Annales de la Fondation Louis de Broglie,
Vol. 7, n° 4, 1982

# FIELD VERSUS PARTICLE DESCRIPTION IN STATISTICAL MECHANICS

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(manuscrit reçu le 5 Mars 1982)

*Abstract : Dynamical field equations arising in statistical mechanics can be regarded as lifts (in many cases Liouville equations) of equations governing the time evolution of a particle or few particles. Three different methods leading to this alternative interpretation of dynamical field equations, their history and usefulness are reviewed.*

*Résumé : Les équations dynamiques de champ qui interviennent en mécanique statistique peuvent être regardées comme des relèvements (dans de nombreux cas des équations de Liouville) d'équations gouvernant l'évolution temporelle d'une ou de quelques particules. On passe en revue trois méthodes différentes menant à cette autre interprétation des équations dynamiques de champ sont analysées, ainsi que leur histoire et leur utilité.*

## I. Introduction

A dynamical theory of macroscopic systems is a theory of the time evolution of an appropriately chosen state variable which contains relevant features of a class of macroscopic systems. The relevant features are the features revealed in interactions of the class of macroscopic systems with a given class of measurement instruments. The class of macroscopic systems for which predictions derived from the dynamical theory agree with the measurements is called the domain of applicability of the dynamical theory. The set of measurements considered is called the experimental basis of the dynamical theory. Classical mechanics, kinetic theory, hydrodynamics represent examples of the dynamical theories of macroscopic systems.

A dynamical field theory of macroscopic systems is a dynamical theory of macroscopic systems in which state variables, denoted f, are cross sections of a bundle (E, p, B). The set B is the base space, E is the total space and $p : E \to B$ is the projection of the bundle (see Ref. 1). For each $x \in B$ the set $p^{-1}(x)$ is called the fiber of the bundle over $x \in B$. The cross section of the bundle is a mapping $B \to E$. Examples of the dynamical field theories of macroscopic systems are : kinetic theory, hydrodynamics. In kinetic theory the base space $B = \mathcal{R}^3 \times \mathcal{R}^3$ is the phase space of one particle, $x \in B$, $x \equiv (\underline{r}, \underline{v})$, $\underline{r}$ denotes the position vector, $\underline{v}$ denotes the velocity. The fiber over $x \in B$ is $\Omega^6$ that is the one dimensional space of 6-forms (volume elements) defined on $\mathcal{R}^6$. The quantity $f(\underline{r}, \underline{v})\, d\underline{r}\, d\underline{v}$ is the number of particles at the infinitesimal volume $d\underline{r}$ about $\underline{r}$ with velocities at the infinitesimal volume $d\underline{v}$ about $\underline{v}$.

Let the time evolution of the field f be governed by the equation

$$\frac{\partial f}{\partial t} = \mathcal{R}(\Lambda, f) \quad , \tag{1}$$



where $\Lambda$ denotes the indetermined quantities introduced by the dynamical theory. Through $\Lambda$ the individuality of the macroscopic systems inside the domain of its applicability is expressed. Let $\mathcal{H}$ denotes the set of all admissible states. Whe shall not, at this point, specify the algebrical and the topological structure of the set $\mathcal{H}$. In kinetic theory the Equation (1) is for example the Boltzmann kinetic equation. In this case $\Lambda$ represents the cross section of a binary collision. In hydrodynamics the Equation (1) is for example the system of the Navier-Stokes-Fourier equations, $\Lambda$ represents the transport coefficients and the functional dependence of the local pressure and the local temperature on the hydrodynamic state variables.

## II. Lift

Let

$$\frac{dx_\alpha}{dt} = F_\alpha(x,t) \quad , \tag{2}$$

$\alpha = 1, \dots n$

govern the time evolution in the base space B ($x \in B$). We look for the time evolution of cross sections of the bundle induced by the time evolution (2) in the base space. If the base space is $B = \mathcal{R}^n$ and the total space $E = B \times \Omega^n$ then the time evolution equation (2) induces the Liouville equation (Ref. 2).

$$\frac{\partial f(x,t)}{\partial t} = - \frac{\partial}{\partial x_\alpha}(f(x,t)F_\alpha(x,t)) \; . \tag{3}$$

If $E = B \times \mathcal{R}$ then the time evolution equation (2) induces (Refs. 3, 4)

$$\frac{\partial f(x,t)}{\partial t} = -F_\alpha(x,t) \frac{\partial}{\partial x_\alpha} f(x,t) \; . \tag{4}$$

We shall say that the time evolution generated by (3) or (4) is a lift of the time evolution generated by (2). (See Ref. 1).

## III. Projection

Now we shall study the inverse problem. We assume that the time evolution of a field f is given, i.e. Equation (1) is given, and we look for the time evolution in the base space, i.e. equation of the type (2), such that its lift is Equation (1). If the Equation (2) can be interpreted as an equation governing the time evolution of a particle (or particles) then we can formulate the projection in other words. We look for an equation governing the time evolution of a particle (or particles) that is equivalent, via the process of lifting, to the given field dynamical equation (1). My objective is :

i) to review three different types of projection of Equation (1) on its base space,

ii) to review the history of the problem (to best of my knowledge the first example of a projection is the Lagrangian formulation of Euler hydrodynamic equations),

iii) to demonstrate its usefulness for obtaining more insight into the physical meaning of the field theories, and, in some cases, for solving the field equations.

## IV. Projection by using the correspondence principle

A quick analysis shows that the projection of Equation (1) on its base space does not in general exist unless we let the right hand side of Equation (2) (i.e. the forces if Equation (2) represents a dynamics of particles) depend on the field state variable f. if we accept this sort of generalized particle dynamics we see immediately that the projection is non unique. A requirement of a correspondence between certain qualitative properties of solutions to the dynamical field equations (1) and qualitative properties of solutions to the projected dynamical equations might be used to single



out one particular projection.

An important property of the time evolution equations (1) or (2) is the time reversibility or irreversibility of their solutions. We shall first introduce the property of the time reversibility in the context of Equation (2).

We introduce in B a transformation $j : B \to B$ such that $j.j$ = identity. If B is a phase space of classical mechanics then j is always introduced by

$$j(\underline{r},\underline{v}) = (\underline{r},-\underline{v}) \quad . \tag{5}$$

The elements $x \in B$ that are invariant with respect to j are called even state variables, the elements $x \in B$ that remain unchanged by applying j except the change of sign are called odd state variables. With the help of the transformation j we can split uniquely the right hand side of Equation (2)

$$F(x,t) = F^+(x,t) + F^-(x,t) \quad , \tag{6}$$

where

$$F^{\pm}(x,t) \overset{def}{=} \frac{1}{2}(F(x,t) \pm jF(jx,t)), \tag{7}$$

$F^+(x,t)$ is called the time irreversible part, $F^-(x,t)$ is called the time reversible part. Solutions to Equation (2) are time reversible if $F^+(x,t) \equiv 0$. Indeed, in this case the transformation j in the space B of state variables compensates exactly the inversion of time (i.e. the flow $B \times \mathbb{R} \to B$ that is generated by Equation (2) remains unchanged if on the left hand side $x \to jx$ and $t \to -t$, $t \in \mathbb{R}$). The Hamiltonian dynamics, with the Hamiltonian function invariant with respect to j, is an example of the time reversible dynamical theory.

Now we introduce the property of the time reversibility in the context of Equation (1). In the space $\mathcal{H}$ of cross sections f of the bundle (E,p,B) we introduce transformation $\mathcal{J}$ by

$$\mathcal{J}f(x) = f(jx) \quad . \tag{8}$$

With the help of $\mathcal{J}$ we again split the right hand side of Equation (1), i.e.

$$\mathcal{R}(\Lambda,f) = \mathcal{R}^{+}(\Lambda,f) + \mathcal{R}^{-}(\Lambda,f) \quad , \tag{9}$$

where

$$\mathcal{R}^{\pm}(\Lambda,f) \overset{\text{def}}{=} \frac{1}{2}(\mathcal{R}(\Lambda,f) \pm \mathcal{J}\mathcal{R}(\Lambda,\mathcal{J}f)) \tag{10}$$

The even and odd state variables and the time reversibility is now introduced in the same way as in the previous paragraph except the transformation j is replaced by $\mathcal{J}$.

We are now in position to formulate the *correspondence principle*.

We require that

$$F^{+}(f;x,t) = 0 \tag{11}$$

for f that are solutions of $\mathcal{R}^{+}(\Lambda,f) = 0$.

In other words, we require that in the class of the field variables f for which the solutions to the field equation (1) are time reversible also the solutions to the projected dynamics are time reversible.

We proceed now to the actual construction of the projected dynamics that satisfies the correspondence principle. We shall assume that $\mathcal{R}^{+}(\Lambda,f)$ has a particular form (gain-loss balance)

$$\mathcal{R}^{+}(\Lambda,f) = \int dy \int dx' \int dy' \; W^{+}(n;x',y';x,y) \tag{12}$$
$$\left\{ \exp\left(- \frac{\delta S}{\delta f(x',t)} - \frac{\delta S}{\delta f(y',t)}\right) - \exp\left( - \frac{\delta S}{\delta f(x,t)} - \frac{\delta S}{\delta f(y,t)} \right.\right.$$



The quantities in (12) that are left indetermined, i.e. the quantities $\Lambda$, are $W^{+}$, S and transformation $(x,y) \overset{T}{\to} (x',y')$. By choosing appropriately these quantities we can obtain the time irreversible parts of all well known kinetic equations including the Boltzmann, the Fokker-Planck and the Enskog kinetic equations. We shall explain the notation introduced in (12). By $n(\underline{r},t)$ we denote $\int d\underline{v}\, f(\underline{r},\underline{v},t)$. The transformation $(x,y) \overset{T}{\to} (x',y')$ transforms two particles before binary interaction to the state $(x',y')$ after the binary interaction. We shall assume that T commutes with j. An example of the transformation T that satisfies this property is the transformation of velocities due to a binary collision. The quantity $W^{+}$ has the meaning of transition probability, we assume thus that $W^{+} > 0$ for all n, x', y', x, y. Moreover, we assume that $W^{+}(n;\, x,y\,;\, x',y') = W^{+}(n;\, j(x'),\, j(y');\, j(x),\, j(y)) = W^{+}(n;\, x',y';\, x,\, y)$. By S(f) we denote a sufficiently regular convex functional of f, we assume moreover that $S(\mathcal{J}f) = S(f)$. By $\frac{\delta S}{\delta f(x,t)}$ we denote the functional derivative of S with respect to f(x,t) (i.e. $\lim_{\varepsilon\to 0} \frac{S(f+\varepsilon\psi) - S(f)}{\varepsilon} = \int dx \frac{\delta S}{\delta f(x,t)} \psi(x)$, where $\psi$ is an arbitrary sufficiently regular function of x).

It can be proved that the only solutions of $\mathcal{R}^{+}(\Lambda,f) = 0$ are the solutions of

$$\frac{\delta S}{\delta f(x',t)} + \frac{\delta S}{\delta f(y',t)} = \frac{\delta S}{\delta f(x,t)} + \frac{\delta S}{\delta f(y,t)} \tag{13}$$

[Proof : We first prove that $\int dx \frac{\delta S}{\delta f(x,t)} \mathcal{R}^{+}(\Lambda,f) \geqslant 0$, where the equality holds if and only if Equation (13) is satisfied.

$$\int dx \frac{\delta S}{\delta f(x,t)} \mathcal{R}^+(\Lambda,f) = \int dx \int dy \int dx' \int dy' \frac{\delta S}{\delta f(x,t)} W^+(\ldots)[\ldots] =$$

$$= \frac{1}{2}\int dx \int dy \int dx' \int dy' \left\{\frac{\delta S}{\delta f(x,t)} + \frac{\delta S}{\delta f(y,t)}\right\} W^+(\ldots)[\ldots] =$$

$$\frac{1}{4}\int dx \int dy \int dx' \int dy' \left\{\frac{\delta S}{\delta f(x,t)} + \frac{\delta S}{\delta f(y,t)} - \frac{\delta S}{\delta f(x',t)} - \frac{\delta S}{\delta f(y',t)}\right\}$$

$$W^+(\ldots)[\ldots] \geqslant 0 \, ,$$

where the equality holds if and only if Equation (13) is satisfied. The last inequality follows from $(\xi - x)(e^{-x} - e^{-\xi}) \geqslant 0$, where the equality holds only for $\xi = x$ ; $\xi$ and $x$ are real numbers. Now, the above result implies that the only solutions of $\mathcal{R}^+(\Lambda,f) = 0$ are the solutions of Equation (15) since if there would be another solution, say $\tilde{f}$, of $\mathcal{R}^+(\Lambda,f) = 0$ then $\int dx \frac{\delta S}{\delta \tilde{f}(x,t)} \mathcal{R}^+(\Lambda,\tilde{f}) = 0$. But this is excluded by the previous result].

Now we shall introduce the Irwing Kirkwood lemma (Ref. 5).

$$\int dx'(\phi(x',x) - \phi(x,x')) = -\frac{\partial}{\partial x_\alpha} K_\alpha(x), \qquad (14)$$

where $x \in \mathcal{R}^n$, $x' \in \mathcal{R}^n$, $\alpha = 1, \ldots n$, $\phi$ is an arbitrary function,

$$K_\alpha(x) = \int dx' \int_0^1 d\eta \; x'_\alpha \; \phi(x(\eta),x'(\eta)),$$
$$x(\eta) = x - \eta x' \qquad (15)$$
$$x'(\eta) = x + (1-\eta)x'.$$

By applying the Irwing-Kirkwood lemma to (12) we obtain



$$F_\alpha(f;x,t) = \frac{1}{f(x,t)} \int dy \int dx' \int dy' \int_0^1 d\eta \; x'_\alpha \times$$
$$W^+(\eta;x(\eta),y(\eta);x'(\eta),y'(\eta)) \times \qquad (16)$$
$$\exp\left\{-\frac{\delta S}{\delta f(x(\eta),t)} - \frac{\delta S}{\delta f(y(\eta),t)}\right\},$$

where

$$x(\eta) = x - \eta x' \; , \quad x'(\eta) = x + (1-\eta)x' \, .$$

We can see easily that $F^+_\alpha(f;x,t) = 0$ for f satisfying (13), i.e. we can prove that the projection (16) of (12) satisfies the correspondence principle.

[Proof : In the right hand side of Equation (16) we make the transformations $\eta \to (1-\eta)$, $x' \to -x'$, $y' \to -y'$.
We obtain :

$$F^+(f;x,t) = -\frac{1}{f(x,t)} \int dy \int dx' \int dy' \int_0^1 d\eta \; x'_\alpha \; W^+(\eta;x(\eta),y(\eta);$$
$$x'(\eta),y'(\eta)) \times \exp\left\{-\frac{\delta S}{\delta f(x'(\eta),t)} - \frac{\delta S}{\delta f(y'(\eta),t)}\right\}.$$

If the field state variable f satisfies (13) then we obtain $F^+(f;x,t) = -F^+(f;x,t)$ and thus $F^+(f;x,t) = 0$]. We do not know however if the correspondence principle singles out the $F^+(f;x,t)$ given by Equation (16).

The time reversible part $\mathcal{R}^-(\Lambda,f)$ of all commonly used kinetic equations (see Ref. 6) is the sum of a part that is clearly the lift of a particle dynamics (thus the projection of this part is readily obtained) and possibly of a part of the type of gain-loss balance whose projection can be obtained by using the Irwing-Kirkwood lemma. Detailed analysis of the Fokker-Planck, Boltzmann and the Enskog kinetic equations can be found in Refs. 7, 8.

An application of the Irwing-Kirkwood lemma can be well illustrated on a simple example. Let us consider a simple model of radioactive decay of a radioactive material,

$$\frac{dx}{dt} = - a\, x \ , \tag{17}$$

where $x \in \mathcal{R}$ is proportional to the number of atoms of the radioactive material, a is a positive constant. The corresponding Liouville equation to (17) is

$$\frac{\partial f(x,t)}{\partial t} = - a \frac{\partial}{\partial x}(x f(x,t)) \ . \tag{18}$$

In order to construct a more realistic model of the radioactive decay process, that takes into account the fact that in every elementary decay the change of x is discrete, we replace Equation (18) by (Ref. 9)

$$\frac{\partial f(x,t)}{\partial t} = a[(x+1)f(x+1,t) - x f(x,t)] \ . \tag{19}$$

By using the Irwing-Kirkwood lemma to Equation (19) we obtain the projection

$$\frac{d\bar{x}}{dt} = - a\, \bar{x} \ , \tag{20}$$

where

$$\bar{x} = \frac{1}{f(x,t)} \int_x^{x+1} dz\, z\, f(z,t) \ , \tag{21}$$

of the field equation (19). By comparing (17) and (20) we see the difference between (18) and (19) from a new angle.

The usefulness of the new representation of the field equation obtained by projecting it on the base space is illustrated by the example of the Vlasov kinetic equation (Ref. 10) and by the argument of Raveché and Green (Ref. 11).

It is well known that the Vlasov kinetic



equation can be regarded as a Liouville equation of a particle moving in a so called mean field that depends on the one particle distribution function $f(\underline{r},\underline{v},t)$. In fact, Vlasov's (Ref. 10) original derivation of the Vlasov equation is based on this point of view. The concept of the mean field has been then also used in constructing various iterative procedures for solving the Vlasov kinetic equation.

Raveché and Green have studied in Ref. 11 truncations of the hierarchy of equations for the densities $n_i(\underline{r}_1,\ldots,\underline{r}_i)$, i = 1, 2, ... obtained from the BBGKY hierarchy by inserting into it $f(\underline{r}_1, \underline{v}_1, \ldots, \underline{r}_i, \underline{v}_i, t) = n_i(\underline{r}_1,\ldots,\underline{r}_i) \exp(N - \frac{1}{2}\beta(v_1^2 + \ldots + v_i^2))$ (the quantity $N$ is determined by $\int d\underline{v}_1 \ldots \int d\underline{v}_i \exp(N - \frac{1}{2}\beta(v_1^2 + \ldots + v_i^2) = 1)$. The problem is to identify the class of truncations that lead to physically reasonable finite system of equations for the densities. Raveché and Green suggested that the truncated system of equations is physically acceptable if the projected equations are the equations for stationary solutions of a Hamiltonian dynamics. The physical meaning of a truncation is thus expressed by Raveché and Green in the projected equations. In Refs. 6, 12 a class of field equations of the type (1) was studied. The problem was to identify the class of the indetermined quantities that make the Equation (1) compatible with equilibrium thermodynamics. The physical meaningful dynamical equation is thus defined as a dynamical equation that is compatible with equilibrium thermodynamics. It has been found that the condition that guarantees the compatibility with equilibrium thermodynamics also guarantees that the projected dynamics is a Hamiltonian dynamics, provided f is restricted to solutions of $\mathcal{R}^+(\Lambda,f) = 0$.

The projections of a dynamical field theory on its base space that fits the setting of this section have probably first appeared in the context of the Smoluchowski equation that is a reduced form of the Fokker-Planck equation (see Ref. 13 for the list of references of the papers of Kramers and Kirkwood that were written in about 1930). The non-standard dissipative force, that depends on the field state variable, in the context of the Smoluchowski equation is called the Brownian force.

In the context of the Vlasov equation the concept of the mean field was introduced by Vlasov (Ref. 10). The mean dissipative forces arising by projecting the Fokker-Planck kinetic equation have been identified by Chandrasekhar (Ref. 14). A systematic study of the Fokker-Planck kinetic equation considered as the Liouville equation of a particle dynamics has been initiated by Fronteau (Ref. 15) and Salmon (Ref. 16). Generalizations to a larger class of kinetic equations has appeared in Refs. 7, 8.

## V. Projection on dynamics of deformation fields

In this section we shall consider the field equations of the type

$$\frac{\partial \rho(\underline{r},t)}{\partial t} = -\frac{\partial}{\partial r_\alpha}(\rho(\underline{r},t)u_\alpha(\underline{r},t)) \qquad (22)$$
$$\frac{\partial u_\alpha(\underline{r},t)}{\partial t} = G_\alpha(\rho,\ \underline{u}\ ;\ \underline{r},\ t)\ , \qquad \alpha = 1,2,3.$$

The base space is $\mathcal{R}^3$ and the fiber associated with each $\underline{r}$ is $\Omega^3 \times \mathcal{R}^3$, where $\Omega^3$ is the space of 3-forms, volume elements, on $\mathcal{R}^3$. The dynamical field equations (22) arise in hydromechanics ; $\rho$ is the mass density, $\underline{u}$ is the velocity of a fluid. In hydromechanics

$$G_\alpha(\rho,\ \underline{u}\ ;\ \underline{r},\ t) = -u_\gamma \frac{\partial}{\partial r_\gamma} u_\alpha + \frac{1}{\rho}\frac{\partial}{\partial r_\gamma}\pi_{\alpha\gamma}\ , \qquad (23)$$



where the pressure tensor $\underline{\underline{\pi}}$ remains indeterminated functional of $\rho$ and $\underline{u}$. The projection of the dynamical field equations on its base space is obtained as follows. The first equation in (22) is the Liouville equation corresponding to

$$\frac{dr_\alpha}{dt} = u_\alpha(\underline{r},t)\ . \qquad (24)$$

Let

$$r_\alpha = X_\alpha(\underline{r}_0,t) \qquad (25)$$

denote the solution of (24) that satisfies the initial condition $r_{0\alpha} = X_\alpha(\underline{r}_0,t_0)$. We shall assume that the mapping X is one-to-one for all $\underline{r}_0$ and all t. The impenetrability of matter implies that this assumption is satisfied. Since $u_\alpha(\underline{r},t) = \left.\frac{\partial X_\alpha(\underline{r}_0,t)}{\partial t}\right|_{t=t}$ we rewrite the second equation in (22) as

$$\frac{\partial^2 X_\alpha(\underline{r}_0,t)}{\partial t^2} = \Psi_\alpha(X,t)\ , \qquad \alpha = 1,2,3\ . \qquad (26)$$

The right hand side of (26) is obtained from $\underline{G}(\rho,\underline{u};\underline{r},t)$ by noting that $\rho(\underline{r},t) = \rho_0 \det\left(\frac{\partial X_\alpha(\underline{r}_0,t)}{\partial r_{0\beta}}\right)$ and by using (25) to rewrite derivatives with respect to $\underline{r}$, that might appear in G, through derivatives with respect to $\underline{r}_0$. The dynamical equation (26) is the projection of the dynamical field equation (22) on the base space. The quantity $\underline{X}(\underline{r}_0,t)$ is called a deformation field. In hydromechanics the dynamical equations (26) are called the Lagrangian form of the Eulerian hydromechanical equations (22).

Usefulness of the Lagrangian equations (26) is well recognized in hydrodynamics. Mathematical advantages have been exploited by Arnold (Ref. 17). Applications of the methods of classical mechanics to Equation

(26) have been considered by Kobussen (Ref. 18) and recently by van Saarloos (Ref. 19).

The dynamical field equations (22), (23) do not represent a full set of hydrodynamic equations. Missing is a time evolution equation for the internal energy. If instead of the internal energy we introduce the total energy $E(\underline{r},t)$ as a new field then the time evolution equation for $E(\underline{r},t)$ can be written as a local form of the conservation of the total energy

$$\frac{\partial E(\underline{r},t)}{\partial t} = - \frac{\partial}{\partial r_\alpha}\left(E(\underline{r},t)w_\alpha(\underline{r},t)\right) . \tag{27}$$

The quantity $\underline{w}$, that includes heat flux remains indetermined. If the usual Fourier constitutive relation for the heat flux is replaced by a more complete Maxwell-Cattaneo constitutive relation (Ref. 20) then we obtain an additional time evolution equation

$$\frac{\partial w_\alpha(\underline{r},t)}{\partial t} = \tilde{G}_\alpha(\rho,\underline{u},E;\underline{r},t) \quad , \tag{28}$$

where the form of $\tilde{G}_\alpha$ has been suggested by Cattaneo (Ref. 20). We note now the pair of the dynamical field equations (27),(28) has the same structure as the pair (22). We can therefore apply the same procedure to arrive at a Lagrangian time evolution equation of the type (26) for a new deformation field $\underline{Y}(\underline{r},t)$. The dynamical field equations (22), (27), (28) that represent the full set of hydrodynamic equations (the function $\underline{G}$ depends now also on E and $\underline{w}$) can be thus projected on the set of two dynamical equations for two deformation fields. Details and some applications can be found in Ref. 21.

## VI. Projection by using the Hopf or alternatively the Klimontovich reformulation of field equations

We can consider dynamical field equations as a base space dynamics (the base space is an infinite



dimensional space $B \equiv \mathcal{H}$) and we can look for the Liouville equation corresponding to this base dynamics. Since there is no natural volume element in infinite dimensional spaces, the Liouville equation cannot be easily constructed. By using the notion of the characteristic functional (Ref. 22) it can be however constructed an equation, known as the Hopf equation (Ref. 23), that plays essentially the same role as the Liouville equation. Similarly as the BBGKY hierarchy is constructed from the Liouville equation corresponding to the finite dimensional base space dynamics we can construct a BBGKY-type hierarchy of time evolution equations from the Hopf equation (see Ref. 24). The dynamical equations from the reduced distribution functions can be then projected, in the same way as in section IV, onto its corresponding base space (that is now finite dimensional). In this way a particle dynamics corresponding to the original field equation is obtained.

This method has been developed mainly for a special case

$$\frac{\partial u_\alpha}{\partial t} = -u_\gamma \frac{\partial u_\alpha}{\partial r_\gamma} - \frac{\partial}{\partial r_\alpha} \frac{1}{4\pi} \int d\underline{r}' \frac{\frac{\partial}{\partial r'_\gamma}(u_\beta(\underline{r}',t)\frac{\partial}{\partial r'_\beta}u_\gamma(\underline{r}',t))}{|\underline{r} - \underline{r}'|} + \gamma \frac{\partial^2}{\partial r_\gamma \partial r_\gamma} u_\alpha \tag{29}$$

of hydromechanical equations (22), (23) that corresponds to incompressible fluid (i.e. $\rho$ = const and thus, from the first equation in (22), we have $\frac{\partial u_\alpha}{\partial r_\alpha} = 0$ ; the pressure tensor $\underline{\underline{\pi}}$ is considered of the form $\pi_{\alpha\beta} = p\delta_{\alpha\beta} + \frac{1}{2}\nu\rho\left(\frac{\partial u_\alpha}{\partial r_\beta} + \frac{\partial u_\beta}{\partial r_\alpha}\right)$ ; the scalar pressure has been eliminated by using $\frac{\partial u_\alpha}{\partial r_\alpha} = 0$ ; $\nu$ is the viscosity transport coefficient (Refs. 24, 25)). We shall briefly review the dynamics of two particles that is associated with Equation (29) (see Ref. 25). Instead of using the Hopf equation Lungren has suggested to use Klimontovich definition of reduced distribution functions (Ref. 26). This method leads finally to the same dynamical equations for the reduced distribution functions as the ones obtained from the Hopf equation (see Refs. 24, 25). The main advantage of the Lungren method is its simplicity. The one particle distribution function $f(\underline{r}_1,\underline{v}_1,t)$, $\underline{r}_1 \in \mathcal{R}^3$, $\underline{v}_1 \in \mathcal{R}^3$, $\underline{r}_1$ is the position vector, $\underline{v}_1$ the velocity, is defined by

$$f(\underline{r}_1,\underline{v}_1,t) = < \delta(\underline{u}(\underline{r}_1,t) - \underline{v}_1) >, \tag{30}$$

where $\delta$ denotes the delta function, <>is an ensemble average that does not need to be specified explicitely. Similarly, the two point distribution functions $f_2(\underline{r}_1,\underline{v}_1,\underline{r}_2,\underline{v}_2,t)$ is defined by

$$f_2(\underline{r}_1,\underline{v}_1,\underline{r}_2,\underline{v}_2,t) = <\delta(\underline{u}(\underline{r}_1,t)-v_1)\,\delta(\underline{u}(\underline{r}_2,t)-\underline{v}_2)> . \tag{31}$$

The equation governing the time evolution of $f(\underline{r}_1,\underline{v}_1,t)$ is obtained as follows. First we note that

$$\frac{\partial f(\underline{r}_1,\underline{v}_1,t)}{\partial t} = <\frac{\partial}{\partial t}\,\delta(\underline{u}(\underline{r}_1,t) - \underline{v}_1)> = <-\frac{\partial u_\alpha(\underline{r}_1,t)}{\partial t}\,\frac{\partial}{\partial v_{1\alpha}}\,\delta(\underline{u}(\underline{r}_1,t) - \underline{v}_1)> . \tag{32}$$

The right hand side of Equation (32) is then rewritten by using Equation (29). After some calculations (for details see Ref. 25) Lungren obtained

$$\frac{\partial f(x_1,t)}{\partial t} = - v_{1\gamma}\frac{\partial f(x_1,t)}{\partial r_{1\gamma}} - \frac{\partial}{\partial r_{2\gamma}}\int dv_2\, v_{2\gamma}\, f_2(x_1,x_2,t) + \frac{\partial}{\partial v_{1\gamma}}\frac{1}{4\pi}\int dx_3\left\{\frac{\partial}{\partial r_{1\gamma}}\frac{1}{|\underline{r}_1-\underline{r}_3|}\right\}\left\{v_{3\alpha}\frac{\partial}{\partial r_{3\alpha}}\right\}^2 f_2(x_1,x_3,t) - \lim_{\underline{r}_3\to\underline{r}_1}\gamma\,\frac{\partial}{\partial v_{1\gamma}}\frac{\partial}{\partial r_{3\alpha}}\frac{\partial}{\partial r_{3\alpha}}\int dv_3\, v_{3\gamma}\, f_2(x_1,x_3,t), \tag{33}$$



where $x_1 \equiv (\underline{r}_1,\underline{v}_1)$, $x_2 \equiv (\underline{r}_2,\underline{v}_2)$, $x_3 \equiv (\underline{r}_3,\underline{v}_3)$.

By using the same method Lundgren also obtained the equation governing the time evolution of the two point distribution function. We shall write only non-dissipative part of the equation (i.e. $\nu \equiv 0$),

$$\frac{\partial f_2(x_1,x_2,t)}{\partial t} = - v_{1\gamma}\frac{\partial}{\partial r_{1\gamma}} f_2(x_1,x_2,t) - v_{2\gamma}\frac{\partial}{\partial r_{2\gamma}} f_2(x_1,x_2,t) + \frac{\partial}{\partial v_{1\gamma}}\left\{\frac{\partial\psi_{12}}{\partial r_{1\gamma}} f_2(x_1,x_2,t) + \frac{1}{\gamma_0}\int_{|\underline{r}_3-\underline{r}_1|>\eta} dx_3\frac{\partial\psi_{13}}{\partial r_{1\gamma}} f_3(x_1,x_2,x_3,t)\right\} + \frac{\partial}{\partial v_{2\gamma}}\frac{\psi_{12}}{\partial r_{2\gamma}} f_2(x_1,x_2,t) + \frac{1}{\gamma_0}\int_{|r_3-r_1|>\eta} dx_3\frac{\partial\psi_{23}}{\partial r_{2\gamma}} f_3(x_1,x_2,x_3,t) \tag{34}$$

where $\eta$ is Kolmogoroff's microscale and $\gamma_0 = \frac{4\pi\eta^3}{3}$,

$$\psi_{ij} = \left\{(v_i-v_j)_\gamma\,\frac{\partial}{\partial(r_i-r_j)_\gamma}\right\}^2 \frac{\frac{\gamma_0}{4\pi}}{|r_i-r_j|} \tag{35}$$

The equation (34) has the same structure as the second equation in the BBGKY hierarchy. We note that the potential energy $\psi$ is now velocity dependent. The dynamics of two particles of which Equation (34) is the corresponding Liouville equation is clearly :

$$\frac{dr_{1\alpha}}{dt} = v_{1\alpha} \qquad \frac{dr_{2\alpha}}{dt} = v_{2\alpha}$$

$$\frac{dv_{2\alpha}}{dt} = -\frac{\partial\psi_{12}}{\partial r_{1\alpha}} - \frac{1}{\gamma_0}\int_{|\underline{r}_3-\underline{r}_2|>\eta} dx_3\,\frac{\partial\psi_{13}}{\partial r_{1\alpha}}\,\frac{f_3(x_1,x_2,x_3,t)}{f_2(x_1,x_2,t)}$$

$$\frac{dv_{2\alpha}}{dt} = -\frac{\partial\psi_{12}}{\partial r_{2\alpha}} - \frac{1}{\gamma_0}\int_{|\underline{r}_3-\underline{r}_1|>\eta} dx_3\,\frac{\partial\psi_{23}}{\partial r_{2\alpha}}\,\frac{f_3(x_1,x_2,x_3,t)}{f_2(x_1,x_2,t)} \tag{36}$$

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