Annales de la Fondation Louis de Broglie
Vol. 6, n° 3, 1981

LIGHT INTENSITY DEPENDENCE OF PHOTON ENERGY

by E. PANARELLA

National Research Council of Canada
Ottawa, Canada K1A OR6

(manuscrit reçu le 15 Septembre 1980)

NDLR : L'article qui va suivre a suscité chez les lecteurs du comité de rédaction quelques réserves ou critiques. Conformément à notre ligne de conduite nous le publions néanmoins volontiers ; dans un prochain numéro nous publierons les remarques relatives à cet article.

Abstract : It is shown that Heisenberg's principle for position and momentum of a photon admits an interpretation in terms of an interaction law which has the same form as a Newtonian potential. The application of the interaction law to a collection of photons leads to the well-known Kirchhoff's formula governing the geometrical distribution of light on a wave pattern. Moreover, since the interaction law prescribes an exchange of momentum (and energy) between interacting photons, it is shown that such exchange is a function of photon number density or light intensity.

Résumé : On montre que le principe d'Heisenberg, appliqué à la position et à l'impulsion d'un photon, admet une interprétation à l'aide d'une loi d'interaction qui a la même forme qu'un potentiel newtonien. L'application de cette loi d'interaction à un ensemble de photons conduit à la formule bien connue de Kirchhoff gouvernant la distribution géométrique de la lumière dans une propagation d'ondes. En outre puisque la loi d'interaction implique un échange d'impulsion (et d'énergie) entre les photons qui interagissent, on montre qu'un tel échange est fonction de la densité du nombre de photons, ou de l'intensité lumineuse.

1. Introduction

In a recent paper by Allen ([1]) the uncertainty principle for position and momentum of a photon has been used to derive a lower bound for the energy of photons in a focussed light beam. It has been shown that the energy of focussed photons cannot be less than a certain minimum value and that some photons, if sharply focussed, are bound to experience an upward energy shift. This result confirms the hypothesis of energy enhancement for focussed photons previously advanced "ad hoc" by Panarella ([2-4]) in order to explain non-linear photoionization events as single-photon processes.

Allen's analysis shows also that the photon energy shift depends only on the geometrical parameters characterizing the focussing. The intensity, or the coherence, or any other physical property of the light does not seem to affect the energy variation. Since the photon energy enhancement presumably originates from one of the following possibilities : a) annihilation in the focal region of a few photons, whose energy is transferred to the surrounding photons, b) inelastic photon-photon scattering, some photons gaining energy in the scattering process at the expense of energy loss from surrounding photons ; and these are clearly non-linear processes in the electromagnetic field equations, observable only at high light intensity (and not just at *any* intensity), it seems clear that Allen's analysis leads to contradiction. This contradiction emerges also from the fact that those same photoionization experiments in the field of laser-induced gas ionization which prompted Allen's analysis had already provided evidence that the intensity and the coherence of light ([4-5]), and not just the focussing geometry, play an important role in the ionization process.

In the following, we shall show that a study of the uncertainty principle suggests Allen's analysis to be a first order approximation of a state of affairs in which the principle, when derived from real experiments, is due to the existence of a particle-particle (or photon-photon) interaction[1]. Since the effect of the interaction becomes more perceptible as the light intensity increases, the energy enhancement of some photons in a focussed beam is shown to depend on light intensity or photon number density, in agreement now with the experimental results. Moreover, when the interaction law is deduced from Heisenberg principle and applied to a collection of photons, the well-known Kirchhoff theorem governing the geometrical distribution of light is obtained. This result shows that photons, as particles, arrange themselves on a wave pattern, with maxima and minima, only because they are constantly under the influence of a law of interaction and not because they are guided by waves.

2. Analysis

The analysis will begin with a study of Heisenberg's

[1] By 'interaction' we mean any mechanism whereby particles can exchange momentum and energy.

principle and the way it was derived. One of the derivations, and indeed the most famous one, was given by Heisenberg himself (6). He considers an electron moving along an axis x. In order to observe the electron and to determine its position, a microscope is used. Any photon used to observe the electron transmits a momentum to the electron which is uncertain by

$$\Delta p_x \simeq \frac{h}{\lambda} \sin\theta \qquad (1)$$

where θ is the microscope angular aperture. On the other hand, since the resolving power of the microscope is

$$\Delta x = \frac{\lambda}{\sin\theta} \qquad (2)$$

one gets

$$\Delta p_x . \Delta x \simeq h \qquad (3)$$

It is clear therefore that Heisenberg's principle can be experimentally formulated only because, in this case, a photon interacts with an electron. If the two particles ignored one another, the photon would have proceeded undisturbed by the electron and the indeterminacy principle could not have been established. That an observer, in turn, picks up the information carried by the scattered photon and makes what is called a "measurement" does not change the physical reality that the photon interacted and transferred momentum to the electron. The question of observability vs. physical reality is therefore irrelevant here.

Of course, photons and electrons are known to interact, and so do electrons and electrons, and protons and protons, and electrons and protons etc. But, when it comes to photons and photons, these are not conceived and permitted to interact in normal circumstances. Unless one considers an experiment whereby electrons, for instance, are used to de-

tect the position of a photon, the Heisenberg uncertainty relation cannot be experimentally proven to apply to photons. The use of electrons (as well as any other particle) is, however, ruled out because the uncertainty principle can be formulated only from a generalization of the outcome of real experiments, such as Compton's. The use of electrons for the detection of the position of a photon would imply the existence of an inverse-Compton effect, involving the transfer of energy from free electrons to photons. This effect has never been found. Therefore, one should look for other ways to prove experimentally that photons obey the uncertainty principle. Otherwise Allen's analysis, based as it is on the uncertainty principle for photons, loses much of its meaning.

We would like to offer here the following experimental proof. Consider Fig. 1 which represents the classical experiment of a beam of photons crossing a narrow slit. After crossing the slit, the x-momentum of each and every photon is changed from zero to anywhere between $-p_x$ and $+p_x$. For reasons that will become clear momentarily, we will here disregard the mathematical model of the photon as a wave-packet and the interpretation of the experiment as a diffraction effect. Disregarding, then, the wave-particle interpretation of the phenomenon, this change of photon momentum can occur only if the photons interact, some of their momentum being transferred to the surrounding photons. The uncertainty of the x-component of the momentum is $\Delta p_x = p\sin\theta = \frac{h}{\lambda}\sin\theta$. The uncertainty of the x-coordinate of the photon is $\Delta x = \frac{\lambda}{\sin\theta}$ (both of these are experimental results and λ is a characteristic length which does not involve the undulatory concept of light but is determined solely from the intensity distribution of light in the diffraction pattern). Multiplying

Δx by Δp_x one gets the same expression for the uncertainties of position and momentum as previously found. This interpretation of interacting photons, therefore, provides the required experimental proof that photons obey the uncertainty principle.

In order now to justify our disregarding of the wave-packet interpretation of the photon we should inquire : 1) whether this interpretation really provides a valid alternative explanation and proof of the uncertainty principle and 2) whether or not the wave-packet model is necessary in order to explain the pattern obtained when photons cross the slit one at a time. In this case, of course, the interaction between photons is absent and the formulation of the uncertainty principle in terms of interacting photons is therefore impossible.

To answer the first question, one should recall that, according to standard quantum theory, a photon is defined as a localized wave-packet formed from the superposition of plane waves of many different frequencies all grouped around some central frequency. It carries the photon's energy-momentum at the group velocity and is completely specified by an amplitude function $\psi(x,t)$ where $|\psi(x,t)|^2$ is interpreted as the probability of finding the photon at coordinate (x,t). If the momentum of the photon is $p = \frac{h\nu}{c}$, the group velocity of the wave-packet is c, and the central wavelength is $\lambda = \frac{h}{p}$ (de Broglie wavelength). This definition of a photon as a wave-packet postulates the existence of a spread of wavelength or momentum. In other words, the spread of momentum of a photon is assumed 'ab initio', or built in the definition, and not derived. Analogously, the definition of the photon as a 'localized' wave-packet implies that the photon must be within an interval Δx. The associated waves therefore must interfere destructively outside this interval.

Recall that the spread of momentum is given by :

$$\Delta p = \Delta \frac{h}{\lambda} = h\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right) \qquad (4)$$

where λ_1 and λ_2 are the shortest and the longest of all possible wavelengths that make up the wave-packet, respectively.

Multiplying both sides of (4) by Δx :

$$\Delta p.\Delta x = h\left(\frac{\Delta x}{\lambda_1} - \frac{\Delta x}{\lambda_2}\right) \qquad (5)$$

At the extremes of Δx all the waves must interfere destructively in order to cancel the probability of finding the photon outside the interval Δx. Therefore the interval Δx must contain at least one wavelength more of the wave λ_1 than of λ_2. Hence

$$\left(\frac{\Delta x}{\lambda_1} - \frac{\Delta x}{\lambda_2}\right) \geqslant 1$$

from which

$$\Delta p.\Delta x \geqslant h \qquad (6)$$

This shows that the mathematically defined photon as a localized wave-packet contains all the ingredients that make it consistent with the uncertainty principle, but certainly does not prove it. The proof can only come from real experiments (7).

As to the second subject of inquiry, whether or not the wave-packet model of the photon is necessary in order to explain the pattern obtained when photons cross the slit one at a time, one should turn one's attention to the fact that no experiment has ever proven that photons are emitted as isolated particles from any source. Quite to the contrary, one always deals with a source of light whose individual atoms cannot radiate independently of each other, because they are constantly interacting with a common radiation field ([8]). This implies that photons are not emitted at random but have certain characteristic bunching properties[2] ([9]). The bunching properties are also essential for the detection process.

Let us consider, in fact, one of the most conventional methods of photon detection, the photographic process. It is known that the photographic grains, when exposed to light, do not become developable unless they absorb at least three or four photons within a time which can be assumed to be of the order of the coherence time of the light[3] ([12]). This means that the photographic grain acts as an R-fold coincidence counter, where R is of the order of 3, 4 or more ([15]). The coherence time $\Delta\tau$, which is the inverse of the light bandwidth, in the typical case of thermal light of peak wavelength $\lambda_0 = 5000$ Å and bandwidth $\Delta\nu = 10^{12}$ sec^{-1} (corresponding to $\Delta\lambda \simeq 10$ Å) is of the order of 1 psec. Photons, therefore, in order to be detected, have to be confined within a distance of the order of the coherence length $c\Delta\tau = 3 \times 10^{-2}$ cm from the target grain. This shows that, in the experiments performed with extremely low light intensity ([16-17]), the assumption that photons were crossing the detection apparatus one at a time was incorrect. If the probability was high that the photons were widely separated, the detection probability then became extremely small. An assumption more in line with reality would be that, no matter what the light intensity, the photons are detected only because they are bunched and hit the target detector as a "clump". We are then dealing in reality with packets of photons and not with packets of waves or wave-packets and the interaction between contiguous photons is not ruled out even at very low light intensities.

[2]The first experiment aimed at determining the Bose-Einstein "clumping" effect of photons was successfully performed by Hanbury-Brown and Twiss ([10-11]).

[3]In general, more than this number will be required since not all the photoelectrons which are produced make a contribution to the latent image formation ([13-14]).

Although we centered our argument around the photographic process of optical photon detection, we see no reason for ruling it out in the case of any other type of fast detection, photoelectric, for instance ([18-19]). Actually, the fact that no photoelectric detector in the optical range has 100 % quantum efficiency shows that a minimum number of photons, larger than one, is required for the release of an electron from a photoemmisive surface.

At any rate, were one able to generate a beam of light whose individual photons were statistically independent, the photons would not be able to interact and any diffraction or interference pattern would then disappear. An experiment along this line has been done. The intensity of a thermal light source was greatly reduced by reducing the number of atoms excited at the source and the photons emitted were then statistically independent. Indeed, the interference pattern disappeared ([20]).

The uncertainty principle refers, therefore, to interacting particles. The principle has never been contradicted. It has the status of a physical law experimentally found. The interpretation, however, of the terms Δp and Δx has always been given in terms of uncertainties of the outcome of the measurement or observation. But, once it is recognized that the observation is not necessary and it is admitted that Heisenberg's relation is a consequence of the interaction between particles, the interpretation of the principle changes. Let us refer again to Fig. 1. Assume that the intensity of light is such that only two photons, at a particular instant of time, cross the slit. Immediately after the slit, because of the mutual interaction and the absence of other photons in the immediate surrounding, the two photons position themselves at a distance equal to the slit width Δx. On the other hand, the original momentum p of each photon changes, soon after the slit, by an amount $\Delta p_x = p\sin\theta$. We have to assume the maximum amount of momentum change in order not to violate the uncertainty principle $\Delta p \Delta x \simeq h$. In other words, each photon receives an amount of momentum Δp_x from the other and the product of $\Delta x = \frac{\lambda}{\sin\theta}$ and $\Delta p_x = \frac{h}{\lambda}\sin\theta$ (these equalities, recall, are experimental results) brings back to the known formula $\Delta x.\Delta p \simeq h$. Based on the new interpretation of its constituent terms, this formula can be written in a more appropriate and concise way :

$$x.p_x \simeq h$$

or

$$p_x \simeq \frac{h}{x} \qquad (7)$$

where p_x is now the change of momentum of a photon in the x-direction caused by the presence of another photon located at a distance x from the first.

To summarize, if the Heisenberg principle is intrinsic in any system (in an atom, for instance, or in a collection of photons), and the principle is a consequence of the interaction between particles, expression (7) represents the law of interaction. This law bears some similarity of form with Newtonian potentials ([21]) because of the inverse function of the distance character.

3. Derivation of Kirchhoff's formula from the interaction law

The formulation of the interaction law leads to inquire whether eq. (7) is capable of explaining known experimental results, the most obvious of which is the arrangement of photons on a wave pattern on a screen. To this end, one has to consider a collection of photons where any two of them can be assumed to be the couple to which Heisenberg's principle applies. Formula (7) has now general validity because it represents a law of momentum transfer due to the Heisenberg's principle[4] between any two particles separated by a distance equal to x. In other words, each particle is subjected to the field (7) of momentum transfer and this field is generated by each and all surrounding particles. The maximum momentum transfer between any two particles (assuming that they are not affected by the presence of other particles) occurs when they approach one another by a distance equal to λ, the momentum exchanged in this case being their own original momentum $p = \frac{h}{\lambda}$. In all other situations, only a fraction of the original momentum is transferred from one particle to the other, depending on the distance x, and this fraction approaches the limit zero when $x \to \infty$. In the light of this interpretation we see that the familiar parameter λ assumes the simple physical meaning of minimum distance of approach between particles, no distance smaller than λ being conceivable. Likewise, since we are dealing with particles endowed with velocity v (or c in the case of photons), the frequency $\nu = \frac{1}{\tau}$ assumes the simple physical meaning of re-

[4]This is in addition to any other particle-particle interaction which may also lead to momentum-energy transfer and may in fact be predominant.

ciprocal of the time it takes for the two particles to travel a distance equal to $\lambda(\tau = \frac{1}{\nu} = \frac{\lambda}{v})$.

If the particle motion is perturbed by the presence of all surrounding particles, let us determine the amount of the perturbation. To this end, let us first write eq. (7) in a more general way :

$$p(r) = \frac{h}{r} \qquad (8)$$

In the optical case, to which we will adhere without loss of generality from now on, the optical disturbance to which a photon is subjected can be calculated by standard means of potential theory [22] by applying Green's theorem to a region R containing the position P(x,y,z) of the photon (test photon) :

$$\int_R (U\nabla^2 V - V\nabla^2 U)dv = \int_S \left(U\frac{\partial V}{\partial n} - V\frac{\partial U}{\partial n}\right)ds \qquad (9)$$

In this identity we take for V the function (8). Since p(r) has a singularity for r = 0, the identity (9) cannot be applied to the whole region R. Therefore, we surround P with a small sphere σ with P as a centre, and remove from R the interior of the sphere. For the resulting region R' we have, since p(r) is harmonic in R' :

$$-\int_{R'}\frac{1}{r}\nabla^2 U dv = \int_S \left(U\frac{\partial}{\partial n}\frac{1}{r} - \frac{1}{r}\frac{\partial U}{\partial n}\right)ds + \int_\sigma \left(U\frac{\partial}{\partial n}\frac{1}{r} - \frac{1}{r}\frac{\partial U}{\partial n}\right)ds \quad (10)$$

where $\frac{\partial}{\partial n}$ signifies a partial derivative in the *outward* normal direction at each point on S. Hence, the last integral may be written

$$\int_\Omega \left(U\frac{1}{r^2} + \frac{1}{r}\frac{\partial U}{\partial r}\right) r^2 d\Omega = \overline{U}.4\pi + \int_\Omega r\frac{\partial U}{\partial r}d\Omega \quad (11)$$

where $\overline{U}$ is a value of U at some point of σ, and the integration is with respect to the solid angle subtended at P by the element of σ. As the radius of σ approaches zero, the limit of the integral over σ in (10) is 4πU(P) and the volume integral on the left converges to the integral over R. We thus arrive at :

$$U(P) = -\frac{1}{4\pi}\int_R \frac{\nabla^2 U}{r}dv + \frac{1}{4\pi}\int_S \frac{\partial U}{\partial n}\frac{1}{r}ds - \frac{1}{4\pi}\int_S U\frac{\partial}{\partial n}\frac{1}{r}ds \quad (12)$$

This is the expression to be satisfied by the second function U in Green's theorem if the first function V is equal to $\frac{h}{r}$.

We would like now to find the form to be taken by the function U in order to satisfy our physical conditions and equation (12). Since we are dealing with a stream of photons moving with velocity c, whose position is changing with time t, the function U representing the optical disturbance at the fixed point P has to depend on time t also :

$$U = U(P,t) \qquad (13)$$

Moreover, since simultaneity of cause and effect (action at a distance) is excluded here, any signal emitted by the moving photons will be transmitted with finite velocity c and will be received by the point P after a time $\frac{r}{c}$. Hence, the integration in (12) must be performed not at time t, but at the retarded time $t - \frac{r}{c}$. In other words, the function U to be inserted in (12) is :

$$U = U(x,y,z,t - \frac{r}{c}) = [U] \qquad (14)$$

where the square brackets indicate the retarded value of the function.

The optical disturbance produced at the point P is given by U(x,y,z,t). If one considers only one photon, travelling with velocity $\vec{c}$, the field of momentum transfer of such a moving source is expressed, in polar coordinates, as [23]

$$p(\vec{r},t) = \frac{h}{\rho - \frac{\vec{c}.\vec{\rho}}{c}} \qquad (15)$$

where $\rho = |\vec{r} - \vec{c}t_0| = c(t-t_0)$ and t_0 represents a time such that a signal emitted by the photon at t_0 will arrive at $\vec{r}$ at time t.

In a cartesian coordinate system, in which the x-axis coincides with the velocity vector $\vec{c}$, expression (15) becomes :

$$p(x,y,z,t) = \frac{h}{c(t-t_0) - \frac{c(x-ct_0)}{c}} = \frac{h}{ct-x} \qquad (16)$$

Therefore, the field of momentum transfer of the photon is a function that satisfies the wave equation :

$$\nabla^2 p = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} \qquad (17)$$

The field of a collection of photons, because of the linearity of wave equation (17), is given by the sum of the fields of each photon. Hence, the function U also satisfies the wave equation.

In summary, it has been verified that the application of Green's theorem to a region R containing the position of the test photon leads to eq. (12), when the function V has been chosen to be the one expressed by (8), the interaction law. For the physical conditions given by a stream of photons moving with velocity c, the

second function to be inserted into Green's formula is U(x,y,z,t), which represents the momentum transferred from all the streaming photons to the fixed test photon, and which must be calculated at the retarded time $t - \frac{r}{c}$ in the integration of (12). It was also proven that the field of momentum transfer (16) for a moving photon satisfies the wave equation (17). Therefore, the function U(x,y,z,t), which represents the field of momentum transfer at point P at time t, is the sum of the fields of each photon, thus satisfying the wave equation

$$\nabla^2 U = \frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} \qquad (18)$$

This and the previous analysis are sufficient to lead, by straightforward but tedious calculations [24], to :

$$U(x,y,z,t) = -\frac{1}{4\pi}\int_S \left\{ [U]\frac{\partial}{\partial n}\frac{1}{r} - \frac{1}{r}\left[\frac{\partial U}{\partial n}\right] - \frac{1}{cr}\frac{\partial r}{\partial n}\left[\frac{\partial U}{\partial n}\right] \right\} ds \qquad (19)$$

This is the well-known general form of Kirchhoff's theorem, as found in any textbook on optics [25]. It expresses the field of momentum transfer produced by a collection of photons, randomly distributed in space and time, at a point (P,t). Given the appropriate boundary conditions, the momentum transferred to the test photon (P_0,t) can then be deduced. The formula does not depend on such physical properties of the photons as energy and momentum, and therefore it is valid for any photon. However, when a particular class of photons is considered, for instance, a monoenergetic group of photons, then a characteristic parameter or length λ appears :

$$\lambda = \frac{h}{p}$$

and this parameter, previously defined as the minimum separation between two isolated photons, affects the photon space and time distribution and specifies the wave function U of Eq. (18). The problem of determining such distribution and finding U(P,t) in this case will not be dealt with here.

In conclusion, the important result has been achieved that the application of law (8) to a collection of streaming photons leads to Eq. (19), i.e. to Kirchhoff's theorem, of fundamental importance in optics as a basis for understanding all diffraction phenomena of light.

4. Discussion

The main result presented in this paper is an interaction law for particles derived from the Heisenberg uncertainty principle. The law is universal, in that it applies to all particles, irrespective of their size, mass, density, initial momentum etc., as well as to non-identical particles. The law states that any two particles exchange momentum, in the amount given by (8), a function of the distance r only. This implies that each and every particle is always in contact with the rest of the world, although the knowledge available of what happens at distance points is negligible. The motion of the particles is affected by the interaction law. In particular, when a stream of photons is considered, it is found that their path is governed by Kirchhoff's theorem, which is the basis for explaining the diffraction phenomena of light in various geometrical configurations of slits and screens.

Unlike the Heisenberg principle, which is a statement of the limitation of the outcome of some measurement, the interaction law makes definite statements. What previously was considered an uncertainty $\Delta p_x = p \sin\theta$ of the extent of momentum change for a particle crossing a slit

now becomes a definite statement as to the momentum change : $p_x = p \sin\theta$. Likewise, what was considered an uncertainty $\Delta x = \frac{\lambda}{\sin\theta}$ of the position of a particle now assumes the precise physical meaning of position x of a particle relative to another particle. The justification for these definite statements rests on the ground that the experimental proof of the Heisenberg principle implies an interaction law which correctly predicts the outcome of the experiments.

As to the reason for not using the general formulation of the Heisenberg principle, in which the $\geqslant$ sign appears, this is because we considered in the paper ideal experiments, where the $\simeq$ sign holds.

It should be mentioned that any exact solution of diffraction problems can be obtained only through Maxwell's equations, which are vectorial equations. Our treatment here has been confined to the derivation of the Kirchhoff integral theorem which applies to each of the Cartesian components of the field vectors ([26]).

Finally, if the Heisenberg interaction is always present, and the effect of the interaction normally manifests as a momentum exchange between particles, with no significant energy exchange, the situation is modified when the number density of particles becomes very high. In the case of focussed high intensity laser beams, significant energy exchange can in fact take place between photons. Allen's formula for the lower bound of photon energy tries to set a quantitative value for the upward energy shift :

$$E \geqslant \frac{hc\tau}{r}$$

where τ is a geometrical parameter characterizing the focussing and r is the uncertainty of the photon position as derived from the conventional interpretation of the Heisenberg principle. In light of the interpretation of the same principle offered here, r has now the meaning of separation or distance between photons. Since r is obviously a function of the photon number density or light intensity I, the energy shift depends on I, in agreement now with the experimental results.

Acknowledgement

We would like to thank P. Savic, J.H.W. Lau and R.P. Gupta for clarifying discussions on the subject of this paper.

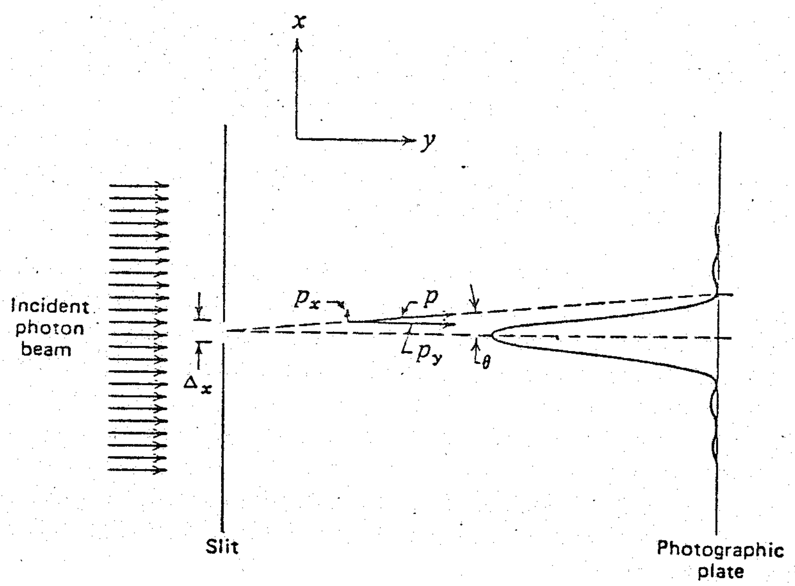

Fig. 1

RÉFÉRENCES

(1) A.D. Allen, Found, Phys. 7, 609 (1977)

(2) E. Panarella, Found. Phys. 4, 227 (1974)

(3) E. Panarella, Phys. Rev. Lett. 33, 950 (1974)

(4) R.P. Gupta and E. Panarella, Lett. Nu. Cim. 22, 484 (1978)

(5) R.P. Gupta, E. Panarella and B.S. Bhakar, "Vacuum thermalization of high intensity laser beams and the uncertainty principle", Proc. 2nd Intern. Conf. on Energy Storage, Compression and Switching, Venice, Italy, 1978

(6) W. Heisenberg, "The Physical Principles of the Quantum Theory", (The University of Chicago Press, Chicago), 1930, p. 31

(7) R.B. Lindsay and H. Margenau, "Foundations of Physics", (John Wiley & Sons, New York) 1949, p. 399

(8) R.H. Dicke, Phys. Rev. 93, 99 (1954)

(9) L. Mandel and E. Wolf, Rev. Mod. Phys. 37, 231 (1965)

(10) R. Hanbury-Brown and R.Q. Twiss, Proc. Roy. Soc. A242, 300 (1957)

(11) R. Hanbury-Brown and R.Q. Twiss, Proc. Roy. Soc. A243, 291 (1958)

(12) W.M. Rosenblum, Journ. Opt. Soc. Am. 58, 60 (1968)

(13) J.C. Dainty and R. Shaw "Image Science" (Academic Press, London) 1974, p. 34

(14) P. Kowaliski, "Applied Photographic Theory" (John Wiley § Sons) 1972, p. 320

([15]) H.J. Zweig and D.P. Gaver, IBM Journ. Res. Dev. 9, 100 (1965)

([16]) G.I. Taylor, Proc. Cambridge Philos. Soc. 15, 114 (1909)

([17]) A.J. Dempster and H.F. Batho, Phys. Rev. 30, 644 (1927)

([18]) R.L. Pfleegor and L. Mandel, Phys. Rev. 159, 1084 (1967)

([19]) L. Csillag, M. Janossy, and Zs. Naray, Acta Phys. Acad. Sc. Hung 32, 275 (1972)

([20]) Yu.P. Dontsov and A.I. Baz, Sov. Phys. JETP 25, 1 (1967)

([21]) O.D. Kellog, "Foundations of Potential Theory" (Frederick Ungar Publishing Co., New York), p. 53

([22]) Ref. 21, p. 218

([23]) P.M. Morse and H. Feshbach, "Methods of Theoretical Physics" (McGraw-Hill, New York) 1953, Part I, p. 841

([24]) B.B. Baker and E.T. Copson, "The Mathematical Theory of Huygens' Principle", (Oxford University Press), 1939, p. 38

([25]) M. Born and E. Wolf, "Principles of Optics", (Pergamon Press, Oxford), 1965, p. 378

([26]) Ref. 25, p. 387.