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AN EXAMPLE OF QUANTIZATION PROCEDURE BASED ON THE EQUATION FOR QUASIPROBABILITY OPERATOR

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Abstract : *Possibility of carrying out quantization procedure based on the equation for quasiprobability operator, proposed by Kuryshkin and co-workers and reflecting principle of dynamical correspondence, is demonstrated. Solution of the equation mentioned above is found out for the concrete physical system-harmonic oscillator. It turns out that the solution obtained coincides, in special cases, with antinormal, normal and Weyl's correspondence rules and leads to results, reasonable enough from the standpoint of generally accepted quantum mechanics.*

Résumé : *On expose la probabilité de contrôler la procédure de quantisation sur la base de l'équation d'opérateur de quasi-probabilité, proposée par Kuryshkin et ses collaborateurs, qui reflète le principe de correspondance en dynamique. On a trouvé la solution de l'équation proposée plus haut pour un système physique concret : l'oscillateur harmonique. Il s'avère que la solution obtenue coïncide dans le cas particulier avec l'anti-normale, la normale et les règles de correspondance de Weyl et ramène à des résultats intelligibles du point de vue commun de la mécanique quantique.*

In the paper of Kuryshkin and co-workers (1) the principle of dynamical correspondence was formulated. This principle is nothing else but a weakened Dirac principle of correspondence of classical and quantum Poisson brackets. Whereas Dirac's principle operates with two arbitrary physical quantities A and B, in the principle (1) one of the quantities must be the Hamiltonian of the system considered. Hence the correspondence rule such obtained is connected with the dynamical characteristics of the physical system considered. In (1) the proposed principle is given also in the form of equation for quasiprobability operator (2), defining the rule of correspondence.

The natural question arises- does the given equation possess any solution whatsoever, or in other words, does the principle of dynamical correspondence, formulated as a possible way of achieving a solution of the problem of correspondence rule, lead to any solution i.e. to any correspondence rule ? If the answer of the question is affirmative then immediately another question arises , whether the obtained rule produces acceptable, from the standpoint of generally accepted quantum mechanics, results or whether the result contradicts experimental data. The current paper deals with a study of these problems.

Thus our aim is to solve the equation for quasiprobability operator :

$$(1)\quad \frac{\partial \hat{F}}{\partial t} + \{H(q,p,t),\hat{F}(q,p,t)\} = \frac{i}{\hbar}\int H(q',p',t)[\hat{F}(q,p,t),\hat{F}(q',p',t)]dq'dp'$$

where $\hat{F}$ parametrically depends on coordinates q, momenta p and time t, H is the Hamiltonian of the physical system considered, {,} and [,] denote classical Poisson bracket and quantum commutator respectively. In the present paper one dimensional harmonic oscillator is taken as the physical system, with the following Hamiltonian

$$(2)\quad H(q,p) = p^2/2m + m\,\omega^2 q^2/2$$

and as the generators of Bose-algebra of quantum operators we take the Hermitian operators $\hat{q}$, $\hat{p}$ with the standard quantum

commutation relation

$$[\hat{q},\hat{p}] = i\hbar\hat{I}$$

where $\hat{I}$ is the unit operator of Bose-algebra. Demanding that the operator $\hat{A}$ should depend only on the operator $\hat{q}$(operator $\hat{p}$) when and only when the corresponding phase-space function A(q,p,t) depends only on coordinate q(momentum p), following representation for quasi-probability operator can be found

$$(3)\quad \hat{F}(q,p,t) = (2\pi\hbar)^{-2}\int u(\xi,\eta,t)\exp\frac{i}{\hbar}[\eta(\hat{q}-q) - \xi(\hat{p}-p)]d\xi\, d\eta$$

Here the function $u(\xi,\eta,t)$ is similar to Cohen's function defining Cohen's generalized correspondence rule (3). In order to fulfill the required properties of quasiprobability operator(2)

$$(4)\quad \hat{F}(q,p,t) = \hat{F}^{+}(q,p,t), \qquad \int\hat{F}(q,p,t)dqdp = \hat{I}\ ,$$

where "+" denotes Hermitian conjugation, the function $u(\xi,\eta,t)$ must satisfy the conditions

$$(5)\quad u^*(\xi,\eta,t) = u(-\xi,-\eta,t), \qquad u(0,0,t) = 1\ .$$

Substituting in equation (1) the operator $\hat{F}$ in representation (3) and Hamiltonian (2), performing the commutation operation in the right-hand side and carrying out the required integrations, we obtain the following equation for the function $u(\xi,\eta,t)$

$$(6)\quad \frac{\partial u(\xi,\eta,t)}{\partial t} = \frac{\eta}{m}\left\{\frac{\partial u}{\partial \xi} - u\left.\frac{\partial u(\xi,0,t)}{\partial \xi}\right|_{\xi=0}\right\} - m\omega^2\xi\left\{\frac{\partial u(\xi,\eta,t)}{\partial \eta} - u\left.\frac{\partial u(0,\eta,t)}{\partial \eta}\right|_{\eta=0}\right\}$$

Solution of this equation is any differentiable function V of the integrals

$$(7)\quad G_1 = (2m)^{-\frac{1}{2}}\eta\,\sin\omega t + (m\omega^2/2)^{\frac{1}{2}}\xi\,\cos\omega t\ ,$$

$$G_2 = (2m)^{-\frac{1}{2}}\eta\,\cos\omega t - (m\omega^2/2)^{\frac{1}{2}}\xi\,\sin\omega t$$

and with the properties

$$(8)\quad V^*(G_1,G_2) = V(-G_1,-G_2),\ V(0,0) = 1\ ,$$

$$\left.\frac{\partial V(G_1,0)}{\partial G_1}\right|_{G_1=0} = \left.\frac{\partial V(0,G_2)}{\partial G_2}\right|_{G_2=0}$$

Demanding the time-independence of the operator A, when the classical function $A(q,p,t)$ does not explicitly depend on time, we obtain $\partial_t V = 0$. Since from the integrals (7) only one time-independent integral $E = G_1^2 + G_2^2$ can be constructed, the solution (8) can be represented as

$$V = V(E),\quad V(0) = 1,\quad V^* = V\ .$$

The set of solutions such obtained, contain, for $V \equiv 1$, Weyl's, $V = \exp\{-H(\xi,\eta)/2\hbar\omega\}$-antinormal, $V = \exp\{H(\xi,\eta)/2\hbar\omega\}$-normal (4) correspondence rules respectively. We will now consider the question of including Kuryshkin's (5) rule of correspondence in this set of solutions.

Kuryshkin's rule is given by the function U, if it possesses the following structure

$$(9)\quad u(\xi,\eta) = \sum_k\int \phi_k^*(x-\xi/2)\phi_k(x+\xi/2)\exp(\tfrac{i}{\hbar}\eta x)\ dx = V(H(\xi,\eta))$$

where $\{\phi_k(\xi)\}$ is the set of functions, defining the rule of correspondence (see refs.(5-6) for details). Putting $\xi=0$ and $\eta=0$ separately in (9) and comparing the expressions obtained, we arrive at an integral identity which, in dimensionless variables $x' = (x/\alpha)$, where $\alpha = (\hbar/m\omega)^{\frac{1}{2}}$ and $\eta' = (m\omega\hbar)^{-\frac{1}{2}}\eta$, looks like

$$(10)\quad \sum_k\int\psi_k^*(x')\psi_k(x')\exp(i\eta'x')dx' = \sum_k\int\psi_k^*(x'-\eta'/2)\psi_k(x'+\eta'/2)dx'$$

where $\psi_k(x') = \phi_k(\alpha x')$.

Expressing the functions ψ_k^*, ψ_k in the left-hand side through their Fourier-transforms, we obtain

$$\sum_k\int\tilde\psi_k^*(p'+\eta')\tilde\psi_k(p')dp' = \sum_k\int\tilde\psi_k^*(x')\psi_k(x'+\eta')dx'\ .$$

According to (7) only the Hermite-functions can satisfy this identity and hence we can take

$$(11)\quad \phi_k(x) = A_k\ \exp(-m\omega x^2/2\hbar)\ H_k((m\omega/\hbar)^{-\frac{1}{2}}x)$$

where H_k is Hermite polynomial, A_k is some complex coefficient. Finally performing the integration in (9), taking into account (11) we obtain

$$(12)\quad u(\xi,\eta) = \sum_k|c_k|^2\ \exp\left\{-\frac{H(\xi,\eta)}{2\hbar\omega}\right\}L_k\left\{\frac{H(\xi,\eta)}{\hbar\omega}\right\}$$

Here L_k is Laguerre polynomial and c_k are coefficients related to A_k.

From (5) we have for them the condition

$$(13)\quad \sum_k|c_k|^2 = 1.$$

Thus the rule of correspondence (5) satisfies equation (6) or (1) under the assumptions made. Requirement of the fulfillment of principle of dynamical correspondence (1) (equation (1)) narrows the class of functions ϕ_k, leaving the incompleteness of Kuryshkin's rule, for the case of harmonic oscillator, in the choice of the coefficients c_k satisfying (13).

Operators of the principal physical quantities, characterizing the physical system considered, can be determined by the general rule

$$\hat A = \int A(q,p,t)\hat F(q,p,t)dqdp\ ,$$

taking into account the representation (3) of F and the kernel (12). From the papers (5-6) of Kuryshkin it is known that his rule of correspondence, in the case of harmonic oscillator, guarantees fulfillment of the ordinary quantum-mechanical relations between the average values of coordinate, momentum, their squares and energy. So here we cite only the expression for the constant denoting the identical shift of all the energy levels, keeping the difference between the levels same as given by generally accepted quantum mechanics and measured experimentally :

$$E_n = \hbar\omega(n+1/2) + \hbar\omega\varepsilon \quad , \varepsilon = 1/2 + \sum_k k|c_k|^2 .$$

Quasiprobability operator for the rule (5) is positive-definite and as such can be interpreted as the operator of joint probability-density of coordinate and momentum. The solution (12) obtained allows us to rewrite F in a form, where its positive-definiteness becomes obvious. With this aim, we shall use the formalism of annihilation and creation operators, introduced in the usual manner :

$$(14) \quad \begin{aligned} \hat{a} &= (m\omega/2\hbar)^{\frac{1}{2}}\hat{q} + i(2\hbar m\omega)^{-\frac{1}{2}}\hat{p}, \quad z = (m\omega/2\hbar)^{\frac{1}{2}}q + i(2\hbar m\omega)^{-\frac{1}{2}}p \\ \hat{a}^{+} &= (m\omega/2\hbar)^{\frac{1}{2}}\hat{q} - i(2\hbar m\omega)^{-\frac{1}{2}}\hat{p}, \quad z^* = (m\omega/2\hbar)^{\frac{1}{2}}q - i(2\hbar m\omega)^{-\frac{1}{2}}p \end{aligned}$$

Technicalities of manipulations in this formalism and certain important relations used by us can be found in the refs.(4,8). Let us write the quasiprobability operator with the kernel (12) in the form

$$(15) \quad \hat{F}(z,z^*) = \pi^{-2}\sum_k |c_k|^2 \int \langle k|\hat{D}(\alpha)|k\rangle \hat{D}(\alpha)\exp(\alpha^* z - \alpha z^*)d^2\alpha ,$$

where $|k\rangle$ is the eigenvector of the number operator $\hat{a}^+\hat{a}$, $\hat{D}(\alpha) = \exp(\alpha\hat{a}^+ - \alpha^*\hat{a})$ is the displacement operator (4), $d^2\alpha = d(\mathrm{Re}\,\alpha)\,d(\mathrm{Im}\,\alpha)$, α, α^* are linear combinations of ξ,η in just the similar manner as z,z^* are combinations of q,p in (14). Using the following easily verifiable property of the quasiprobability operator

$$\hat{D}(z)\hat{F}(0,0)\hat{D}^{-1}(z) = \hat{F}(z,z^*)$$

and the theorem stating that from the representation

$$\hat{G}(\hat{a},\hat{a}^+) = \int g(\alpha,\alpha^*)\hat{D}(\alpha)d^2\alpha ,$$

it follows that

$$g(\alpha,\alpha^*) = \pi^{-1}\,\mathrm{Tr}[\hat{G}(\hat{a},\hat{a}^+)\hat{D}^+(\alpha)]$$

we rewrite the operator (15) in the form

$$(16) \quad \hat{F}(z,z^*) = \pi^{-1}\sum_k \hat{D}(z)|k\rangle\langle k|\hat{D}^+(z) .$$

Positive-definiteness of the operator (16) is evident from its structure. We mention that the eigenstates of the operator (16) are the vectors $\hat{D}(z)|k\rangle$ and the corresponding eigenvalues are the numbers $\pi^{-1}|c_k|^2$.

Thus it is shown that the equation (1) for quasiprobability operator, being equivalent form of writing down the principle of dynamical correspondence (1) (weakened Dirac principle of correspondence of classical and quantum Poisson brackets), possesses solution. This fact is demonstrated in the example of harmonic oscillator. The solution obtained contains normal and antinormal rules, Weyl's and Kuryshkin's rule of correspondence.

Principle of dynamical correspondence limits the arbitrariness in Kuryshkin's rule, in the problem considered, upto a choice of the coefficients c_k. In other words, principle of dynamical correspondence (1) is a step forward towards the concretization of the rule (5-6).

Representation of the quasiprobability operator in the formalism of creation and annihilation operators (16) makes its positive-definiteness obvious and gives a new perspective of development of the rule (5-6) in this formalism.

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