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POYNTING'S THEOREM AND SOURCES

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Abstract: *When sources of electric or magnetic fields are present, it is very convenient and useful to follow a convention that involves a partial-force concept; i.e., the word "force" and the symbol* "F" *refer to only the "electromagnetic" part of the force on a source element; the "mechanical" force that holds the source element in place is ignored. A result of this convention is that* "F" *is generally not zero on a fixed charged body or source element. Of course this result is contrary to Newton's second law of motion* $F = dp/dt$. *But this contradiction ordinarily causes no problem because an observer at rest with respect to sources realizes that in one instance* F *is a partial force, whereas in the other,* F *is the total force. He therefore automatically and perhaps subconsciously corrects for this discrepancy. However, this contradiction has on occasion caused serious problems in the interpretation of electromagnetic energy flow and storage as described below. The electromagnetic energy continuity equation (Poynting's theorem) for a frame* S' *in which all sources are at rest is the fourth component of the electromagnetic "momentum-energy continuity equation"*

$$\mathrm{Div}'_\nu T'_{\mu\nu} = f'_\mu, \qquad (i)$$

where $T'_{\mu\nu}$ *is the "electromagnetic stress-momentum-energy four-tensor" and where* f'_μ *is the "electromagnetic force-density four-vector." Thus Poynting's theorem appears to be manifestly covariant. However, when (i) is transformed to a general frame S, the first and fourth components of* f'_μ *intermix in the usual way. But if sources are present,* f'_1 *may represent a partial force (or a nonzero fictitious rate of change of momentum) whereas* f'_4 *may represent real power. Therefore the usual algebraic simplifications (that ordinarily cause the mathematical form of the fourth-component equation in S to be the same as in S') cannot be carried to completion. Instead, the mathematical form of the electromagnetic energy continuity equation cannot be simplified beyond*

$$\mathrm{div}(\gamma^2\mathbf{S} + \gamma^2\mathbf{v}.\mathbf{T}) + (\partial/\partial t)(\gamma^2 W - \gamma^2\boldsymbol{\beta}.\mathbf{S}/c)$$
$$= -[\gamma^2(\mathbf{J} - \rho\mathbf{v}).\mathbf{E} - \gamma^2\boldsymbol{\beta}.(\mathbf{J}\mathrm{X}\mathbf{H})], \qquad (ii)$$

where $\boldsymbol{\beta} \equiv \mathbf{v}/c$, $\gamma \equiv (1-\beta^2)^{-\frac{1}{2}}$, $\mathbf{S}$ *is the Poynting vector,* $\mathbf{T}$ *the Maxwell stress three-tensor,* W *the Poynting energy density,* $\mathbf{J}$ *the current density,* $\mathbf{E}$ *the electric field, and* $\mathbf{H}$ *the magnetic field in Heaviside-Lorentz units in free space. Therefore we see that the sometimes useful partial-force convention contains a built-in trap; i.e., the intermixing of the first and fourth components of (i) to obtain (ii) renders the general observer incapable of intuitively correcting for the contradiction built into the partial-force concept. A large number of serious problems have arisen from this situation. The oldest and most famous such problem arose when Heaviside set the electromagnetic momentum of a charged sphere (whose electromagnetic energy at rest was* U_0*) equal to the mechanical momentum of a body to find that the electromagnetic mass of such a body is* $(4/3)U_0/c^2$*. Less well known is that if Heaviside had set the total electromagnetic energy of such a moving body minus its electrostatic energy equal to the kinetic energy of the body, he would have found the electromagnetic mass to be* $(5/3)U_0/c^2$*. If instead of selecting a uniformly charged spherical shell for his calculation, he had chosen a charged parallel-plate capacitor, then he would have found the electromagnetic mass of the capacitor to be 3, 2,* $\sqrt{3}$*, 0, -1, or* $\sqrt{-1}$ *(all in units of* U_0/c^2*), depending on (a) the orientation of the capacitor with respect to the velocity vector and (b) the choice of mechanical analog equation used to define electromagnetic mass. Another well known problem is that an observer at rest with respect to a charged sphere concludes that no energy flows in the field; but if this observer acquires a uniform velocity with respect to this sphere, and if he applies the Poynting vector, then he concludes that the concomitant field energy flow has a component* perpendicular *to the relative velocity. Several ingenious schemes have been invented to account for or explain away such paradoxes; e.g., Poincaré stresses and von Laue energy currents. But these ingenious schemes serve only to compound a basic error. All of these paradoxes disappear if one takes into account the existence of the partial-force concept and accepts the concomitant mathematical form of Poynting's theorem (ii) as transformed to a general frame. The interpretation of the three basic terms in (ii) is the same as the interpretation of the corresponding terms in Poynting's theorem.*

<u>Résumé</u>: Quand les sources des champs électriques ou magnétiques sont présentes, il est très pratique et très utile de suivre une convention qui introduit un concept de force partielle; c'est-à-dire que le mot "force" et le symbole "F" se rapportent seulement à la partie "électromagnétique" de la force sur un élément de la source. Cette convention entraîne que "F" n'est en général pas nul pour un corps chargé fixe ou un élément de source. Bien entendu ce résultat est contraire à la

seconde loi du mouvement de Newton: $F = dp/dt$. Mais cette contradiction ne pose en général pas de problème parce qu'un observateur au repos par rapport aux sources se rend compte que dans un cas **F** est une force partielle, alors que dans l'autre **F** est la force totale. Il corrige donc automatiquement, et peut-être inconsciemment, ce désaccord. Toutefois cette contradiction a parfois causé de sérieux problèmes dans l'interprétation du flux et de la densité d'énergie électromagnétique, comme on le décrira ci-dessous. L'équation de conservation de l'énergie électromagnétique (théorème de Poynting) dans un référentiel S' dans lequel toutes les sources sont au repos est la quatrième composante de "l'équation de conservation de l'impulsion-énergie électromagnétique"

$$\mathrm{Div}'_{\nu} T'_{\mu\nu} = f'_{\mu} \qquad (i)$$

où $T'_{\mu\nu}$ est le tenseur "contrainte-impulsion-énergie électromagnétique" et f'_{μ} le "quadrivecteur de densité de force électromagnétique." Donc le théorème de Poynting apparaît comme manifestement covariant. Cependant, quand (i) est transformée dans un référentiel quelconque S, les premières et quatrième composantes de f'_{μ} se combinent de la façon habituelle. Mais s'il y a des sources, f'_1 peut représenter une force partielle (ou un taux de variation fictif d'impulsion non nul) alors que f'_4 peut représenter une puissance réelle. C'est pourquoi les simplifications algébriques habituelles (qui d'habitude font que la forme mathématique de la quatrième composante de l'équation dans S est la même que dans S') ne marchent pas jusqu'au bout. La forme mathématique de l'équation de conservation de l'énergie électromagnétique ne peut alors être simplifiée au-delà de

$$\mathrm{div}(\gamma^2 \mathbf{S} + \gamma^2 \mathbf{v}.\mathbf{T}) + (\partial/\partial t)(\gamma^2 W - \gamma^2 \boldsymbol{\beta}.\mathbf{S}/c)$$
$$= -[\gamma^2(\mathbf{J} - \rho\mathbf{v}).\mathbf{E} - \gamma^2\boldsymbol{\beta}.(\mathbf{J}\mathrm{X}\mathbf{H})] \qquad (ii)$$

où $\boldsymbol{\beta} = \mathbf{v}/c$, $\gamma = (1-\beta^2)^{-1/2}$, **S** est le vecteur de Poynting, **T** le tenseur des contraintes de Maxwell à 3 dimensions, W la densité d'énergie de Poynting, **J** la densité de courant, **E** le champ électrique, et **H** le champ magnétique en unités de Heaviside-Lorentz dans le vide. Nous voyons donc que la convention de force partielle, quelquefois utile, comporte un piège; en effet le mélange des premières et quatrième composantes de (i) pour obtenir (ii) rend l'observateur dans un référentiel quelconque incapable de corriger intuitivement la contradiction inhérente au concept de force partielle. Un grand nombre de sérieux problèmes proviennent de cette situation. Le plus ancien et le plus fameux d'entre eux s'est posé quand Heaviside posa que l'impulsion électromagnétique d'une sphère chargée (dont l'énergie électromagnétique au repose était U_0) était égale à l'impulsion mécanique d'un corps, et en déduisit que la masse électromagnétique d'un tel corps est $(4/3)U_0/c^2$. Il est moins connu que, si Heaviside avait posé que c'était l'énergie électromagnétique totale d'un tel corps en mouvement moins son énergie électrostatique qui était égale à l'énergie cinétique du corps, il aurait trouvé que la masse électromagnétique était $(5/3)U_0/c^2$. Si, au lieu de choisir une surface sphérique uniformément chargée pour son calcul, il avait choisi un condensateur plan, il aurait alors trouvé pour la masse électromagnétique du condensateur 3, 2, $\sqrt{3}$, 0, -1, ou $\sqrt{-1}$ (tous en unités U_0/c^2) suivant (a) l'orientation du condensateur par rapport à sa vitesse et (b) le choix de l'équation mécanique analogue utilisés pour définir la masse électromagnétique. Un autre problème bien connu est qu'un observateur au repos par rapport à une sphère chargée conclut qu'aucune énergie ne s'écoule dans le champ; mais si cet observateur acquiert une vitesse uniforme par rapport à cette sphère, et s'il se sert du vecteur de Poynting, alors il conclut que le flux d'énergie concomitant possède une composante perpendiculaire à la vitesse relative. Plusieurs modèles ingénieux ont été inventés pour rendre compte ou expliquer ces *paradoxes; par exemple les tensions de Poincaré, et les courants d'énergie de Von Laue. Mais ces schémas ingénieux servent seulement à augmenter une erreur fondamentale. Tous ces paradoxes disparaissent si on tient compte de l'existence du concept de force partielle et si on accepte la forme mathématique correspondante du théorème de Poynting (ii) telle qu'elle devient dans un référentiel quelconque. L'interprétation des trois termes de base dans (ii) est la même que l'interprétation des termes correspondants dans le théorème de Poynting.*

1. INTRODUCTION

The concept of the electromagnetic field as the seat of electromagnetic *energy* was proposed in qualitative and intuitive terms by Faraday(1). The concept was put into quantitative terms in successive steps by W. Thomson(2) and Maxwell(3), who also proposed that the electromagnetic field of radiation possesses *momentum*. The *flow* of energy in the electromagnetic field of sources at rest (e.g., during the discharge of a capacitor) and in the field of radiation was described quantitatively by Poynting(4), who derived an electromagnetic energy continuity theorem from Maxwell's field equations. Although Poynting's theorem preceded the theory of relativity by some 20 years, the mathematical covariance of the theorem was assured when Minkowski(5) constructed a stress-momentum-energy four-tensor $T_{\mu\nu}$ from Maxwell's stress three-tensor, Poynting's energy-current-density vector, and the Maxwell-Poynting energy-storage-density scalar and showed that Poynting's theorem is the fourth component of the manifestly covariant equation

$$\frac{\partial T_{\mu\nu}}{\partial x_\nu} = f_\mu , \tag{1}$$

where f_μ is the electromagnetic force-density four-vector.

However, in spite of the obvious mathematical covariance of (1), Poynting's theorem as traditionally interpreted and applied has long been known(6) to lead to inconsistencies and anomalies. For example, Panofsky and Phillips(7) state "Paradoxical results may be obtained if one tries to identify the Poynting vector with the energy flow per unit area at any particular point." And subsequently(8) they state "The second difficulty arises from the fact that the expression

$$G^i = \int T^{4i} dv, \tag{21-62}$$

is basically not covariant unless the field is 'free' in the sense of Sec. 21-4" (i.e., unless the field is not attached to sources).

Most authors appear to be in general agreement that the difficulties associated with Poynting's theorem are in some way related to the instability of the model of a charged body, i.e., the fact that the space components of f_μ are not zero even, for example, at a point within the charge of an *electrostatic* system. As a specific example, they identify the problem of the anomalous *4/3 factor* in the electromagnetic mass of the classical electron with the instability of the model(9). This relationship was first recognized by Poincaré(10), as related by Pauli(11). "It will, in any case, be necessary to introduce forces which hold the Coulomb repulsive forces of the electron charge on itself in equilibrium, and such forces are not derivable from Maxwell-Lorentz electrodynamics. Poincaré already recognized the need for this and purely formally introduced a scalar cohesive pressure p, on whose nature he could not make any statement." Pauli continued, "Generally speaking, the problem of the electron has to be formulated as follows: The energy-momentum tensor S_{ik} of Maxwell-Lorentz electrodynamics has to have terms added to it in such a way that the conservation laws

$$\frac{\partial T_i^{\;k}}{\partial x^k} = 0$$

for the total energy-momentum tensor become compatible with the existence of charges."

Becker and Sauter[12] show in detail how one can accomplish this goal by constructing a tensor "of mechanical or other origin" which when added to the tensor $T_{\mu\nu}$ provides for stability. They then showed that the combined tensor and the associated combined force-density four-vector removed some of the anomalies associated with Poynting's theorem.

A different approach has been suggested by Rohrlich[13,14], who pointed out that when performing the integration to obtain the momentum-energy of the classical electron it is necessary to select an integration hypersurface that is related to the motion of the electron in a relativistically invariant manner. He then showed that the appropriate choice is a hypersurface orthogonal to the four-velocity at the intersection of the world line with the surface. Since the charge is assumed to be concentrated at a point, the surface may be chosen arbitrarily outside the region of intersection. Hence the hypersurface may be assumed to be plane. The hyperplane thus selected is equivalent to an integration for all observers at constant time t' (i.e., constant time as measured in a rest frame) regardless of the motion of the observer concerned. Rohrlich thereby obtained a four-vector for the momentum-energy of the classical electron, while staying strictly within relativistic electromagnetic theory.

In following Rohrlich, Panofsky and Phillips[8] wrote "It is thus appropriate to question whether we can substitute a fully covariant expression

for Eq. (21-62) which will give the correct relativistic transformation properties for the mechanical aspects of the electromagnetic field by itself." And Jackson[15] wrote "The discussion of the previous section has one puzzling aspect . . . .. A noncovariant electromagnetic contribution to the self-energy or momentum of a charged particle is balanced by a noncovariant contribution from the Poincaré stresses, so the result is properly covariant . . . .. Nevertheless it is legitimate to ask whether the purely electromagnetic contributions of self-energy and momentum *can be defined* to have the proper Lorentz transformation properties."

Yet another different (but equivalent) approach is that of Cavalleri and Salgarelli[16], who describe an "asynchronous" formulation of mechanics. In the asynchronous formulation, a general observer would measure lengths and perform integrations over an extended body at one instant of time as measured in a frame at rest with respect to the body. (This approach presumes that any relative motion of various parts of the body with respect to each other is small enough that the criterion for simultaneity does not vary significantly among the frames in which various parts of the body are at rest.) Grøn[17] has discussed in some detail the asynchronous formulation, including its application to thermodynamics.

One may summarize the foregoing approaches as follows. Poincaré, Pauli, and Becker and Sauter sought to construct a new tensor providing for stability even if they had to include nonelectromagnetic forces (and nonelectromagnetic momentum and energy); in their formulation, neither the electromagnetic nor the nonelectromagnetic momentum and energy constitute four-vectors; only the sum is covariant. Rohrlich, Jackson, and Panofsky and Phillips sought to stay within pure electromagnetic theory, thereby separating the covariance considerations from the stability question; in effect, they retained the conventional electromagnetic tensor $T_{\mu\nu}$ but substituted a four-vector representation of the volume element for *dv* in (21-62). The asynchronous approach appears to be equivalent.

The present treatment is a basic reexamination of the problem. It arose independently of all the aforementioned approaches and differs from them in the following ways. The problem addressed by the papers cited above is essentially the noncovariance of the momentum-energy of static sources referred to a frame that is moving with respect to them. The general problem addressed herein is in the area of electrodynamics in the sense that work is being done in macroscopic systems that are moving with respect to the laboratory frame; the role of the coupling of the electric and magnetic fields is examined; and the principal problem addressed is the noncovariance of Poynting's theorem itself—an idea that has the ring of heresy. So the philosophic approach herein is to find the basic reason for this noncovariance, to modify the equation to make it covariant, and then to interpret and apply the modified equation in the same way that Poynting's theorem has been traditionally interpreted and applied. Because the present approach involves the entirety of Poynting's theorem, the power density term is also modified; and it appears convenient to define some new field terms with unusual transformation properties. Specifically, the present approach starts with a detailed examination of the philosophic basis of each subcomponent term in (1). The conclusions from this examination are then applied to the transformation process. The final result is a consistent or covariant set of definitions (momentum, energy, and power densities), a new stress-momentum-energy four-tensor $\mathfrak{T}_{\lambda\mu}$, a new mathematical form for Poynting's theorem, a new mathematical form for the corresponding momentum continuity theorem, and a new power-density four-vector $\mathfrak{p}_{\lambda}$.

A corollary result (that was actually built into the approach) is that the criterion of simultaneity may be applied with equal validity with respect to any frame. Another result of the present approach is that when the proposed new tensor $\mathfrak{T}_{\lambda\mu}$ is substituted for $T_{\mu\nu}$, the conservation laws discussed by Pauli are satisfied, and equation (21-62) of Panofsky and Phillips becomes covariant even in the presence of sources.

Two of the results from the present paper (substitute expressions for **S** and *W* for electrostatic and magnetostatic systems) have already been derived by Rohrlich([13,14]), and they also follow from the asynchronous formulation of mechanics proposed by Cavalleri and Salgarelli([16]).

Since Poynting's theorem is the fourth component of (1), it is *by definition* covariant. How then can it be noncovariant? The answer is that Poynting's theorem is indeed *mathematically* covariant, but (as discussed in detail in Sect. 3) the physical *interpretation* of the terms is not consistent with the rest of physics. For example, the term that is interpreted as energy density contains a component that is not real energy density; and similarly for the term for density of energy flow and power transfer.

2. CONVENTIONS AND DEFINITIONS

For reference purposes we list here some conventions and definitions on which subsequent sections are based.

We use Heaviside-Lorentz units in vacuo. All mechanical supports and structures are assumed to be nonconducting and nonpolarizable; all charge is Maxwellian; and all fields occupying the same space are assumed to be coupled by Maxwell's curl equations. A source is considered to be a macroscopic structure that produces a field. The source is considered to be at rest if the basic structure is at rest. For example, a convection current whose configuration is stationary is considered to be a source at rest.

Throughout the present paper the term *rest observer (frame)* is understood to refer to a Lorentz observer (frame) with respect to whom (which) all sources are at rest (except in a few discussions in which we permit sources to move slowly). All observers and frames are assumed to be Lorentz. All coordinate systems are assumed to have the same orientation in three-space, and the three-space origins of all coordinate systems are assumed to coincide at $t = t' = 0$. Frame S' is assumed to move with speed $v = \beta c$ along the x axis of frame S.

We use the Poincaré Euclidean metric and do not distinguish between covariant and contravariant indices. Roman indices represent ordinary space directions and run from 1 to 3; Greek indices run from 1 to 4. We use the Einstein summation convention.

The electromagnetic stress-momentum-energy four-tensor $T_{\mu\nu}$ is defined to be

$$T_{\mu\nu} \equiv \begin{bmatrix} \mathbf{T} & -i\mathbf{S}/c \\ -i\mathbf{S}/c & W \end{bmatrix} . \qquad (2)$$

The Poynting energy density W is defined to be

$$W \equiv \tfrac{1}{2}(E^2 + H^2) . \qquad (3)$$

The Poynting vector $\mathbf{S}$ is defined to be

$$\mathbf{S} \equiv c\mathbf{E}\mathbf{X}\mathbf{H}. \qquad (4)$$

The symbol $\mathbf{T}$ may be considered to be either the Maxwell stress three-tensor T_{jk} or the corresponding dyadic

$$\mathbf{T} \equiv \mathbf{EE} + \mathbf{HH} - W\mathbf{I}. \qquad (5)$$

The symbol $\mathbf{I}$ represents the identity dyadic or idemfactor. And the electromagnetic force-density four-vector f_μ is defined to be

$$f_\mu \equiv (\rho\mathbf{E} + \mathbf{J}\mathbf{X}\mathbf{H}/c,\ i\mathbf{J}.\mathbf{E}/c) . \qquad (6)$$

Substituting (2)-(6) into the fourth component of (1) we obtain the usual three-space form of Poynting's theorem:

$$\mathrm{div}\mathbf{S} + \partial W/\partial t = -\mathbf{J}.\mathbf{E}. \qquad (7)$$

Consider now the equation

$$\frac{\partial T'_{\mu\nu}}{\partial x'_\nu} = f'_\mu \qquad (8)$$

for an observer O' at rest in a particular frame S' with respect to which all sources are at rest and in which the fields are coupled by Maxwell's curl equations. No anomalies are encountered with the use of Poynting's theorem by observer O'.

We next ask the question: If observer O' uses the equation

$$\frac{\partial T'_{4\nu}}{\partial x'_\nu} = f'_4 \qquad (9)$$

or

$$\mathrm{div}'\mathbf{S}' + \partial W'/\partial t' = -\mathbf{J}'.\mathbf{E}' \qquad (10)$$

to describe electromagnetic energy continuity, what equation should observer O use, to be consistent with O'?

For reasons that will become apparent in the next section, we make a detailed answer to this question. We start by making a Lorentz transformation of (9) in the usual way, obtaining

$$\gamma \frac{\partial T_{4\nu}}{\partial x_\nu} - i\gamma\beta \frac{\partial T_{1\nu}}{\partial x_\nu} = \gamma f_4 - i\gamma\beta f_1, \qquad (11)$$

which by the use of (2)-(6) may be rewritten in the form

$$\mathrm{div}(\gamma \mathbf{S} + \gamma \mathbf{v}.\mathbf{T}) + (\partial/\partial t)(\gamma W - \gamma \boldsymbol{\beta}.\mathbf{S}/c)$$

$$= -\gamma \mathbf{J}.\mathbf{E} + \gamma\rho \mathbf{v}.\mathbf{E} + \gamma \boldsymbol{\beta}.(\mathbf{J}\mathbf{X}\mathbf{H}), \qquad (12)$$

where $\gamma \equiv (1 - \beta^2)^{-\frac{1}{2}}$.

To simplify (12) we write the transformed first component of (8) and then treat the transformed first and fourth components as two simultaneous equations, obtaining

$$\frac{\mathrm{div}\mathbf{S}}{\gamma} + \frac{1}{\gamma}\frac{\partial W}{\partial t} = -\frac{\mathbf{J}.\mathbf{E}}{\gamma}. \qquad (13)$$

Then remembering that (13) is term-for-term equal to (10), and remembering also that the two frames differ by a rotation in the (x,ict) plane through an angle whose cosine is γ, we multiply each term in (13) by γ to get the corresponding quantity for observer O, thus obtaining the same mathematical form as (10).

This procedure is a detailed, somewhat unusual but nevertheless conventional, way of demonstrating the mathematical covariance of Poynting's theorem. Where then does the basic difficulty with Poynting's theorem in the presence of sources arise?

3. THE NATURE OF THE PROBLEM

The basic problem is one of conventions, definitions, and interpretations. A four-vector is an ensemble of four quantities that transform (under a rotation of the coordinate system) in the same manner as the coordinates of a fixed point in four-space. The object f_μ is defined mathematically to transform as a four-vector; in other words, *to be* a four-vector. However, the mere fact that an object possesses the requisite *mathematical* properties of a four-vector does not necessarily mean that this object also has the proper *physical* definition and interpretation to be applied to the real world. Does f_μ satisfy the *physical* requirements of a four-vector?

A mathematical four-vector represents a real physical quantity only if the components of that quantity actually intermix in nature (under a rotation of the coordinate system) in the appropriate way; in other words, only if the components relate to each other in nature in the same way as the coordinates of a point in four-space relate to each other. The momentum and energy of a particle constitute a four-vector that does apply to the real world because momentum and energy do intermix in the appropriate way. The rate of transfer of momentum per unit volume and the rate of transfer of energy per unit volume for an extended body likewise constitute a four-vector that applies to the real world. The entity f_4' is defined to be the rate of transfer of energy per unit volume from the electromagnetic field to a source element at rest at the point of interest P; *however*, f_k' *is defined in such a way that it is not in general the rate of transfer of momentum per unit volume from the field to the source element at* P. Therefore, when f_k' and f_4' intermix upon transformation of the frame of reference, real energy and fictitious momentum (or fictitious energy) intermix, leading to a composite quantity that does not have any ordinary physical interpretation.

Consider the following example: a parallel-disk capacitor at rest being charged by a uniform string of positive charge of density ρ' in a manner analogous to that of a Van de Graaff generator. Let the string of charge have a small uniform velocity $\mathbf{u}'$ along the axis of the capacitor. In keeping with the conventions stated in Sect. 2, we consider the source of the electric field between the plates and the source of the magnetic field to be stationary. The mechanical force along the string is doing work on the charge, which in turn is doing work on the field, at the rate $-\rho'\mathbf{u}'.\mathbf{E}' = -\mathbf{J}'.\mathbf{E}'$. This is real work; the concomitant potential energy is directly observable in terms of $\int \frac{1}{2}E'^2 dV'$; and this stored energy can be retrieved. However, the mechanical force along the string is not transferring momentum to the charge; the charge is not transferring momentum to the field; no momentum is being stored; and no momentum can be retrieved. Nevertheless, according to the conventional partial-force concept in electromagnetism, the space components of f'_μ are

$$f'_k = \rho'\mathbf{E}' + \mathbf{J}'\mathrm{X}\mathbf{H}'/c,$$

which are not zero in the string of charge. Thus we have in this example an anomaly: at a point P in the string of charge, f'_4 represents real work or energy transfer between field and source, but f'_k does not represent real momentum transfer between field and source. *Therefore f'_k fails the physical test for a legitimate companion to f'_4* in the ensemble f'_μ. We thus conclude that f'_μ as conventionally defined does not satisfy the *physical* compatibility requirements of a four-vector ensemble because real energy and nonmomentum do not relate to each other in nature in the same way as the coordinates of a point in four-space.

Consider for this example the inverse of the transformation from (9) to (11) for the point P:

$$f_4 = \gamma(f'_4 + i\beta f'_1) = \gamma(\rho'\mathbf{u}'.\mathbf{E}'/c + i\rho'\boldsymbol{\beta}.\mathbf{E}'). \qquad (14)$$

Since f'_1 is not associated with the time rate of transfer of real *momentum* per unit volume at P in S' (because f'_1 is balanced by the equal and opposite mechanical force of the string) the quantity $\beta f'_1$ in (14) cannot be associated with the time rate of transfer of real *energy* per unit volume at P in S. Therefore f_4 consists of a mixture of energy (f'_4) and nonenergy ($\beta f'_1$).

This inconsistency between the definitions and interpretations of the time and space components of f'_μ (that f'_4 is the density of the rate of transfer of real energy and that f'_k is the density of the rate of transfer of nonreal or fictitious momentum) causes no significant difficulties for a rest observer; he intuitively takes into account the fact that f'_k is only a partial electric force, balanced by an equal and opposite partial mechanical force, hence does not lead to a finite rate of transfer of momentum. *However, for the nonrest observer, serious difficulties do arise because he is not able to handle intuitively the mixture of energy and nonenergy contained in f_4 in* (14) *because both energy and nonenergy are represented by the single term f_4 or* $\mathbf{J}.\mathbf{E}/c$. Therefore the nonrest observer implicitly interprets both parts of f_4 in (14) as real energy.

Even for purely electrostatic systems, the nonrest observer is likely to interpret $\rho'\boldsymbol{\beta}.\mathbf{E}'$ (or $\rho\mathbf{v}.\mathbf{E}/c$) as real power transfer density. An example is the explanation of Laue[18] for the null result of the Trouton-Noble[19] experiment, in which a charged capacitor was suspended by a torsion fiber so that it was free to rotate under the influence of the apparent torque couple exerted on the plates by their mutual electromagnetic force. (The motion of the earth through the ether provided the magnetic part of the force.) Laue's explanation of the null result of Trouton and Noble was that the torque exerted by the magnetic force was absorbed by the "increasing angular momentum" of the "elastic energy current" associated with $\rho \mathbf{v}.\mathbf{E}$. Laue's explanation has been described by Pauli[20] and by Becker and Sauter[21].

4. PROPOSED SOLUTION

The essence of the problem is this: if sources are present, then f_k' is conventionally *defined* to represent something other than the rate of transfer of momentum; but when f_μ' is transformed from S' to S and the usual intermixing of the first and fourth components occurs, then f_4 is conventionally *interpreted* to represent the rate of transfer of real energy even though part of f_4 is $\beta f_1'$ and even though $\beta f_1'$ cannot represent the rate of transfer of real energy in S because f_1' does not represent the rate of transfer of real momentum in S'.

This contradiction in the conventional definition and interpretation of the components of f_μ and all the consequences of that contradiction are what led Pauli to write[11] "We therefore see that *the Maxwell-Lorentz electrodynamics is quite incompatible with the existence of charges, unless it is supplemented by extraneous theoretical concepts.*" (The emphasis is Pauli's.)

The "mechanical" stress-momentum-energy tensor, such as that of Becker and Sauter[12], is one such "extraneous theoretical concept" referred to by Pauli; the Poincaré stresses[10] are another manifestation of the same extraneous theoretical concept; and the Laue energy currents[18] are yet another such concept.

Ordinarily, when a transformation of a four-vector equation is made from one frame of reference to another, the transformed first and fourth component equations are merged as two simultaneous equations to reduce the number of variables and thereby to simplify the transformed equations as described below (12) and (13). This procedure leads to the same mathematical form in both frames involved in a transformation; and of course the same mathematical form is the essence of covariance. The proposed solution of the problems and paradoxes associated with Poynting's theorem is to recognize that the merging of real energy and fictitious momentum terms to create a single "energy" term is an illegitimate procedure because it erradicates the distinction between real and fictitious quantities. *Maintaining this distinction between real and fictitious quantities requires that the mathematical form of Poynting's theorem be more complex in a nonrest frame than in a rest frame.*

To understand precisely how the interpretation of terms is distorted by the merging process, we consider it in some detail. We transform the four-vector equation

$$A_\mu' = B_\mu' \tag{14a}$$

from a rest frame S' to a general frame S. In the usual way we obtain the first and fourth components:

$$\text{(1st Component)} \qquad \gamma A_1 + i\gamma\beta A_4 = \gamma B_1 + i\gamma\beta B_4 \tag{14b}$$

$$\text{(4th Component)} \qquad \gamma A_4 - i\gamma\beta A_1 = \gamma B_4 - i\gamma\beta B_1 . \tag{14c}$$

Equations (14b) and (14c) represent side-for-side the identical physical entities represented by the corresponding components of (14a). There is no problem with interpretation so far. Rewriting (14b)

$$\gamma A_1 = \gamma B_1 + i\gamma\beta B_4 - i\gamma\beta A_4 \qquad (14d)$$

and substituting (14d) into (14c) we obtain

$$\gamma A_4 - i\gamma\beta B_1 + \gamma\beta^2 B_4 - \gamma\beta^2 A_4 = \gamma B_4 - i\gamma\beta B_1. \qquad (14e)$$

Equation (14e) is still side-for-side the same as the fourth component of (14a), and there is still no problem with interpretation. So far, we have merged two equations containing four variables to obtain a single equation containing three variables. However, if we add $i\gamma\beta B_1$ to both sides of (14e), then we eliminate another variable, leaving only two variables, as follows:

$$\gamma A_4 + \gamma\beta^2 B_4 - \gamma\beta^2 A_4 = \gamma B_4. \qquad (14f)$$

Further algebraic manipulation leads to

$$A_4 = B_4. \qquad (14g)$$

But we choose to stop at (14e) *because adding* $i\gamma\beta B_1$ *(or* $i\gamma\beta f_1$*) to both sides of (14e) causes irreversible damage to the physical interpretation of (14e).* This step amounts to adding $i\gamma\beta f_1$ or $i\gamma\rho\mathbf{v}.\mathbf{E}$ (which is *nonphysical* power density) to both sides of an equation relating two *real* power-density terms. Of course, if $\rho = 0$, then this step *is* legitimate. But if $\rho \neq 0$, then $\rho\mathbf{v}.\mathbf{E}$ is merely an artifact of the motion of the observer; and since this fact is obscured by the removal of this term from each side of (14e), a general observer is unable to compensate for the artifact content of A_4 and B_4.

Therefore, returning to (12), which in effect keeps separate accounts of real and fictitious quantities, we multiply both sides by γ [for the same reason as (13)] to obtain

$$\mathrm{div}(\gamma^2\mathbf{S} + \gamma^2\mathbf{v}.\mathrm{T}) + (\partial/\partial t)(\gamma^2 W - \gamma^2\boldsymbol{\beta}.\mathbf{S}/c)$$
$$= -\left[\gamma^2(\mathbf{J} - \rho\mathbf{v}).\mathbf{E} - \gamma^2\boldsymbol{\beta}.(\mathbf{J}\mathrm{X}\mathbf{H})\right] \qquad (15)$$

or

$$\mathrm{div}(\gamma^2\mathbf{S} + \gamma^2\mathbf{v}.\mathrm{T}) + (\partial/\partial t)(\gamma^2 W - \gamma^2\boldsymbol{\beta}.\mathbf{S}/c)$$
$$= -\left[\gamma^2\mathbf{J}.(\mathbf{E} + \boldsymbol{\beta}\mathrm{X}\mathbf{H}) - \rho\mathbf{v}.\mathbf{E}\right], \qquad (15a)$$

which is the proposed electromagnetic energy continuity statement for a general observer, whether sources are present or not.

Perhaps it should be emphasized that a term-by-term substitution of the ordinary transformations of $\mathbf{J}'$, ρ', $\mathbf{E}'$, $\mathbf{H}'$, *div'*, and $\partial/\partial t'$ into (10) leads also to (15).

We interpret the three major terms in (15) as follows. For a given point *P* in the field (coupled by Maxwell's equations), the right-hand side of (15) is the power per unit volume being transferred to the field from a source element at *P*; the time-derivative term is the rate of increase of field energy per unit volume stored at *P*; and the divergence term is the rate at which energy per unit volume is flowing away from *P* to other parts of the field.

Each side of (15) is valid for (i) any orientation of the coordinate system (in three-space), (ii) any constant speed of the system being observed with respect to the laboratory (or the orientation of the coordinate system in four-space), and (iii) any direction of the relative velocity. Furthermore, the left-hand side is (iv) independent of the system of electromagnetic units.

Note that, if $\beta = 0$, then (15) reduces to Poynting's theorem as it should. A transformation of (15) to another frame S'' need not stop at the analog of (12); hence may be shown in a straightforward way to lead to the same mathematical form as that of (15).

From (15) [as interpreted] we write revised definitions for electromagnetic-energy-current density $\mathfrak{S}$,

$$\mathfrak{S} \equiv \gamma^2 \mathbf{S} + \gamma^2 \mathbf{v}.\mathbf{T} \ , \qquad (16)$$

for electromagnetic-energy-storage density $\mathfrak{W}$,

$$\mathfrak{W} \equiv \gamma^2 W - \gamma^2 \boldsymbol{\beta}.\mathbf{S}/c, \qquad (17)$$

and for electromagnetic-power-transfer density,

Electromagnetic-power Density

$$\text{for } O \equiv \gamma^2 \left[\mathbf{J}.\mathbf{E} - \rho \mathbf{v}.\mathbf{E} - \boldsymbol{\beta}.(\mathbf{J}\mathrm{X}\mathbf{H}) \right]. \qquad (18)$$

The argument of Abraham[22] with respect to the vector $\mathbf{S}$ can be applied to the energy-current-density vector $\mathfrak{S}$ to obtain an electromagnetic-momentum-density vector: $\mathfrak{S}/c^2$.

5. INTERPRETATION OF THE PROPOSED ELECTROMAGNETIC-POWER-TRANSFER-DENSITY TERM

For a rest observer O', the quantity $\mathbf{J}'.\mathbf{E}'$ is indeed a valid definition of electromagnetic-power-transfer density, and it is easy to appreciate intuitively the physical significance of the product $\mathbf{J}'.\mathbf{E}'$. On the other hand, the physical significance of the second and third terms inside the brackets of (18) is not immediately obvious. But if observer O is to be consistent with O', then O must use (18) or some equivalent expression.

What is the intuitive physical interpretation of the terms in (18)? Essentially it is this: the product $\mathbf{J}.\mathbf{E}$ as conventionally defined includes convection currents and induced electric fields that arise solely from the motion of the observer; hence even for electrostatic or magnetostatic systems, the product $\mathbf{J}.\mathbf{E}$ implies nonzero power transfer; this apparent power transfer is fictitious; the second and third terms subtract such fictitious power that exists in $\mathbf{J}.\mathbf{E}$ to give the net real power transfer density. The following paragraph illustrates this concept in some detail.

Because of the vector identity

$$\mathbf{A}.(\mathbf{B}\mathrm{X}\mathbf{C}) \equiv \mathbf{B}.(\mathbf{C}\mathrm{X}\mathbf{A}),$$

the third term in (18) may be rewritten

$$\boldsymbol{\beta}.(\mathbf{J}\mathrm{X}\mathbf{H}) \equiv \mathbf{J}.(\mathbf{H}\mathrm{X}\boldsymbol{\beta}) = \mathbf{J}.\mathbf{E}_n,$$

where $\mathbf{E}_n$ is the induced electric field associated with the magnetic field $\mathbf{H}$. Since

$$\rho \mathbf{v}.\mathbf{E}_n = \rho \mathbf{v}.(\mathbf{H}\mathrm{X}\mathbf{v}/c) \equiv 0,$$

we may add this term to the right-hand side of (18), obtaining

Electromagnetic-power Density

$$\text{for } O = \gamma^2 (\mathbf{J}.\mathbf{E} - \rho \mathbf{v}.\mathbf{E} - \mathbf{J}.\mathbf{E}_n + \rho \mathbf{v}.\mathbf{E}_n)$$

$$= \gamma^2 (\mathbf{J} - \rho \mathbf{v}).(\mathbf{E} - \mathbf{E}_n) \qquad (19)$$

Thus we see that the effect of the second and third terms in (18) is to subtract from $\mathbf{J}.\mathbf{E}$ the first-order terms (in β) that arise solely from the motion of the observer.

It is instructive to derive an alternate (but equivalent) power-transfer-density expression in an apparently different way which permits a direct comparison with elementary concepts from relativity theory. To do so, we distinguish between the contributions of "separated charges" and "currents in neutral conductors" to the quantities **J**, ρ, **E**, and **H**. The quoted terms in the preceding sentence are defined in the language of *O'*, an observer at rest with respect to all sources.

Letting a subscript *q* denote a quantity that owes its existence strictly to separated charges and letting a subscript *i* denote a quantity that owes its existence strictly to currents in neutral conductors, we write the modified standard transformation equations as follows:

$$\rho_i = \beta . J_i/c \tag{20}$$

$$\mathbf{J}_q = \rho_q \mathbf{v} \tag{21}$$

$$\mathbf{E}_i = \mathbf{H}_i X \beta \tag{22}$$

$$\mathbf{H}_q = \beta X \mathbf{E}_q . \tag{23}$$

Using (20)-(23) and the standard transformation equations for **J'** and **E'** we write

$$\begin{aligned} \mathbf{J}'.\mathbf{E}' &= J_x'E_x' + J_y'E_y' + J_z'E_z' \\ &= \gamma(J_xE_x + J_yE_y + J_zE_z - \rho v E_x - \beta J_y H_z + \beta J_z H_y) \\ &= \gamma[(J_{ix} + J_{qx})E_{qx} + J_{iy}(E_{qy} + E_{iy}) + J_{iz}(E_{qz} + E_{iz}) \\ &\quad -(\rho_q + \rho_i)vE_{qx} - \beta J_{iy}(H_{iz} + H_{qz}) + \beta J_{iz}(H_{iy} + H_{qy})] \\ &= \gamma(\mathbf{J}_i.\mathbf{E}_q - \beta^2 \mathbf{J}_i.\mathbf{E}_q) = \mathbf{J}_i.\mathbf{E}_q/\gamma . \end{aligned} \tag{24}$$

The somewhat complicated procedure that led to (24) is analogous to the well known simple procedure that leads to

$$dt' = dt/\gamma$$

and also to other common equations for the transformation of the fourth component of a time-like vector between a rest frame *S'* and a general frame *S*.

Therefore, on the one hand, it follows that the quantity $\mathbf{J}_i.\mathbf{E}_q$ transforms in the same way as the fourth component of a time-like four-vector. On the other hand, the quantity **J.E** (although it does transform mathematically like the fourth component of a four-vector) does not necessarily transform like a time-like four-vector. Indeed the ensemble f_μ does not possess ordinary time-like or space-like characteristics typical of four-vectors representing physical quantities. [In other words, the *space* components of a *time-like* vector for a body (such as momentum-energy) vanish in a rest frame; and the *time* component of a *space-like* vector (such as the interval between two scratches on a bar) vanishes in a rest frame.] Furthermore, **J.E** does not represent strictly real power transfer if moving sources are present.

If a rest observer *O'* defines electromagnetic-power-transfer density to be **J'.E'** and if observer *O* is to be consistent with *O'* in what he defines as electromagnetic-power-transfer density, then the analysis from (20) to (24) requires that observer *O* define electromagnetic-power-transfer density to be $\mathbf{J}_i.\mathbf{E}_q$:

$$\textit{Electromagnetic-power Density for } O \equiv \mathbf{J}_i.\mathbf{E}_q \tag{25}$$

or some equivalent definition. The expression **J.E** is not a consistent definition of electromagnetic-power-transfer density as long as **J** is interpreted to include convection currents (moving charges) as well as conduction currents, as long as $\mathbf{E}$ is interpreted to include the "induced electric field" associated with moving magnetic sources, and as long as $\mathbf{E}$ is interpreted not to include the microscopic field associated with the "mechanical" restraining forces within the body on which the charge resides.

Although the immediately preceding analysis and conclusion [based on (24)] were made and arrived at in a different manner from the procedure that led to (18), a direct substitution of (20)-(23) into (18) leads to the same expression as given by (25) for electromagnetic-power-transfer density for observer O.

Thus we have three equivalent expressions for electromagnetic-power-transfer density for a general observer: (18), (19), and (25).

We note that the quantity $(\mathbf{J} - \rho\mathbf{v})$ from (19) is not precisely equivalent to the quantity $\mathbf{J}_i$ from (25) because the x component of $(\mathbf{J} - \rho\mathbf{v})$ transforms as follows:

$$J_x - \rho v = \gamma(J_x' + \rho' v) - \gamma(\rho' + \beta J_x'/c)v = \gamma(J_x' - \beta^2 J_x')$$

$$= \gamma J_x'(1 - \beta^2) = J_{ix}(1 - \beta^2).$$

Similarly, the quantity $(\mathbf{E} - \mathbf{E}_n)$ from (19) is not precisely equivalent to the quantity $\mathbf{E}_q$ from (25) because the transverse components of $(\mathbf{E} - \mathbf{E}_n)$ transform as follows:

$$(\mathbf{E} - \mathbf{E}_n)_\perp = (\mathbf{E} - \mathbf{H}\mathrm{X}\boldsymbol{\beta})_\perp = \gamma(\mathbf{E}' - \boldsymbol{\beta}\mathrm{X}\mathbf{H}')_\perp + \gamma\left[\boldsymbol{\beta}\mathrm{X}(\mathbf{H}' + \boldsymbol{\beta}\mathrm{X}\mathbf{E}')\right]_\perp$$

$$= \gamma\mathbf{E}_\perp' + \gamma\left[\boldsymbol{\beta}\mathrm{X}(\boldsymbol{\beta}\mathrm{X}\mathbf{E}')\right]_\perp = \gamma\mathbf{E}_\perp'(1 - \beta^2)$$

$$= E_{q\perp}(1 - \beta^2).$$

Perhaps it should be emphasized that $\mathbf{J}_i$ and $\mathbf{E}_q$ have unusual transformation properties: fields and quantities that are not coupled in S' are not coupled upon transformation to S in the process of obtaining $\mathbf{J}_i$ and $\mathbf{E}_q$. That is to say, $\mathbf{E}_q$ is the result of transforming only that which is *electric* field in S' at P—any independent or noncoupled magnetic field at P is ignored in this particular type of transformation because there is no energy transfer between a stationary conductor and a magnetostatic field, regardless of the motion of the observer. Similarly, $\mathbf{J}_i$ is the result of transforming only that which is *current density* in S' at P—any independent or noncoupled charge density at P is ignored in this particular type of transformation because there is no energy transfer between a static charge and an electric field, regardless of the motion of the observer. The basic reason that the unorthodox transformation properties eliminate the paradoxes is that the object f_μ is itself unorthodox: It includes real energy and fictitious momentum in its components.

It is easier to apply (25) than (18) to simple situations to gain an intuitive appreciation of the proposed definition of electromagnetic-power-transfer density. The basic physical interpretation of $\mathbf{J}_i \cdot \mathbf{E}_q$ is that whatever a rest observer interprets as real power transfer density at a point P a general observer will interpret the power transfer density at P to be γ times as much.

Consider a specific example illustrating this concept: a line segment of a neutral resistive conductor parallel to $\mathbf{v}$. Since the electric field parallel to $\mathbf{v}$ is the same for all observers and since $J_{ix} = \gamma J_x'$, it follows that $\mathbf{J}_i \cdot \mathbf{E}_q = \gamma \mathbf{J}' \cdot \mathbf{E}'$ for this example, a result which is consistent with elementary concepts from the theory of relativity.

Consider another specific example: two small static distributions of charge with opposite signs at opposite ends of a rod parallel to the x' axis. Let P be located in or at the positive charge. According to O', no work is being done at P because $\mathbf{J}' = 0$. However, observer O, using $\mathbf{J}.\mathbf{E}$ for electromagnetic-power transfer density, concludes that work is being done at P because

$$\mathbf{J}.\mathbf{E} = \rho\mathbf{v}.\mathbf{E} \neq 0.$$

The proposed definition (25) leads to

$$\mathbf{J}_i.\mathbf{E}_q = (\gamma\mathbf{J}').\mathbf{E}' = 0.$$

Consider one more simple example (which was used in a different context in Sect. 3): an element of charge dq being pulled slowly with velocity $\mathbf{u}$ by a string across the space between the two parallel plates of a charged capacitor. At the location of dq the power transfer density is $\mathbf{J}'.\mathbf{E}' = \rho'\mathbf{u}'.\mathbf{E}'$ according to O'. Transforming to frame S, we write

$$\mathbf{J}_i.\mathbf{E}_q = (\gamma\mathbf{J}').\mathbf{E}' = \gamma\mathbf{J}'.\mathbf{E}',$$

which again agrees with elementary concepts from relativity theory but does not agree with conventional electromagnetic theory involving Poynting's theorem and sources.

6. A PROPOSED ELECTROMAGNETIC POWER-DENSITY FOUR-VECTOR AND POWER-DENSITY FOUR-TENSOR

The interpretation of f_4' as the density of the time rate of transfer of energy in the classical equation of continuity is valid; but it is clear that f_k' as usually defined may not legitimately be interpreted in general as the density of the time rate of transfer of momentum. One can easily find a new expression for a compatible companion for f_4', i.e., one can use f_4' to

construct a complete four-vector all of whose components may be legitimately interpreted in the conventional way as the density of the time rate of transfer of momentum-energy in systems in which there is no relative motion among the sources. To do so, we start by defining for a rest frame S' the density of the time rate of transfer of momentum-energy to the electromagnetic field:

$$\mathfrak{p}_\lambda' \equiv -(\mathbf{0},\ i\mathbf{J}'.\mathbf{E}'/c). \qquad (26)$$

Next we transform $\mathfrak{p}_\lambda'$ to a general frame S:

$$\mathfrak{p}_\lambda \equiv -\{\gamma\boldsymbol{\beta}(\mathbf{J}'.\mathbf{E}')/c, i\gamma\mathbf{J}'.\mathbf{E}'/c\}. \qquad (27)$$

And then we substitute the ordinary transformations of $\mathbf{J}'$ and $\mathbf{E}'$ into (27) to obtain

$$\mathfrak{p}_\lambda = -\{\gamma^2\left[\mathbf{J}.\mathbf{E} - \rho\mathbf{v}.\mathbf{E} - \boldsymbol{\beta}.(\mathbf{J}\times\mathbf{H})\right]\boldsymbol{\beta}/c,$$
$$i\gamma^2\left[\mathbf{J}.\mathbf{E} - \rho\mathbf{v}.\mathbf{E} - \boldsymbol{\beta}.(\mathbf{J}\times\mathbf{H})\right]/c\}. \qquad (28)$$

By inspection we see that (28) may be rewritten

$$\mathfrak{p}_\lambda = f_\mu\beta_\mu\beta_\lambda, \qquad (29)$$

where $\beta_\mu \equiv (\gamma\boldsymbol{\beta}, i\gamma)$. One can see further by inspection that the fourth component of (28) or (29) leads to the same redefinition of power transfer density as (18).

Note that the net effect of taking the inner product $f_\mu\beta_\mu$ is to annul the influence of the fictitious momentum transfer rate represented by f_k' (because $f_\mu\beta_\mu = f_\mu'\beta_\mu' = f_4'\beta_4'$), thus eliminating the capability of f_k' to do mischief by intermixing with f_4' upon transformation to a nonrest frame, thereby misleading a nonrest observer as to the real power density.

The *space* part of $\mathfrak{p}_\lambda$ in (28) may be understood as follows. Assume that an observer O' is charging a capacitor by means of an electrical battery while everything is at rest in his laboratory to which frame S' is attached. The capacitor is hanging by a fine wire so that it is free to swing. Another observer O using a frame S with respect to which S' has a velocity $\boldsymbol{\beta}$ notes that the electromagnetic *momentum* must be increasing between the capacitor plates as well as the energy U (because this energy has a mass equivalent to U/c^2 and a velocity equal to $\boldsymbol{\beta}c$). Since observer O does not see the capacitor lag or swing back (i.e., he does not see it tend to slow down as its mass is increased) he concludes that the source of the *energy* (or the source of $\mathfrak{p}_4$) must also be the source of a *force* (or the source of $\mathfrak{p}_k$) that gives this energy the momentum that it needs to keep up with the rest of the moving laboratory (i.e., to keep $\boldsymbol{\beta}$ constant). This is the essence of Trouton's first experiment(23) to detect the motion of the earth through the ether. Of course he did not observe the capacitor to slow down when he charged it.

We note in passing that, although "paradoxical results may be obtained if one tries to identify the Poynting vector with the energy flow per unit area at any particular point,"(7) no such results are obtained with $\mathfrak{p}_\lambda$. The transfer, flow, and storage of energy predicted by (15) and (28) do no violence to the intuition. If we imagine that Trouton used one parallel-plate capacitor to charge another, then detailed calculations by observer O of power transfer, energy flow, energy storage, forces, and torques over Trouton's laboratory (S') reveals an interesting but otherwise unremarkable distribution of values.

Applying either the procedure leading to (15) or the procedure leading to (28) to the *space* components

of (1) we find the general-frame form of the equation of continuity to be

$$\mathrm{Div}_\mu \beta_\lambda T_{\mu\nu}\beta_\nu = f_\mu \beta_\mu \beta_\lambda, \tag{30}$$

which may be rewritten in the form

$$\mathrm{Div}_\mu \mathfrak{T}_{\lambda\mu} = \mathfrak{p}_\lambda, \tag{31}$$

where

$$\mathfrak{T}_{\lambda\mu} \equiv \begin{bmatrix} \gamma^2(\mathbf{S}/c+\boldsymbol{\beta}.\mathsf{T})\boldsymbol{\beta} & i\gamma^2(W-\mathbf{S}.\boldsymbol{\beta}/c)\boldsymbol{\beta} \\ i\gamma^2(\mathbf{S}/c+\boldsymbol{\beta}.\mathsf{T}) & -\gamma^2(W-\mathbf{S}.\boldsymbol{\beta}/c) \end{bmatrix} = \begin{bmatrix} \mathfrak{S}\boldsymbol{\beta}/c & i\mathfrak{W}\boldsymbol{\beta} \\ i\mathfrak{S}/c & -\mathfrak{W} \end{bmatrix} \tag{32}$$

is a generalized stress-momentum-energy tensor that eliminates the inconsistencies associated with $T_{\mu\nu}$.

Equation (15) [the electromagnetic energy continuity equation] follows immediately from (31); and the corresponding momentum continuity equation also follows from (31). Note that the $(k,4)$ components of $\mathfrak{T}_{\lambda\mu}$ represent momentum density, that the $(4,k)$ components represent energy current density, and that the $(4,4)$ component represents energy storage density.

If $\mathfrak{T}_{4i}$ is substituted for T^{4i} in (21-62) of Panofsky and Phillips (Sect. 1 herein), then the resulting equation *is* covariant whether the field is "free" or not; and the resulting G^i is therefore a four-vector.

Furthermore, if one uses $\mathfrak{T}_{\lambda\mu}$ instead of $T_{\mu\nu}$, then Pauli's statement, "The energy-momentum tensor S_{ik} of Maxwell-Lorentz electrodynamics has to have terms added to it in such a way that the conservation laws

$$\frac{\partial T_i^{\ k}}{\partial x^k} = 0$$

for the total energy-momentum tensor become compatible with the existence of charges," no longer holds because

$$\frac{\partial \mathfrak{T}_{\lambda\mu}}{\partial x_\mu} = 0$$

is valid for systems in which no work is being done.

7. DISCUSSION

The interpretations of the three terms in Poynting's theorem are very appealing to the intuition because Poynting's theorem applies to electromagnetic work and energy the same classical concept of continuity as that applied to fluids, heat, neutrons, ... Per unit time and per unit volume at a given point *P* in space, the amount of work performed on the field by some external source of energy is equal to the increase in energy stored in the field at *P* plus the net amount of energy flowing to other parts of the field. Equation (15) may be interpreted in the same way. By substituting (16), (17), (18), and (25) into (15) we obtain an equation that even resembles in appearance Poynting's theorem and thereby implies the same interpretation:

$$-\mathbf{J}_i \cdot \mathbf{E}_q = \frac{\partial \mathfrak{W}}{\partial t} + \mathrm{div}\,\mathfrak{S}. \tag{33}$$

However, an examination of (16), (17), and (18) reveals that the energy current density, energy storage density, and power transfer density at a point in space involve nonlocal terms: the β's and γ's. Ever since Poynting's theorem was assumed to apply to moving

sources and stationary sources with uncoupled overlapping fields, physicists have believed that one could ascertain the electromagnetic energy storage density and energy current density at a point in free space by a measurement of only $\mathbf{E}$ and $\mathbf{H}$ at that point. It now appears that this attractive concept is not valid within present conventions if moving sources are present. One needs to know what is happening somewhere else: in particular, one needs to know how a given field is produced and how fast the sources are moving. If some conventions are changed (e.g., the partial force concept from electrostatics) then it may be possible to eliminate the β's and γ's from the definitions of electromagnetic energy and momentum densities; but such considerations are outside the scope of the present paper.

The mathematical expressions for the momentum and energy densities implied by (16) and (17) are the same as those obtained by Rohrlich(13,14), even though he used a different physical argument, model, and logic, as described in Sect. 1; he was working specifically with the point electron whereas the present work is concerned specifically with macroscopic Maxwellian charge; and he was working with a static system whereas the present work concerns dynamic systems. The essential point here is that the requirements of *physical* covariance (as distinguished from abstract mathematical covariance) were imposed on both derivations (Rohrlich's and that herein). Quantitative discussions of Rohrlich's results, when arbitrarily applied to certain macroscopic charged bodies, have been given(24,25). Several other authors have successfully resolved the discrepancies that arise when the classical concepts of electromagnetic momentum and energy are applied to the classical electron; a brief history of these efforts has been published(25). The asynchronous formulation of mechanics proposed by Cavalleri and Salgarelli(16) is analogous to the approach of Rohrlich and gives the same results.

Note added in proof. A detailed and thoughtful discussion of the philosophic basis for the choice of simultaneity criterion (synchronous vs. asynchronous) has recently been presented by H. Arzeliès,

The present results are consistent with those of Arzeliès, but the present philosophic approach is different from both other approaches (synchronous and asynchronous) as discussed in the next five paragraphs.

The asynchronous approach[13-16] involves integration by observer O at constant t' or constant proper time. As previously shown[25], this requirement is equivalent to a substitution of a four-vector representation $d\sigma_\nu$ for the volume element dv in (21-62) of Sect. 1; viz.,

$$G_\mu = \int T_{\mu\nu} d\sigma_\nu,$$

where

$$d\sigma_\nu \equiv \gamma dv(\gamma\boldsymbol{\beta}, i\gamma) = (\gamma^2\boldsymbol{\beta}, i\gamma^2)dv.$$

For the special case of an electrostatic system it has been shown[25] that this substitution of $d\sigma_\nu$ for dv leads to

$$\mathbf{G} = \int (\gamma^2/c^2)\left[c\mathbf{E}\times\mathbf{H} + \tfrac{1}{2}(E^2 - H^2)\mathbf{v} - c(\mathbf{E}\times\mathbf{H})\right]dv.$$

Thus we see that integration by observer O at constant proper time subtracts out *identically* the original Poynting vector $\mathbf{S}$ from the expression for the electromagnetic momentum of the system, as viewed by a moving observer O, and substitutes an entirely different expression for it. Therefore, as far as electrostatic systems are concerned, the major differences between the present approach and that involving integration at constant proper time may be summarized as follows.

Present Approach. The composition of the Poynting energy flow vector $\mathbf{S}$ and energy density W are described in detail by the sequence of equations (14a)-(14g); i.e., for static systems referred to a general frame, $\mathbf{S}$ and W include subcomponents that arise from the fictitious momentum and energy associated with balanced partial forces. The present approach is to stop at (14e) because adding $i\gamma\beta f_1$ or $i\gamma\rho\mathbf{v}\cdot\mathbf{E}$ to both sides of (14e) does irreparable harm to the physical interpretation of the terms. Stopping at (14e) is equivalent to choosing (29) because the taking of the inner product $f_\mu\beta_\mu$ also annuls the influence of the f_k components. The result in either case is that one obtains unconventional expressions for electromagnetic momentum and energy densities which are then used in (21-62) in the conventional way:

$$G_\lambda = \int \mathfrak{T}_{\lambda 4} dv.$$

Asynchronous Approach. Although the step between (14e) and (14f) destroys the *interpretation* of the terms involved, it is still possible to *retrieve the basic information* from (14f) or (14g) by requiring all observers to perform the integration at constant proper time, which is equivalent to taking the inner product $T_{\mu\nu}d\sigma_\nu$ in the equation

$$G_\mu = \int T_{\mu\nu} d\sigma_\nu.$$

This procedure subtracts out the classical result and substitutes a new result which is the same as that of the present approach. For intuitive purposes one might think of the constant-proper-time requirement as a photographic snapshot of the situation by observer O'; such a snapshot, when used by observer O, precludes the combination (motion of system S and the balanced partial forces applied to static elements of a system in S') from introducing any apparent transfer of momentum-energy, even for observer O, thereby eliminating the classical problems.

Comparison. For observer O to obtain the correct answers for electromagnetic momentum and energy of moving static systems, it is essential that the effect of f'_k (or the corresponding components of $Div'_\nu T'_{\mu\nu}$) be annulled in one way or another. There are two ways to do this: (i) Stop at (14e) while transforming (8) from S' to S or take the inner product $f_\mu \beta_\mu$ to obtain the densities as illustrated by (29) or (ii) perform the integration at constant proper time or integrate with a four-vector representation $d\sigma_\nu$ of the volume element, thereby effectively taking the inner product $T_{\mu\nu}\beta_\nu$.

So, in effect, for static systems the present approach may be said to involve correction *before* the integration (i.e., at the "density" stage) whereas the asynchronous approach involves correction *during* the integration (or at the total momentum-energy stage).

8. APPLICATION OF (15) TO ELECTROSTATIC SYSTEMS

The present paper grew out of the discovery that the electromagnetic mass m_{em} of a charged parallel plate capacitor is $2U_0/c^2$ (where U_0 is the electrostatic field energy of the capacitor at rest). This value arises (i) from application of the Poynting vector to calculate the electromagnetic *momentum* of the capacitor moving in a direction parallel to the plates and (ii) from equating this value to $m_{em}v$. If the direction of motion is perpendicular to the plates, then m_{em} becomes zero. But if the electromagnetic *energy* U of the moving capacitor is calculated from the Poynting energy density by equating $\frac{1}{2}m_{em}v^2$ with $U-U_0$, then the electromagnetic mass becomes $3U_0/c^2$ for motion parallel to the plates or $-U_0/c^2$ for motion perpendicular to the plates. Other combinations of orientation, electromagnetic field integration, and electromagnetic mass equation give $\sqrt{3}$

U_0/c^2 or $\sqrt{-1}\ U_0/c^2$. This appeared to be an intolerable state of affairs for physics: electromagnetic mass not only appeared to be a tensor of rank two but also appeared to be dependent on the choice of mechanical defining equation.

The electromagnetic mass of a *sphere* was calculated by Heaviside (1885) to be $(4/3)U_0/c^2$ from the Poynting vector and the definition of mechanical linear momentum. This value has received a great deal of attention, but another value appears to have been ignored: the Poynting *energy density* together with the definition of *kinetic energy* at low speeds leads to $(5/3)U_0/c^2$.

We now consider the electromagnetic energy, momentum, and electromagnetic mass of any electrostatic system according to the proposed substitute for Poynting's theorem (15). If the field in S' is purely electrostatic, then (15) may be reduced to a simpler form by the use of (23) and the vector identity

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) \equiv \mathbf{B}(\mathbf{A}.\mathbf{C}) - \mathbf{C}(\mathbf{A}.\mathbf{B}). \tag{34}$$

Substituting (23) and (34) together with $\mathbf{J}_i.\mathbf{E}_q = 0$ into (15) we obtain

$$\mathrm{div}\left[\tfrac{1}{2}\gamma^2(E^2 - H^2)\mathbf{v}\right] + (\partial/\partial t)\left[\tfrac{1}{2}\gamma^2(E^2 - H^2)\right] = 0 \tag{35}$$

or

$$\mathrm{Div}\left[\tfrac{1}{2}\gamma(E^2 - H^2)u_\nu\right] = 0 \tag{36}$$

or

$$\mathrm{Div}_\nu(-\tfrac{1}{4}\gamma F^2_{\kappa\lambda}u_\nu) = 0, \tag{37}$$

where $u_\nu \equiv (\gamma\mathbf{v},\ i\gamma c)$ and where $F_{\kappa\lambda}$ is the conventional electromagnetic field tensor.

Equations (35-37) are just different forms of the same fourth-component equation. We now write the equation of continuity as a complete four-vector equation

$$\mathrm{Div}_\nu(-\tfrac{1}{4}F^2_{\kappa\lambda}u_\nu)\beta_\mu = 0. \tag{38}$$

Comparing (35) with (15) or (7) and making the same interpretations of the terms we obtain an energy-current-density vector

$$\mathfrak{S} = \tfrac{1}{2}\gamma^2(E^2 - H^2)\mathbf{v}, \tag{39}$$

which may be compared with the Poynting vector [defined by (4)]; and we obtain an energy storage density

$$\mathfrak{W} = \tfrac{1}{2}\gamma^2(E^2 - H^2), \tag{40}$$

which may be compared with the conventional expression [defined by (3)].

Since the quantity $(E^2 - H^2)$ is Lorentz invariant, it follows immediately that the electromagnetic momentum-energy of a charged body constitutes a four-vector. For example, the energy contained in a volume element dV according to observer O is

$$dU = \mathfrak{W}dV = \tfrac{1}{2}\gamma^2(E^2 - H^2)dV = \tfrac{1}{2}\gamma^2(E'^2 - H'^2)dV'/\gamma = \gamma dU'.$$

The vector $\mathbf{E}\times\mathbf{H}$ for a moving uniformly charged spherical shell (or point charge) is not parallel to $\boldsymbol{\beta}$ at a general point in space. Instead, for points generally in front of (behind) a moving positive charge, $\mathbf{E}\times\mathbf{H}$ has a component toward (away from) the velocity axis. If one

interprets $c\mathbf{E}\times\mathbf{H}$ as the energy current density, then energy must be flowing generally *toward* the forward velocity axis, which therefore must be an energy *sink* (and generally *away from* the aft velocity axis, which therefore must be an energy *source*). Of course, the forward (aft) velocity axis is *not* actually an energy sink (source). This prediction of Poynting's vector is one of the "paradoxical results" referred to by Panofsky and Phillips(7). Note that no such paradoxical prediction is made by the energy current density vector $\mathfrak{S}$ of (39) because $\mathfrak{S}$ is everywhere parallel to $\boldsymbol{\beta}$. According to $\mathfrak{S}$, the stored field energy contained within each and every volume element dV' (fixed with respect to the charge) moves at the uniform velocity $\mathbf{v}$ in accordance with intuitive expectations.

9. APPLICATION OF (15) TO PERMANENT MAGNETS

For a system consisting of permanent magnets (but with no separated charges or macroscopic currents) one may substitute (22) and (34) along with $\mathbf{J}_i\cdot\mathbf{E}_q = 0$ into (15) to obtain

$$\mathrm{div}\left[\tfrac{1}{2}\gamma^2(H^2 - E^2)\mathbf{v}\right] + (\partial/\partial t)\left[\tfrac{1}{2}\gamma^2(H^2 - E^2)\right] = 0 \tag{41}$$

or

$$\mathrm{Div}\left[\tfrac{1}{2}\gamma(H^2 - E^2)u_\nu\right] = 0 \tag{42}$$

or

$$\mathrm{Div}_\nu(\tfrac{1}{4}\gamma F^2_{\kappa\lambda}u_\nu) = 0 \tag{43}$$

or

$$\mathrm{Div}_\nu(\tfrac{1}{4}F^2_{\kappa\lambda}u_\nu)\beta_\mu = 0. \tag{44}$$

The symmetry of (41)-(44) with (35)-(38) is obvious. Since the quantity $(H^2 - E^2)$ is Lorentz invariant, it follows immediately that the electromagnetic momentum-energy of a permanent magnet constitutes a four-vector. Rohrlich[26] has previously derived the energy and momentum densities implied by (41) for permanent-magnet systems. Here, as in the case of electrostatic systems, his physical and mathematical models are so different from those used herein that it is not immediately apparent that they would lead to the same answer. But what they have in common is the property that matters: each model in its own way requires both mathematical and physical covariance.

10. APPLICATION OF (15) TO RADIATION

Poynting's theorem has long been known to be convariant for radiation in the absence of sources. The reason that Poynting's theorem is physically covariant for radiation whereas it is not physically covariant for sources is that, if no sources are present, then

$$f'_\mu = f_\mu = (\mathbf{0},\ 0)$$

for all frames. Hence f'_μ here does not contain the contradiction wherein a nonzero force acts on a source element, producing no acceleration; and similarly, for $\partial T'_{\mu\nu}/\partial x'_\nu$. Therefore, the procedure described between (12) and (13) may be followed so that (15) reduces to the same mathematical form as Poynting's theorem. It follows, therefore, as a matter of course, that (15) is also covariant for radiation.

Poynting's theorem may alternatively be justified for radiation on the following somewhat intuitive basis. Poynting's theorem is a combination of Maxwell's two curl equations (the source equations are ignored in the usual proof). Radiation is described essentially completely by the two curl equations (i.e., the electric field may be thought of as being induced by a changing magnetic field, and vice versa). Hence it is

reasonable to expect Poynting's theorem to describe correctly energy continuity for radiation in the absence of sources.

For the special case of radiation, certain other mathematical simplifications occur. For example, since $E = H$, it follows that

$$W_{rad} = E^2; \tag{45}$$

and the energy current density is

$$\mathbf{S}_{rad} = cE^2\mathbf{i}, \tag{46}$$

where **i** is a unit vector in the direction of propagation.

As a specific example of the application of (15) or (17) to radiation, consider a plane polarized wave traveling along the x' axis with $\mathbf{E}' = E'\mathbf{j}$ and $\mathbf{H}' = H'\mathbf{k}$. Observer O' writes for the energy dU' within a volume element dV'

$$dU' = E'^2 dV'. \tag{47}$$

Observer O, using (17), (45)-(47), and the standard electromagnetic field transformation equations, writes

$$\begin{aligned} dU &= (\gamma^2 W - \gamma^2 \boldsymbol{\beta}.\mathbf{S}/c)\,dV = (\gamma^2 E^2 - \beta\gamma^2 E^2)\,dV \\ &= \gamma^2 E_y^2 (1-\beta)\,dV = \gamma^2\left[\gamma^2 (E'_y + \beta H'_z)^2\right](1-\beta)(dV'/\gamma) \\ &= \gamma^3 (1+\beta)^2 (1-\beta) E'^2 dV' = \left[(1+\beta)/(1-\beta)\right]^{\frac{1}{2}} dU', \end{aligned}$$

which is the energy version of the ordinary longitudinal symmetric relativity Doppler effect, hence is the same result obtained by a conventional application of Poynting's theorem.

11. RECAPITULATION, CONCLUSIONS, AND EPILOGUE

The major equations of electromagnetic theory are Lorentz invariant. Yet almost from the inception of relativity theory, the electromagnetic momentum-energy of sources has been believed not to constitute a four-vector. How can this be? Logic demands that a covariant theory lead to covariant quantities. Perhaps a major factor in the general acceptance of this logic inconsistency in electromagnetic theory has been the apparent success of the ingenious *ad hoc* theories (in particular, the Poincaré stresses and the Laue elastic energy currents) in explaining away numerical discrepancies in quantitatively correct terms, while simultaneously providing for the stability of the system. It was no accidental coincidence that these *ad hoc* theories succeeded in providing just the right amount of correction to account for the discrepancies, because these discrepancies, the instability of the system, and the *ad hoc* theories all have the same ultimate basis: the conventions, definitions, and interpretations involving quantities that are called by the names *force* and *stress* and that are treated in transformations as bona fide forces and stresses but which are *not* associated with a transfer of momentum or energy.

It is an amusing pastime to calculate in complete detail for specific configurations the amounts of fictitious energy and momentum introduced by the intermixing of the first and fourth components of f'_μ (and $Div'_\nu T'_{\mu\nu}$) upon transformation to a general frame and then to observe that these amounts turn out to be exactly equal to the magnitudes of fictitious energy and momentum associated with the Poincaré stresses (or pressure) or with Becker and Sauter's tensor of mechanical or other origin. Similar considerations hold for fictitious torque and the Laue elastic energy currents.

An even more interesting pastime for observer O is to calculate in detail the amount of *work* required to produce a charged body with a specific configuration at rest in S'. For some reason such calculations appear to be missing from the common literature. What makes such calculations so amusing is that they present some striking examples of nonconservation of energy. Application of (31) removes such embarrassing results.

Perhaps it is worth noting that Poincaré, Becker and Sauter, and a line of others were close to the truth when they kept attempting to relate the *instability* of the electrostatic system with the anomalous *4/3* factor in the electromagnetic mass of a charged spherical shell (or of the electron). Although it is now clear that there is no intrinsic connection between the stability of a charged body and the nature of its electromagnetic mass, momentum, and energy, there *is* an *extrinsic* connection: the unorthodox convention of using the word *force* and the symbol F in a manner that is inconsistent with Newton's second law. In the proposed resolution of this problem, the question of stability does not arise because, if one treats the partial force for what it is, there is no exchange of energy or momentum between the electromagnetic system and the stabilizing system; each obeys the Lorentz transformation in the same way.

The approach of Rohrlich or the application of the asynchronous formulation from the mechanics of Cavalleri and Salgarelli eliminates the mischief of the partial-force convention for static systems by requiring that length measurements and integrations over extended bodies be performed at constant proper time. This requirement is equivalent to taking the inner product $T_{\mu\nu}d\sigma_\nu$ *during* the integration, as a substitute for (21-62). For static systems the present approach is equivalent to taking the inner product of $f_\mu\beta_\mu\beta_\lambda$ *before* the integration, as a substitute for f_μ. Thus we see that each procedure, in its own way, effectively negates the influence of the partial forces f_k. In a sense, the constant-t' requirement retroactively undoes the damage introduced by the step from (14e) to (14f), which added the fictitious energy $i\gamma\rho\mathbf{v}.\mathbf{E}$ to both sides of the energy continuity equation.

It appears to be appropriate to acknowledge that such objects as f_μ do transform mathematically in the proper way (because they are *defined* to do so); hence Poynting's theorem is a *mathematically* true statement for all frames. It represents a mathematically covariant description of the continuity of *something*. If no sources are present (or if all sources are stationary and if all overlapping electric and magnetic fields are coupled by Maxwell's curl equations) then that *something* is electromagnetic energy; but if moving sources or uncoupled overlapping electric and magnetic fields are present, then that *something* appears to have no useful physical interpretation.

In spite of the aforementioned problems associated with f_μ, it still appears that the technique of working with just the *electromagnetic* force acting on source elements is convenient and useful in solving many problems in electrostatics, magnetostatics, and in some areas of electrodynamics. Equation (15),

$$\mathrm{div}(\gamma^2\mathbf{S} + \gamma^2\mathbf{v}.\mathrm{T}) + (\partial/\partial t)(\gamma^2 W - \gamma^2\boldsymbol{\beta}.\mathbf{S}/c)$$

$$= -\gamma^2(\mathbf{J} - \rho\mathbf{v}).(\mathbf{E} - \mathbf{H}\mathrm{X}\boldsymbol{\beta}), \qquad (15)$$

by automatically keeping separate accounts of real and fictitious quantities, permits one to do that. Each major term in (15) is interpreted in the same way as the corresponding term in Poynting's theorem.

Since the energy current density and storage density terms from (15) contain nonlocal quantities, it follows that the energy current density and storage density cannot in general be expressed strictly in terms of only local quantities *as usually interpreted*. Equation (15) is valid if all sources have the same velocity $\boldsymbol{\beta}$ with respect to S. If relative motion exists among the sources, then the energy and momentum transfer densities at some point P in the field can be calculated only after one somehow resolves the field at P according to the various contributing sources and radiation. He may then sum over all the contributions. It appears that the most convenient way of performing this summation is in terms of the independently transformed quantities $\mathbf{E}_q$ and $\mathbf{H}_i$.

Most of the considerations herein have been directed at systems which are at rest with respect to a common frame of reference S', which itself may have any velocity less than c with respect to another frame S. Currents in neutral conductors have been permitted, and convection currents in stationary configurations have also been permitted. Of course this limitation of no relative motion among sources restricts the usefulness of the result because many systems of interest involve relative motion among different parts of the system. Such systems will be the subject of another investigation. For present purposes it is sufficient to observe that the present treatment permits slow relative motion among different sources comprising the system (with no limitation other than c on the motion of the system as a whole).

The approach here has been concerned primarily with the flow of energy in dynamic fields (where electromagnetic work is being done) involving sources; and secondarily, with the electromagnetic momenta and energies of moving electrostatic or magnetostatic systems. Systems involving uncoupled fields (e.g., overlapping electrostatic and magnetostatic fields) and systems involving electromagnetic momentum associated with circuits but not with work (e.g., a constant current in an isolated superconductor) are outside the scope of the present paper. These will be treated in separate works.

The present paper has attempted to show that Pauli's tensor S_{ik} need not necessarily "have terms added to it in such a way that the conservation laws . . . become compatible with the existence of charges"; rather, that the tensor S_{ik} itself can (and indeed must) be reconstructed within pure electromagnetic theory such that the conservation laws become compatible with the existence of charges. When such is done, there is no longer any need for Poincaré stresses or for the Becker and Sauter tensor of mechanical or other origin or for Laue elastic energy currents or for any other such *ad hoc* hypotheses about the existence of physical quantities that are not experimentally observable.

Perhaps it should be noted here that Poynting himself(4) did not apply his theorem to moving sources or to uncoupled fields. He applied it only to radiation and to stationary sources with coupled fields. Therefore there are no paradoxes in his own applications of his theorem. He seemed to realize the conditions under which his theorem was valid. Poynting's interpretation of the theorem was in terms of energy storage and energy flow. Abraham(22) later added the interpretation of electromagnetic field momentum for charged bodies; and he and others applied the theorem to moving sources. Even before Poynting deduced his theorem, J.J. Thomson(27) utilized the concepts of W. Thomson(2) and Maxwell(3) regarding electric and magnetic field energy to develop the concept of electromagnetic mass.

The case for Poynting's theorem as an electromagnetic energy continuity statement was argued eloquently by Stratton(28). "The classical interpretation of Poynting's theorem appears to rest to a considerable degree on hypothesis. Various alternative forms of the theorem have been offered from time to time, but none of these has the advantage of greater plausibility or greater simplicity to recommend it, and it is significant that thus far no other interpretation has contributed anything of value to the theory. The hypothesis of an energy density in the electromagnetic field and a flow of intensity **S** = **E**X**H** has, on the other hand, proved extraordinarily fruitful. A theory is not an absolute truth but a self-consistent analytical formulation of relations governing a group of natural phenomena. By this standard there is every reason to retain the Poynting-Heaviside viewpoint until a clash with new experimental evidence shall call for its revision."

The present paper is an attempt to show that an alternative form of the theorem does have the advantage of greater plausibility if not greater simplicity. For example, the direction of electromagnetic energy flow for a moving electrostatic system as predicted by the proposed form of the theorem is consistent with the intuition and with the rest of physics. On the one hand, it appears that Poynting's theorem, together with the ordinary concepts of classical mechanics, does not constitute a self-consistent formulation because this combination predicts (i) that electromagnetic mass is a tensor of rank two, (ii) that it assumes different values in different equations from mechanics, and (iii) that energy and momentum are not conserved quantities. On the other hand, it appears that the proposed substitute form of the theorem, together with the concepts of classical mechanics, does provide a self-consistent formulation for certain classes of systems. Hence the present paper is a call for the revision of Poynting's theorem.

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