$$\frac{M_o}{H} = \frac{\mu_o \rho \beta^2}{\lambda k T} \qquad (3.34)$$

where λ is a constant to be determined, is a well-known connecting formula between magnetic and thermal effects. We shall assume that (3.34) holds in our

magnetisable electronic fluid context and relates the quantities shown in (3.34) which have been defined earlier. By (3.5) we have that the magnetisation $\underline{M}_o$ carried by the particle is given by,

$$\underline{M}_o = \rho \beta \underline{k}\,. \qquad (3.35)$$

If we use the magnitude of $\underline{M}_o$ in (3.34), we get the relation,

$$H = \frac{\lambda k T}{\mu_o \beta}\,. \qquad (3.36)$$

However, if our electromagnetic scheme is to converge onto quantum mechanics it is necessary that the temperature which has emerged through relation (3.34) should have the same functional form as the temperature which occurs in equation (1.12) and is more precisely defined by equation (1.10). We have seen that $\nu \ell n \rho$ is the stream function for the $v_1\underline{i} - v_2\underline{j}$ flow field in the relations (3.9) and (3.10). We can use this fact to express the temperature of equation (1.10) in the form,

$$kT = m\nu \frac{\partial v_2}{\partial x}\,, \qquad (3.37)$$

and using the Cauchy-Riemann equations (1.39) and (1.40), we can put equation (3.37) into the form,

$$kT\underline{k} = \frac{m\nu}{2} \nabla_\wedge \mathbf{v}\,. \qquad (3.38)$$

Thus a quantum mechanically compatible form for the magnetic induction field $\tilde{\underline{B}}$ is from (3.31), (3.32) and (3.36)

$$- \underset{\sim}{\tilde{B}} = \mu_o \underline{H} = \lambda \frac{kT}{\beta} \underline{k}$$

$$= \frac{\lambda m \nu}{2\beta} \nabla_\Lambda \mathbf{v} \; . \qquad (3.39)$$

As it is essential that our electromagnetic structure in compatible with the form (3.39), we shall now feed that form into the structure together with the values we know that ν and β must have in the case of an electron. These values are from (1.10) $\nu = \hbar/2m$ and one Bohr magneton for β. That is

$$\beta = - \hbar e'/2m. \qquad (3.40)$$

Thus we shall now work with

$$\underline{\tilde{B}} = \frac{\lambda m}{2e'} \nabla_\Lambda \mathbf{v}. \qquad (3.41)$$

If we substitute this functional form for $\underline{\tilde{B}}$ into (3.30), we obtain for $\underline{J}^{(t)}$,

$$\mu_o \underline{J}^{(t)} = \frac{\lambda m}{e'} \nabla_\Lambda (\nabla_\Lambda \mathbf{v}). \qquad (3.42)$$

Using the Cauchy Riemann equations (1.39) and (1.40), (3.42) can be written in the form,

$$\mu_o \underline{J}^{(t)} = \frac{\lambda m}{e'} \frac{\partial^2}{\partial x^2} (v_1 \underline{i} - v_2 \underline{j}), \qquad (3.43)$$

and so by using (3.26), we obtain $\underline{J}^{(r)}$ in the form,

$$\mu_o \underline{J}^{(r)} = - \frac{\lambda m}{e'} \frac{\partial^2}{\partial x^2} (v_1 \underline{j} + v_2 \underline{i}). \qquad (3.44)$$

By (3.4) and the Cauchy-Riemann equations,

$$e' \underline{E}^{(r)} = - \frac{\lambda m}{\mu_o \sigma} \nabla \left(\frac{\partial v_2}{\partial x}\right)$$

$$= - \frac{\lambda}{\mu_o \sigma \nu} \nabla (kT) \qquad (3.45)$$

having used (3.37).

Thus the equilibrium condition,

$$\mathcal{F}_1 = 0, \qquad (3.46)$$

will hold if,

$$\frac{\mu_o \sigma \nu}{\lambda} = 1. \qquad (3.47)$$

However, from (3.28) and (3.40)

$$\mu_o \sigma \nu = 1. \qquad (3.48)$$

(3.47) and (3.48) together give for the value of the constant λ,

$$\lambda = 1. \qquad (3.49)$$

The reader will recognise that this value for λ gives what is regarded as the quantum form for Curie's law (3.34). If we now substitute the expression (3.41) for $\tilde{B}$ into the final equilibrium condition from (3.23),

$$\mathfrak{F}_3 = 0, \qquad (3.50)$$

we get, using (3.38),

$$\rho\bar{\mathbf{V}} \wedge \frac{\lambda mkT\underline{k}}{\nu} - kT\nabla\rho = 0$$

or

$$\underline{k}\wedge\bar{\mathbf{V}} = -\nu\nabla\ell n\rho \qquad (3.51)$$

where we have used (3.49). Thus we recover the condition (3.8) that $\nu\ell n\rho$ is the stream function for the $\bar{\mathbf{V}}$ flow. We have now established that the quantum form for the temperature distribution is a possible solution of the electrodynamic set ot equilibrium equations $\mathfrak{F}_1 = \mathfrak{F}_2 = \mathfrak{F}_3 = 0$ and, indeed, that these equations are mutually consistent with the set of parameters involved having values related by (3.40), (3.48) and (3.49). The equation of motion (3.2) or (3.20) reduces to,

$$\rho m \frac{d\bar{\mathbf{V}}}{dt} = -\underline{J}^{(t)} \wedge \underline{B}_o - \rho\nabla V_1 . \qquad (3.52)$$

where by (3.5), (3.26) and (3.40) we have

$$\underline{L}_1 = \underline{J}^{(t)} \wedge \underline{B}_o = \rho\nabla(kT) . \qquad (3.53)$$

By rearranging the expression for L_1 and using (3.30) and (3.35) it can be put into the form,

$$\underline{L}_1 = \underline{M}_o \wedge \nabla \wedge \underline{\tilde{B}} = -\underline{M}_o \wedge \nabla \wedge \underline{M}\mu_o , \qquad (3.54)$$

using (3.32) and (3.33). Thus the equation of motion can be expressed in the yet more transparent form,

$$\rho m \frac{d\bar{\mathbf{V}}}{dt} = \mu_o \underline{M}_o \wedge \nabla \wedge \underline{M} - \rho\nabla V_1 . \qquad (3.55)$$

and in this form we can see clearly that the extra '*quantum*' force is just the force exerted by the induced magnetic field $\underline{M}$ in virtue of its inhomogeneity on the dipole magnetisation M_o of the particle. A point of interest which arises from this work is the relation (3.48), $\sigma\mu_o\nu = 1$, connecting $\nu = \hbar/2m$ to the conductivity of the electronic fluid. From this we see that $\mu_o\nu$ in the resistivity of the fluid medium and this identification of $\mu_o\nu$ is rather similar to the identification of ν with a viscosity coefficient which can be found in the work of Nelson ([18]) and others.

Conclusions

This analysis removes the mystery from the extra term which occurs as a force in representations of Schrödinger quantum mechanics in the form of a Newton like equation of motion. The occurence of this special quantum force is the feature of Schrödinger quantum theory which has been regarded as essentially distinguishing the Schrödinger type of theory from the classical type of theory. However, as has been shown here that this extra term can be explained on a classical basis and this suggests that quantum mechanics is not so philosophically distinct from what is often called classical physics. It seems that analysis of the local properties of the analytically continued configuration variables of Schrödinger quantum theory in terms of their representations in a real two dimensional space leads to some surprising results. In particular, it seems that the internal structure of the electronic fluid is in essence an example of perfect plasma containment.

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