Annales de la Fondation Louis de Broglie,
Vol. 9, n° 1, 1984

ADAPTIVE CONTROLLERS FOR A CLASS OF UNCERTAIN SYSTEMS

by M. CORLESS and G. LEITMANN
University of California
Berkeley, California, U.S.A.

*****

Séminaire de la Fondation L. de Broglie du 16 Mai 1983

*****

Abstract : *We consider a particular class of uncertain dynamical systems. Since some of the uncertain elements may result in unstable behavior, we seek controllers which guarantee that all possible responses of the system are uniformly bounded and approach a desired response. Toward that end, we present a class of adaptive controllers.*

Résumé : *Nous considérons une classe particulière de systèmes dynamiques incertains. Comme quelques uns des éléments incertains peuvent entraîner un comportement instable, nous cherchons des commandes qui garantissent que toutes les réponses possibles du système sont uniformément bornées et approchent une réponse désirée. A la fin de l'article, nous présentons une classe de commandes adaptatives.*

Based on research supported by the National Science Foundation under Grant ECS-8210324 and carried out while G. Leitmann was the recipient of a U.S. Senior Scientist Award of the Alexander von Humboldt Foundation.

1. Introduction

In order to control the behavior of a system in the "real" world, the system analyst seeks to capture the system's salient features in a mathematical model. This abstraction of the "real" system usually contains uncertain elements, for example, uncertainties due to parameters, constant or varying, which are unknown or imperfectly known, or uncertainties due to unknown or imperfectly known inputs into the system. Despite such imperfect knowledge about the chosen mathematical model, one often seeks to devise controllers which will "steer" the system in some desired fashion, for example, so that the system response will approach or track a desired reference response ; by suitable definition of the system (state) variables such a problem can usually be cast into that of stabilizing a prescribed state.

Two main avenues are open to the analyst seeking to control an uncertain dynamical system. He may choose a stochastic approach in which information about the uncertain elements as well as about the system response is statistical in nature ; for example, see Refs. 1 and 2. Loosely speaking, when modelling via random variables, one is content with desirable behavior on the average. The other approach to the control of uncertain systems, and the one for which we shall opt in the present discussion, is deterministic ; see Refs. 3-33. Available, or assumed, information about uncertain elements is deterministic in nature. Here one seeks controllers which assure the desired response of the dynamical system.

In this paper, the mathematical model is embodied in ordinary differential equations, the state equations of the system. For each of the systems under consideration there exists a state feedback control which assures that the zero state is globally uniformly asymptotically stable. However, these controls depend on constants in the system description which are not known ; for example, such constants are the values of unknown constant disturbances or unknown bounds on time-varying parameters or inputs. We propose controllers which may be regarded as adaptive versions of the feedback controls mentioned above ; in place of the unknown constants, one employs quantities which change or adapt as the state of

the system evolves. Under some circumstances, these adaptive quantities may be considered to be estimates of the unknown constants. The method of devising these adaptive controllers is based on the constructive use of Lyapunov theory as suggested, in a somewhat different context, in Refs. 9, 10, 11, 34, 35 and 36 ; see Ref. 21.

2. Systems Under Consideration

The class of systems under consideration here is a subclass of those considered in Refs. 20 and 21. The systems are described by

$$\dot{x}(t) = f(t,x(t)) + \Delta f(t,x(t)) + B(t,x(t))u(t) \qquad (1)$$

where $t \in R$, $x(t) \in R^n$ is the state, and $u(t) \in R^m$ is the control. The functions $f, \Delta f : R \times R^n \to R^n$ and $B : R \times R^n \to R^{n\times m}$ are unknown but assumed to satisfy certain conditions (Assumptions A1 - A3).

Roughly speaking, the problem treated here is that of specifying $u(t)$ so that all solutions of (1) are bounded and converge to the zero state.

We suppose that the following assumptions are satisfied :

Assumption A1. (i) The function f is Caratheodory (see Appendix) and $f(t,0) = 0$[1] for all $t \in R$.

(ii) There exist a C^1 function $V : R \times R^n \to R_+$ and continuous nondecreasing functions $\gamma_1, \gamma_2, \gamma_3 : R_+ \to R_+$ which satisfy

$$\gamma_i(0) = 0, \quad r > 0 \Rightarrow \gamma_i(r) > 0, \quad i = 1,2,3 \qquad (2)$$

$$\lim_{r\to\infty} \gamma_1(r) = \infty, \qquad (3)$$

[1] We use "0" to denote a zero vector.

such that for all $(t,x) \in R \times R^n$

$$\gamma_1(\|x\|) \leqslant V(t,x) \leqslant \gamma_2(\|x\|), \quad (4)$$

$$\frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x)f(t,x) \leqslant -\gamma_3(\|x\|). \quad (5)$$

Assumption A1 asserts that there exists a Lyapunov function V which guarantees that the zero state is a g.u.a.s. (globally uniformly asymptotically stable) equilibrium point of the system described by

$$\dot{x}(t) = f(t,x(t)) \;; \quad (6)$$

see Refs. 36, 37, and 38.

Assumption A2. There exists a positive-definite symmetric matrix $F \in R^{m\times m}$ and a strongly Caratheodory function $B^0 : R \times R^n \to R^{n\times m}$ such that (see Appendix)

$$B(t,x) = B^0(t,x)F \quad \forall \quad (t,x) \in R \times R^n \quad (7)$$

and, at each $t \in R$, $\alpha(t,x(t))$ is *known*, where[2]

$$\alpha(t,x) = B^0(t,x)^T \frac{\partial V}{\partial x}(t,x)^T . \quad (8)$$

One may consider the matrix B(t,x) as reflecting uncertainty in the manner in which the control enters the system description.

Assumption A3. (i) There are constants $d \in R^m$, $D \in R^{m\times p}$ and strongly Caratheodory functions $h : R \times R^n \to R^p$ and $e : R \times R^n \to R^m$ such that, for all $(t,x) \in R \times R^n$,

$$\Delta f(t,x) = B^0(t,x)[d + Dh(t,x) + e(t,x)] . \quad (9)$$

[2] Superscript T denotes transpose.

The function Δf represents the effects of potentially destabilizing uncertainties on the system. The next part of Assumption A3 concerns the amount of information available on Δf.

(ii) There exists a *known* strongly Caratheodory function $\pi : R \times R^n \times R^k_+ \to R_+$ such that for some $\beta \in R^k_+$ (possibly unknown),

$$\frac{\|e(t,x)\|}{\lambda_{min}(F)} \leqslant \pi(t,x,\beta) \quad (t,x) \in R \times R^n , \quad (10)$$

where $\lambda_{min}(F)$ denotes the smallest eigenvalue of F. Also, at each $t \in R$, $h(t,x(t))$ and $x(t)$ are *known*.

(iii) For each $(t,x) \in R \times R^n$, the function $\pi(t,x,.) : R^k_+ \to R_+$ is nondecreasing with respect to each component of its argument β, $-\pi(t,x,.)$ is convex, and $\partial\pi/\partial\beta$ exists and is strongly Caratheodory.

The following conditions, which are *not* assumptions, will affect the choice of some of the parameters in the proposed controllers.

Condition C1. For each $d \geqslant 0$, there exists $b_1(d) \geqslant 0$ such that, for all $(t,x) \in R \times R^n$,

$$\|x\| \leqslant d \Rightarrow \|\alpha(t,x)\| \leqslant b_1(d) . \quad (11)$$

Condition C2. For each $d \geqslant 0$, there exists $b_2(d) \geqslant 0$ such that, for all $(t,x) \in R \times R^n$,

$$\|x\| \leqslant d \Rightarrow \|h(t,x)\| \leqslant b_2(d) \quad (12)$$

Substituting (7) and (9) into (1) yields

$$\dot{x}(t) = f(t,x(t)) + B^{\circ}(t,x(t))[Fu(t) + d + Dh(t,x(t)) + e(t,x(t))]. \qquad (13)$$

Now, following Refs. 23 and 25 it may readily be shown that, if one could utilize the control given by

$$u(t) = v + Kh(t,x(t)) + p(t,x(t)) \qquad (14)$$

where

$$v = -F^{-1}d, \qquad K = -F^{-1}D \qquad (15)$$

and $p : R \times R^n \to R^m$ satisfies

$$\alpha(t,x) \neq 0 \Rightarrow p(t,x) = -\frac{\alpha(t,x)}{\|\alpha(t,x)\|}\pi(t,x,\beta) \qquad (16)$$

and assures existence of solutions to

$$\dot{x}(t) = f(t,x(t)) + B^{\circ}(t,x(t))[Fp(t,x(t)) + e(t,x(t))], \qquad (17)$$

then the resulting controlled system would be described by (17), which has the zero state as a g.u.a.s. equilibrium point and hence has the desired properties. However, utilization of the control that satisfies (14) - (16) requires knowledge of v, K, and β, which we have not assumed.

In the next section we present a class of controllers whose utilization requires only the assumed information and guarantees the desired performance.

3. Proposed Controllers

To assure that the systems presented in the previous section have the desired behavior we propose the utilization of one of the following controllers, each of which consists of three parts and is given by

$$u(t) = u^{(1)}(t) + u^{(2)}(t) + u^{(3)}(t). \qquad (18)$$

The first part of each controller is given by

$$u^{(1)}(t) = \hat{v}(t) - \bar{\ell}_1\ell_1\tilde{\alpha}(t), \qquad (19)$$

$$\dot{\hat{v}}(t) = -\ell_1\tilde{\alpha}(t), \qquad (20)$$

where $\hat{v}(t_0) \in R^m$ is arbitrary,

$$\ell_1 > 0, \quad \bar{\ell}_1 \geq 0, \qquad (21)$$

$\bar{\ell}_1 > 0$ if C1 is not satisfied, and

$$\tilde{\alpha}(t) = \alpha(t,x(t)). \qquad (22)$$

Note that $u^{(1)}(t)$ is also given by

$$u^{(1)}(t) = \hat{v}(t) + \bar{\ell}_1\dot{\hat{v}}(t) \qquad (23)$$

or

$$u^{(1)}(t) = -\bar{\ell}_1\ell_1\tilde{\alpha}(t) - \ell_1\int_{t_0}^{t}\tilde{\alpha}(\tau)d\tau + \hat{v}(t_0), \qquad (24)$$

and hence may be termed a PI (proportional plus integral) controller.

The second part of each controller is given by

$$u^{(2)}(t) = \hat{K}(t)h(t,x(t)) - \bar{\ell}_2\tilde{\alpha}(t)h(t,x(t))^T\Gamma h(t,x(t)), \qquad (25)$$

$$\dot{\hat{K}}(t) = -\tilde{\alpha}(t)h(t,x(t))^T\Gamma, \qquad (26)$$

where $\hat{K}(t) \in R^{m\times p}$, $\hat{K}(t_0)$ being arbitrary, $\Gamma \in R^{p\times p}$ is positive-definite and symmetric,

$$\bar{\ell}_2 \geq 0, \qquad (27)$$

and $\bar{\ell}_2 > 0$ if either C1 or C2 is not satisfied.

Note that $u^{(2)}(t)$ is also given by

$$u^{(2)}(t) = [\hat{K}(t) + \bar{\ell}_2\dot{\hat{K}}(t)]h(t,x(t)) \quad (28)$$

or

$$u^{(2)}(t) = [-\bar{\ell}_2\Omega(t) - \int_{t_0}^{t} \Omega(\tau)d\tau + \hat{K}(t_0)]h(t,x(t)) \quad (29)$$

where

$$\Omega(t) = \tilde{\alpha}(t)h(t,x(t))^T \Gamma. \quad (30)$$

The third part of each controller is given by

$$u^{(3)}(t) = -\pi(t,x(t),\hat{\beta}(t))\tilde{s}(t), \quad (31)$$

$$\dot{\hat{\beta}}(t) = L^{(3)}\frac{\partial\pi}{\partial\beta}(t,x(t),\hat{\beta}(t))^T\|\tilde{\alpha}(t)\|, \quad (32)$$

$$\dot{\varepsilon}(t) = -\ell_4\varepsilon(t), \quad (33)$$

where

$$\hat{\beta}_i(t_0) > 0, \qquad i = 1,2,\dots,k, \quad (34)$$

$$\varepsilon(t_0) > 0,$$

$L^{(3)} \in R^{k\times k}$ is diagonal with positive elements,

$$\ell_4 > 0, \quad (35)$$

and

$$\tilde{s}(t) = s(t,x(t),\hat{\beta}(t),\varepsilon(t)),$$

with $s : R \times R^n \times (0,\infty)^{k+1} \to R^m$ being any strongly Caratheodory function which assures that

$$\tilde{s}(t)\|\tilde{\alpha}(t)\| = \|\tilde{s}(t)\|\tilde{\alpha}(t), \quad (36)$$

so that $\tilde{s}(t)$ has the same direction as $\tilde{\alpha}(t)$, and

$$\|\tilde{\mu}(t)\| > \varepsilon(t) \Rightarrow \tilde{s}(t) = \frac{\tilde{\alpha}(t)}{\|\tilde{\alpha}(t)\|}, \quad (37)$$

where

$$\tilde{\mu}(t) = \pi(t,x(t),\hat{\beta}(t))\tilde{\alpha}(t). \quad (38)$$

A particular example of such a function $\tilde{s}$ is given by

$$\tilde{s}(t) = \text{sat}(\tilde{\mu}(t)/\varepsilon(t)) \quad (39)$$

where

$$\text{sat}(\eta) = \begin{cases} \eta, & \|\eta\| \leq 1 \\ \dfrac{\eta}{\|\eta\|}, & \|\eta\| > 1 ; \end{cases} \quad (40)$$

see Ref. 21.

4. Properties of Systems with Proposed Controllers

Before stating a theorem, let us consider any system described by equation (1), satisfying Assumptions A1, A2, A3, and subject to any corresponding controller given in Section 3. By defining the parameter "estimate" at t

$$\hat{q}(t) = (\hat{v}^T(t),\ \hat{k}_1(t),\ \hat{k}_2(t),\dots,\ \hat{k}_m(t),\ \hat{\beta}^T(t),\ \varepsilon(t))^T$$

where $\hat{k}_i(t)$, $i = 1,2,\dots, m$, are the rows of $\hat{K}(t)$, and by appropriately defining (see Ref. 21) $\bar{f}^{(1)} : R \times R^n \times Q \to R^n$ and $\bar{f}^{(2)} : R \times R^n \times Q \to R^r$, where

$$Q = R^m \times R^{mp} \times (0,\infty)^{k+1},$$

$$r = (m + 1)p + k + 1,$$

such a controlled system can be described by

$$\dot{x}(t) = \bar{f}^{(1)}(t,x(t),\hat{q}(t))$$
$$\dot{\hat{q}}(t) = \bar{f}^{(2)}(t,x(t),\hat{q}(t)). \quad (41)$$

This is a system whose state space is $R^n \times Q$.

By a solution of (41) we shall mean an absolutely continuous function $(x(.),\hat{q}(.)) : [t_0,t_1) \to R^n \times Q$, where $t_1 \in (t_0,\infty]$, which satisfies (41) for all $t \in [t_0,\infty)$ except possibly on a set of Lebesgue measure zero.

Defining the parameter vector

$$q = (v^T, k_1, k_2, \ldots, k_m, \beta^T, 0)^T,$$

where k_i, $i = 1,2,\ldots,m$ are the rows of K, we are now ready to state a theorem.

Theorem 4.1. Consider any system described by (1), satisfying Assumptions A1, A2, A3, and subject to any corresponding controller given in Section 3. Such a controlled system can be described by (41) and has the following properties:

P1. Existence of Solutions. For each $(t_0, x_0, \hat{q}_0) \in R \times R^n \times Q$ there exists a solution $(x(.),\hat{q}(.)) : [t_0,t_1) \to R^n \times Q$ of (41) with $(x(t_0),\hat{q}(t_0)) = (x_0,\hat{q}_0)$.

P2. Uniform Stability of $(0,q)$. For each $\eta > 0$, there exists $\delta > 0$ such that, if $(x(.),\hat{q}(.))$ is any solution of (41) with $\|x(t_0)\|$, $\|\hat{q}(t_0) - q\| < \delta$, then $\|x(t)\|$, $\|\hat{q}(t) - q\| < \eta$ for all $t \in [t_0,t_1)$.

P3. Uniform Boundedness of Solutions. For each $r_1,r_2 > 0$ there exist $d_1(r_1,r_2)$, $d_2(r_1,r_2) \geq 0$ such that, if $(x(.),\hat{q}(.))$ is any solution of (41) with $\|x(t_0)\| \leq r_1$ and $\|\hat{q}(t_0) - q\| \leq r_2$, then $\|x(t)\| \leq d_1(r_1,r_2)$ and $\|\hat{q}(t) - q\| \leq d_2(r_1,r_2)$ for all $t \in [t_0,t_1)$.

P4. Extension of Solutions. Every solution of (41) can be extended into a solution defined on $[t_0,\infty)$.

P5. Convergence of $x(.)$ to Zero. If $(x(.),\hat{q}(.)): [t_0,\infty) \to R^n \times Q$ is a solution of (41) then

$$\lim_{t\to\infty} x(t) = 0.$$

Proof. Substituting equation (18) into (13), the controlled system under consideration can be described by

$$\begin{aligned}\dot{x}(t) = f(t,x(t)) &+ B^0(t,x(t))[Fu^{(1)}(t) + d] \\ &+ B^0(t,x(t))[Fu^{(2)}(t) + Dh(t,x(t))] \\ &+ B^0(t,x(t))[Fu^{(3)}(t) + e(t,x(t))]. \quad (42)\end{aligned}$$

It may readily be shown that any system described by (42) and satisfying Assumptions A1, A2, A3, is a class 4 system as defined in Ref. 21. Also $u^{(1)}$, $u^{(2)}$, $u^{(3)}$ as defined in Section 3 yield a class 4 controller (as defined in Ref. 21) for the system described by (42). Hence, by Theorem 4.1 of Ref. 21, the controlled system under consideration has properties P1 - P5.

5. An Application

Consider an uncertain system described by

$$\dot{z}(t) = A(t,z(t))z(t) + w(t,z(t)) + Bu(t) \qquad (43)$$

where $t \in R$, $z(t) \in R^n$, and $u(t) \in R^m$. The functions $A : R \times R^n \to R^{n\times n}$, $w : R \times R^n \to R^n$ and the matrix $B \in R^{n\times m}$ are unknown but assumed to satisfy certain conditions ((i) - (iv)).

Given a "desirable" state response $z^*(.)$ which satisfies

$$\dot{z}^*(t) = A^*z^*(t) + B^*u^*(t), \qquad (44)$$

where $z^*(t) \in R^n$, $u^*(t) \in R^{m^*}$, $A^* \in R^{n\times n}$, and $B^* \in R^{n\times m^*}$, we wish to obtain a controller which guarantees that

$$\lim_{t\to\infty} [z(t) - z^*(t)] = 0 . \qquad (45)$$

It is assumed that the following conditions are

satisfied :

(i) For some positive-definite symmetric $F \in R^{m\times m}$ and some *known* $B^\circ \in R^{n\times m}$

$$B = B^\circ F . \qquad (46)$$

(ii) The pair (A^*, B°) is stabilizable.

(iii) There exists a constant $d \in R^m$ and a Caratheodory function $\tilde{d} : R \times R^n \to R^m$ such that

$$w(t,z) = B^\circ[d + \tilde{d}(t,z)] \quad \forall \quad (t,z) \in R \times R^n \qquad (47)$$

where

$$\|\tilde{d}(t,z)\| \leqslant c_1 \quad \forall \quad (t,z) \in R \times R^n \qquad (48)$$

for some $c_1 \geqslant 0$.

(iv) There exists a constant $H \in R^{m\times n}$ and a Caratheodory function $\tilde{H} : R \times R^n \to R^{m\times n}$ such that

$$A(t,z) = A^* + B^\circ[H + \tilde{H}(t,z)] \quad \forall \quad (t,z) \in R \times R^n \qquad (49)$$

where

$$\|\tilde{H}(t,z)\| \leqslant c_2 \quad \forall \quad (t,z) \in R \times R^n \qquad (50)$$

for some $c_2 \geqslant 0$.

(v) For some $M \in R^{m\times m}$

$$B^* = B^\circ M . \qquad (51)$$

(vi) The function $u^*(.)$ is Lebesgue measurable and bounded on bounded intervals.

(vii) The matrices A^* and B^* are *known* and for each t, $u^*(t)$, $z^*(t)$, and $z(t)$ are *known*.

To obtain a problem formulation in the form considered in Section 2, we let

$$x(t) = z(t) - z^*(t) ; \qquad (52)$$

hence

$$z(t) = z^*(t) + x(t) , \qquad (53)$$

and, utilizing (52), (43), and (44),

$$\dot{x}(t) = A^*x(t) + [A(t,z(t)) - A^*]z(t) + w(t,z(t)) - B^*u^*(t) + Bu(t). \qquad (54)$$

The requirement that (45) be satisfied is now equivalent to requiring that $x(.)$ converges to the zero vector.

Since the pair (A^*, B°) is known and stabilizable, there exists a known matrix $L \in R^{m\times n}$ such that, the matrix $\bar{A}$, given by

$$\bar{A} = A^* + B^\circ L \qquad (55)$$

is strictly stable. Thus, utilizing (54) and (55),

$$\dot{x}(t) = \bar{A}x(t) + [A(t,z(t)) - A^*]z(t) + w(t,z(t)) - B^\circ Lx(t) - B^*u^*(t) + Bu(t) . \qquad (56)$$

Equation (56) is in the form of equation (1) with

$$f(t,x) = \bar{A}x , \qquad (57)$$

$$\Delta f(t,x) = [A(t,z^*(t) + x) - A^*](z^*(t) + x) + w(t,z^*(t) + x) - B^\circ Lx - B^*u^*(t) , \qquad (58)$$

$$B(t,x) = B . \qquad (59)$$

We now check to see if Assumptions A1, A2, and A3 are satisfied.

As a consequence of (57), A1(i) is satisfied. Since $\bar{A}$ is strictly stable, A1(ii) is satisfied by taking any positive-definite symmetric $Q \in R^{n\times n}$ and letting

$$V(t,x) = 1/2\, x^T P x \quad \forall\ (t,x) \in R \times R^n \tag{60}$$

where $P \in R^{n\times n}$ is the unique positive-definite symmetric solution of

$$P\bar{A} + \bar{A}^T P + Q = 0 ; \tag{61}$$

see Refs. 36, 37, and 38.

Taking any pair, B^0 and F, which satisfy the requirements of condition (i), letting

$$B^0(t,x) = B^0 \quad \forall\ (t,x) \in R \times R^n , \tag{62}$$

and utilizing (59), (46), (62), (8), and (60), one obtains for all $(t,x) \in R \times R^n$,

$$B(t,x) = B^0(t,x)F , \tag{63}$$

$$\alpha(t,x) = B^{0T} P x . \tag{64}$$

Since F is symmetric and positive-definite, B^0 and P are known, and x(t) is known at each t, it follows from (63), (62), and (64) that Assumption A2 is satisfied.

In addition, it follows from (64) that condition C1 is satisfied.

Substituting (47), (49), and (51) into (58) yields

$$\Delta f(t,x) = B^0[d + H(z^*(t) + x) - Lx - Mu^*(t) + \tilde{d}(t,z^*(t) + x) + \tilde{H}(t,z^*(t) + x)(z^*(t)+x)]; \tag{65}$$

hence, letting,

$$D = [H \quad -I_m]^{3}, \tag{66}$$

$$h(t,x) = [(z^*(t) + x)^T, (Lx + Mu^*(t))^T]^T , \tag{67}$$

$$e(t,x) = \tilde{d}(t,z^*(t) + x) + \tilde{H}(t,z^*(t) + x)(z^*(t) + x), \tag{68}$$

$\Delta f(t,x)$ may be written as

$$\Delta f(t,x) = B^0(t,x)[d + Dh(t,x) + e(t,x)]$$

and part (i) of Assumption A3 is satisfied.

Utilizing (68), (48), and (50), one obtains for all $(t,x) \in R \times R^n$

$$\begin{aligned} \| e(t,x)\| &= \| \tilde{d}(t,z^*(t) + x) + \tilde{H}(t,z^*(t) + x)(z^*(t) + x)\| \\ &\leq \| \tilde{d}(t,z^*(t) + x)\| + \| \tilde{H}(t,z^*(t) + x)\| \| z^*(t) + x\| \\ &\leq c_1 + c_2 \| z^*(t) + x\| ; \end{aligned} \tag{69}$$

thus,

$$\frac{\| e(t,x) \|}{\lambda_{min}(F)} \leq \pi(t,x,\beta) \quad \forall\ (t,x) \in R \times R^n \tag{70}$$

where $\pi : R \times R^n \times R_+^2 \to R_+$ is given by

$$\pi(t,x,\beta) = \beta_1 + \beta_2 \| z^*(t) + x\| , \tag{71}$$

$$(\beta_1,\beta_2) = \beta^T ,$$

and

$$\beta_1 = \frac{c_1}{\lambda_{min}(F)} , \quad \beta_2 = \frac{c_2}{\lambda_{min}(F)} . \tag{72}$$

[3] I_m denotes the identity matrix in $R^{m\times m}$

As a consequence of (70), (71), (67), and condition (vii), part (ii) of Assumption A3 is satisfied.

Part (iii) of Assumption A3 follows from (71) and the convexity of linear functions.

Note that, if $u^*(.)$ is bounded and A^* is strictly stable, then $z^*(.)$ is bounded, and, utilizing (67), condition C2 is satisfied.

Thus, the error system as described by (54) belongs to the class of systems considered in Section 2. In view of this, we propose the utilization of one of the controllers presented in Section 3. Hence, we let

$$u(t) = u^{(1)}(t) + u^{(2)}(t) + u^{(3)}(t), \tag{73}$$

where $u^{(1)}(t)$ is given by (19), (20), and (21) with

$$\tilde{\alpha}(t) = B^{oT}Px(t) \;; \tag{74}$$

$u^{(2)}(t)$ is given by (25), (26), (27), and (67) with $\bar{\ell}_2 > 0$ if either $u^*(.)$ is unbounded or A^* is not strictly stable, and, letting

$$\begin{bmatrix} \ell_1^{(3)} & 0 \\ 0 & \ell_2^{(3)} \end{bmatrix} = L^{(3)}, \tag{75}$$

$u^{(3)}(t)$ is given by

$$u^{(3)}(t) = -[\hat{\beta}_1(t) + \hat{\beta}_2(t)\|z(t)\|]\tilde{s}(t), \tag{76}$$

$$\dot{\hat{\beta}}_1(t) = \ell_1^{(3)}\|\tilde{\alpha}(t)\|, \quad \hat{\beta}_1(t_0) > 0, \tag{77}$$

$$\dot{\hat{\beta}}_2(t) = \ell_2^{(3)}\|z(t)\|\|\tilde{\alpha}(t)\|, \quad \hat{\beta}_2(t_0) > 0, \tag{78}$$

$$\dot{\varepsilon}(t) = -\ell_4\varepsilon(t), \quad \varepsilon(t_0) > 0\;, \tag{79}$$

where

$$\tilde{s}(t) = s(t,x(t),\hat{\beta}(t),\varepsilon(t)),$$

with $s : R \times R^n \times (0,\infty)^3 \to R^m$ being any strongly Caratheodory function which assures the satisfaction of (36) and (37) with

$$\tilde{\mu}(t) = [\hat{\beta}_1(t) + \hat{\beta}_2(t)\|z(t)\|]\tilde{\alpha}(t), \tag{80}$$

and

$$\ell_1^{(3)}, \ell_2^{(3)}, \ell_4 > 0\;. \tag{81}$$

As a consequence of Theorem 4.1, utilization of a controller as specified above assures that $x(.)$ is bounded and $\lim_{t\to\infty} x(t) = 0$.

<u>Remark</u>. The problem, in which one requires that

$$\lim_{t\to\infty} z(t) = 0\;, \tag{82}$$

is a particular case of the type of problem treated above. In this case

$$z^*(t) \equiv 0\;, \tag{83}$$

and hence, $z^*(.)$ satisfies (44) for $u^*(t) \equiv 0$ and *any* $A^* \in R^{n\times n}$. Also, since $\Delta f(t,x)$ can be written as

$$\Delta f(t,x) = B^o[d + (H-L)x + \tilde{d}(t,x) + \tilde{H}(t,x)x]\;, \tag{84}$$

one may let

$$D = H - L\;, \tag{85}$$

$$h(t,x) = x \tag{86}$$

and utilize (86), rather than (67), in the controller construction.

<u>Simulated Example</u>. For numerical simulation, we have taken a system described by

$$\dot{z}_1(t) = z_2(t) ,$$

$$\dot{z}_2(t) = -\sin z_1(t) - 1 - \cos t + 0.5\, u(t) .$$

The desirable state response was taken to satisfy

$$\dot{z}^*_1(t) = z^*_2(t) ,$$

$$\dot{z}^*_2(t) = -z^*_1(t) - 2\, z^*_2(t) + u^*(t) .$$

Letting L = 0 and

$$Q = \begin{bmatrix} 2 & 2 \\ 2 & 6 \end{bmatrix} ,$$

P was given by

$$P = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} .$$

The function $\tilde{s}$ was taken to be given by (39), (40), and (80), and the rest of the controller parameters were taken to be :

$$\ell_1 = 10 , \quad \bar{\ell}_1 = 0 ,$$

$$\Gamma = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix} , \quad \bar{\ell}_2 = 0 ,$$

$$\ell^{(3)}_1, \ell^{(3)}_2 = 5 , \quad \ell_4 = 0.05 .$$

The results of simulations with $u^*(t)$ given by

$$u^*(t) = \begin{cases} 1 , & 10n \leqslant t < 10n + 5 , \\ -1 , & 10n + 5 \leqslant t < 10n + 10 , \end{cases} \quad n \text{ an integer},$$

and with initial conditions

$$z_1(0),\ z_2(0),\ z^*_1(0),\ z^*_2(0) = 0 ,$$

$$\hat{v}(0) = 0 , \quad \hat{K}(0) = 0 ,$$

$$\hat{\beta}_1(0),\ \hat{\beta}_2(0),\ \epsilon(0) = 0.01 ,$$

are presented graphically in Figs. 1-6. The desired state response is presented in Fig. 1 and the difference between the system response and the desired response is presented in Fig. 2. Note that the discontinuities in the control function seen in Fig. 3 are due to the $\hat{k}_3u^*$ term in the $u^{(2)}$ portion of the control where

$$(\hat{k}_1, \hat{k}_2, \hat{k}_3) = \hat{K} ;$$

this term is discontinuous because u^* is a square-wave. As seen in Figs. 4, 5, and 6, the various adaptive parameters seem to tend toward constants.

6. Appendix

In what follows, D is a subset of R^p.

(i) Caratheodory Function. A function $f : R \times D \to R^q$ is Caratheodory iff: for each $t \in R$, $f(t,.)$ is continuous ; for each $z \in D$, $f(.,z)$ is Lebesgue measurable ; and, for each compact subset C of $R \times D$, there exists a Lebesgue integrable function $m_C(.)$ such that, for all $(t,z) \in C$,

$$\|f(t,z)\| \leqslant m_C(t) .$$

(ii) Strongly Caratheodory Function. A function $f : R \times D \to R^q$ is strongly Caratheodory iff it satisfies (i) with $m_C(.)$ replaced by a constant m_C.

REFERENCES

1. Kushner, H.J., On the Status of Optimal Control and Stability for Stochastic Systems, IEEE International Convention Rec., Vol. 14, pp. 143-151, 1966.
2. Astrom, K.J., Introduction to Stochastic Control Theory, Academic Press, New York, New York, 1970.
3. Johnson, C.D., Optimal Control of the Linear Regulator with Constant Disturbances, IEEE Transactions on Automatic Control, Vol. AC-13, pp. 416-421, 1968.
4. Sobral, M., Jr., and Stefanek, R.G., Optimal Control of the Linear Regulator Subject to Disturbances, IEEE Transactions on Automatic Control, Vol. AC-15, pp. 498-500, 1970.
5. Davison, E.J., and Smith, H.W., Pole Assignment in Linear Time-Invariant Systems with Constant Disturbances, Automatica, Vol. 7, pp. 489-498, 1971.
6. Davison, E.J., and Goldenberg, A., Robust Control of a General Servomechanism Problem : The Servo Compensator, Automatica, Vol. 11, pp. 461-471, 1975.
7. Desoer, C.A., and Wang, Y.T., On the Minimum Order of a Robust Servocompensator, IEEE Transactions on Automatic Control, Vol. AC-23, pp. 70-73, 1978.
8. Porter, B., and Grujic, Lj.T., Continuous-Time Tracking Systems Incorporating Lur'e Plants with Multiple Nonlinearities, International Journal of Systems Science, Vol. 11, pp. 837-840, 1980.
9. Rang, E.R., Adaptive Controllers Derived by Stability Considerations, Minneapolis-Honeywell Regulator Company, Memorandum N°. MR-7905, 1962.
10. Butchart, R.L., and Shakcloth, B., Synthesis of Model Reference Adaptive Control by Liapunov's Second Method, Proceedings of the 2nd IFAC Symposium on Theory of Self-Adaptive Control Systems, Teddington, England, Plenum Press, pp. 145-152, 1966.
11. Parks, P.C., Liapunov Redesign of Model Reference Adaptive Control Systems, IEEE Transactions on Automatic Control, Vol. 11, pp. 362-367, 1966.

12. Winsor, C.A., and Roy, R.J., Design of Model Reference Adaptive Control Systems by Liapunov's Second Method, IEEE Transactions on Automatic Control, Vol. AC-13, p. 204, 1968.
13. Lindorff, D.P., and Carroll, R.L., Survey of Adaptive Control using Liapunov Design, International Journal of Control, Vol. 18, pp. 897-914, 1973.
14. Landau, I.D., A Survey of Model Reference Adaptive Techniques - Theory and Applications, Automatica, Vol. 10, pp. 353-379, 1974.
15. Monopoli, R.V., Model Reference Adaptive Control with an Augmented Error Signal, IEEE Transations on Automatic Control, Vol. AC-19, pp. 474-484, 1974.
16. Narendra, K.S., and Valavani, L.S., Stable Adaptive Controller Design - Direct Control, IEEE Transactions on Automatic Control, Vol. AC-23, pp. 570-583, 1978.
17. Narendra, K.S., Lin, Y.H., and Valavani, L.S., Stable Adaptive Controller Design, Part II : Proof of Stability, IEEE Transactions on Automatic Control, Vol. AC-25, pp. 440-448, 1980.
18. Morse, A.S., Global Stability of Parameter-Adaptive Control Systems, IEEE Transactions on Automatic Control, Vol. AC-25, pp. 433-439, 1981.
19. Parks, P.C., Stability and Convergence of Adaptive Controllers - Continuous Systems, IEEE Proceedings, Vol. 128, Part D, pp. 195-200, 1981.
20. Corless, M., and Leitmann, G., Adaptive Control of Systems Containing Uncertain Functions and Unknown Functions with Uncertain Bounds, Journal of Optimization Theory and Applications, Vol. 41, n° 1, pp. 155-168, 1983.
21. Corless, M., and Leitmann, G., Adaptive Control for Uncertain Dynamical Systems, Mathematical Theory of Dynamical Systems and Microphysics, Edited by A. Blaquière and G. Leitmann, Academic Press, New York, New York, 1984.

22. Monopoli, R.V., Engineering Aspects of Control System Design via the Direct Method of Liapunov, NASA, Report N° CR-654, 1966.

23. Gutman, S., and Leitmann, G., Stabilizing Feedback Control for Dynamical Systems with Bounded Uncertainty, Proceedings of the IEEE Conference on Decision and Control, Gainesville, Florida, 1976.

24. Gutman, S., and Leitmann, G., Stabilizing Control for Linear Systems with Bounded Parameter and Input Uncertainty, Proceedings of the 7th IFIP Conference on Optimization Techniques, Nice, France, Springer-Verlag, Berlin, Germany, 1976.

25. Gutman, S., Uncertain Dynamical Systems - Lyapunov Min-Max Approach, IEEE Transactions on Automatic Control, Vol. 24, pp. 437-443, 1979.

26. Leitmann, G., On the Efficacy of Nonlinear Control in Uncertain Linear Systems, Journal of Dynamic Systems, Measurement, and Control, Vol. 103, pp. 95-102, 1981.

27. Corless, M., and Leitmann, G., Continuous State Feedback Guaranteeing Uniform Ultimate Boundedness for Uncertain Dynamic Systems, IEEE Transactions on Automatic Control, Vol. 26, pp. 1134-1144, 1981.

28. Leitmann, G., Deterministic Control of Uncertain Systems, Acta Astronautica, Vol. 7, pp. 1457-1461, 1980.

29. Thorp, J.S., and Barmish, B.R., On Guaranteed Stability of Uncertain Systems via Linear Control, Journal of Optimization Theory and Applications, Vol. 35, pp. 559-579, 1981.

30. Barmish, B.R., Corless, M., and Leitmann, G., A New Class of Stabilizing Controllers for Uncertain Dynamical Systems, SIAM Journal of Control and Optimization, Vol. 21, pp. 246-255, 1983.

31. Molander, P., Stabilization of Uncertain Systems, Lund Institute of Technology, Report N°. LUTFD2/(TFRT-1020) /1-111, 1979.

32. Barmish, B.R., Petersen, I.R., and Feuer, A., Linear Ultimate Boundedness Control of Uncertain Dynamical Systems, submitted for publication.

33. Ryan, E.P., A Variable Structure Approach to Feedback Regulation of Uncertain Dynamical Systems, International Journal of Control, Vol. 38, to appear, 1984.

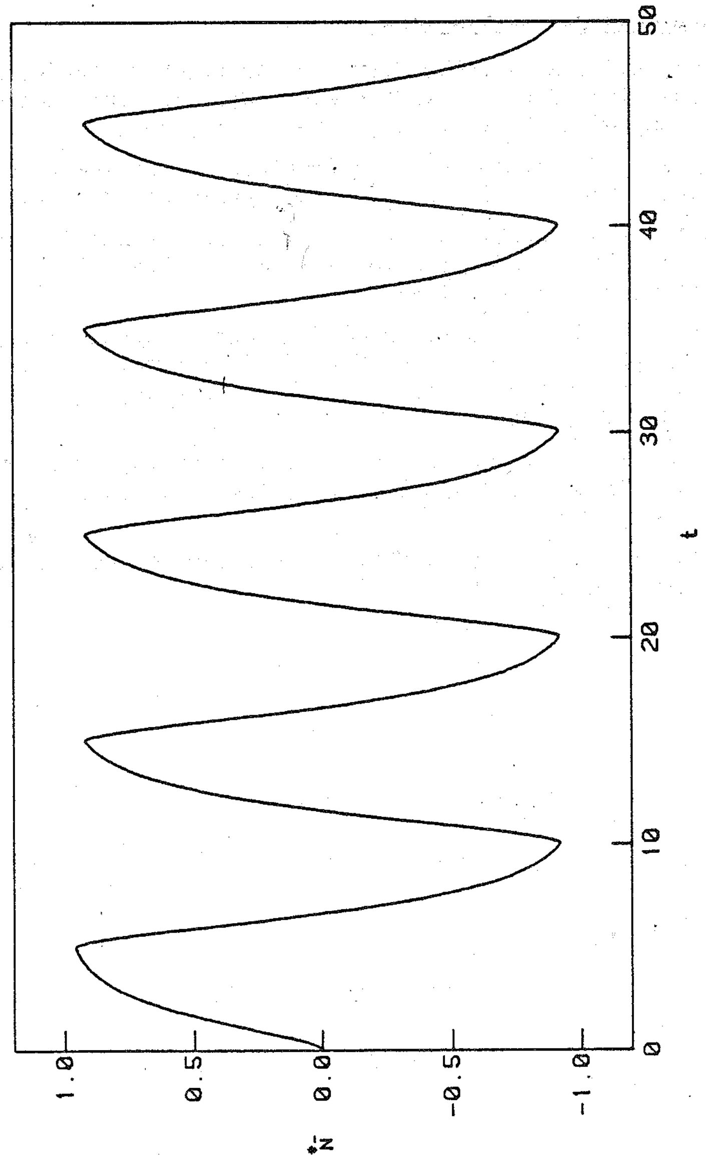
34. Lur'e, A.M., Some Nonlinear Problems in the Theory of Automatic Control (in Russian), Gostekhizdat, Moscow, 1951.

35. Letov, A.M., Stability of Nonlinear Regulating Systems (in Russian), Izdatel'stvo Technichesko-Teoretischeskoi Literatury, Moscow, 1955.

36. Kalman, R.E., and Bertram, J.E., Control System Analysis and Design via the Second Method of Lyapunov, I : Continuous-Time Systems, Journal of Basic Engineering, Vol. 82, pp. 371-393, 1960.

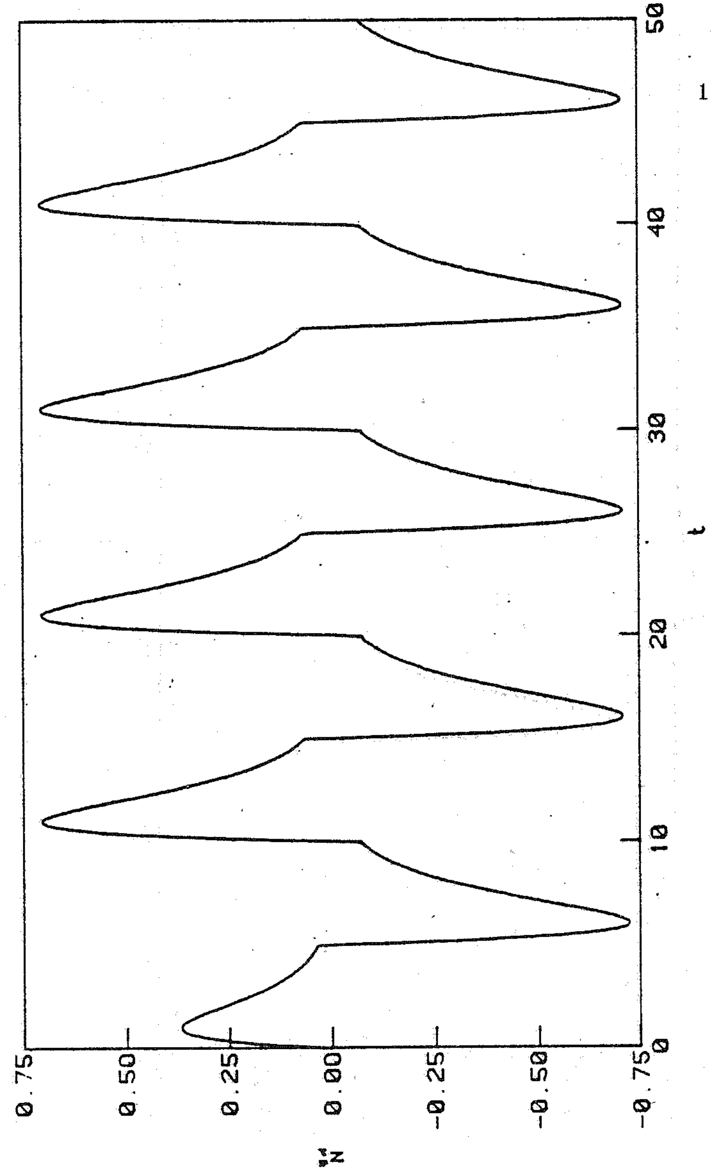
37. La Salle, J.P., and Lefschetz, S., Stability by Liapunov's Direct Method with Applications, Academic Press, New York, New York, 1961.

38. Hahn, W., Stability of Motion, Springer-Verlag, Berlin, Germany, 1967.



1(i)

Figure 1. Desired state response.



1(ii)

Figure 1. Desired state response.

2(i)

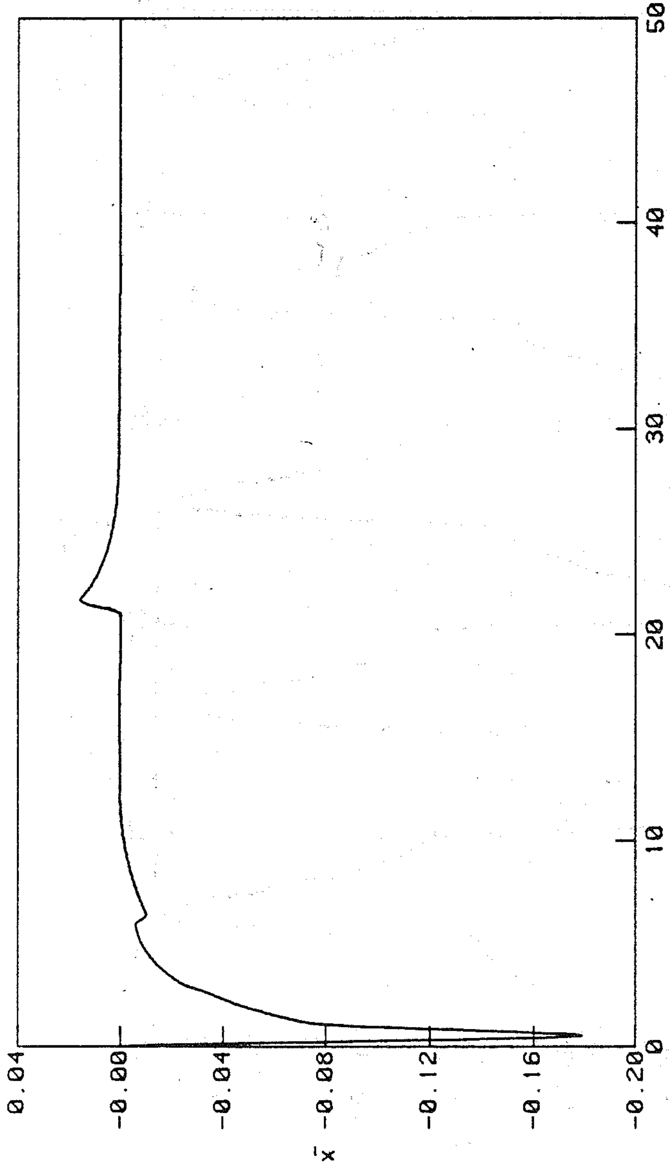


Figure 2. Difference between system and desired responses.

2(ii)

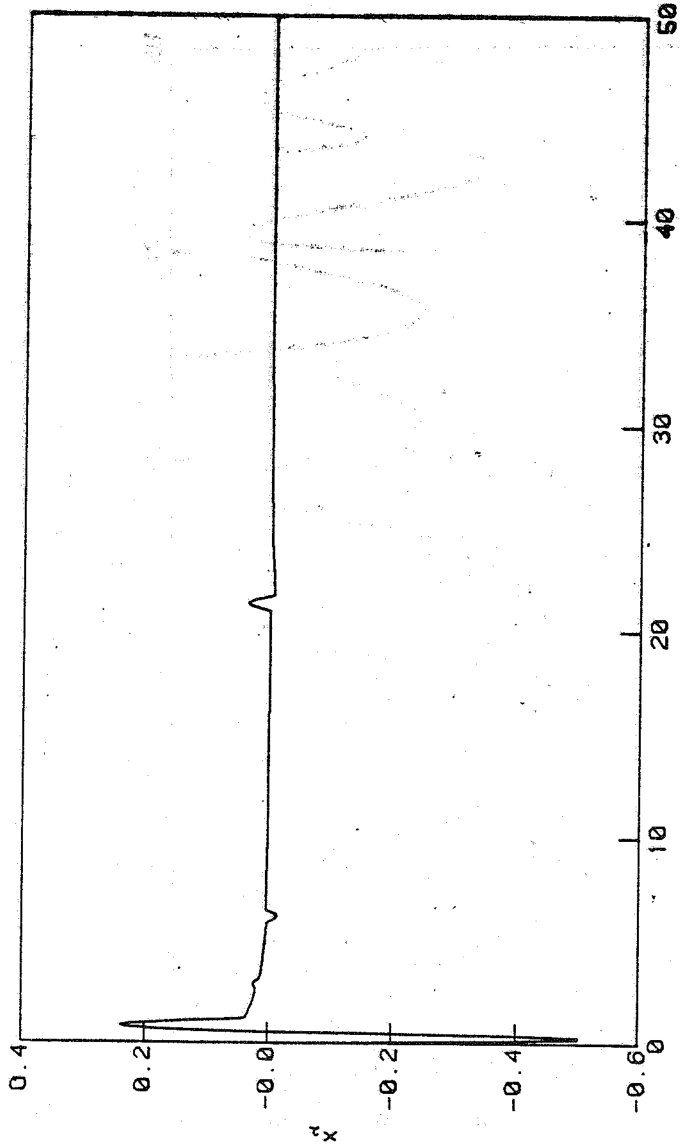


Figure 2. Difference between system and desired responses.

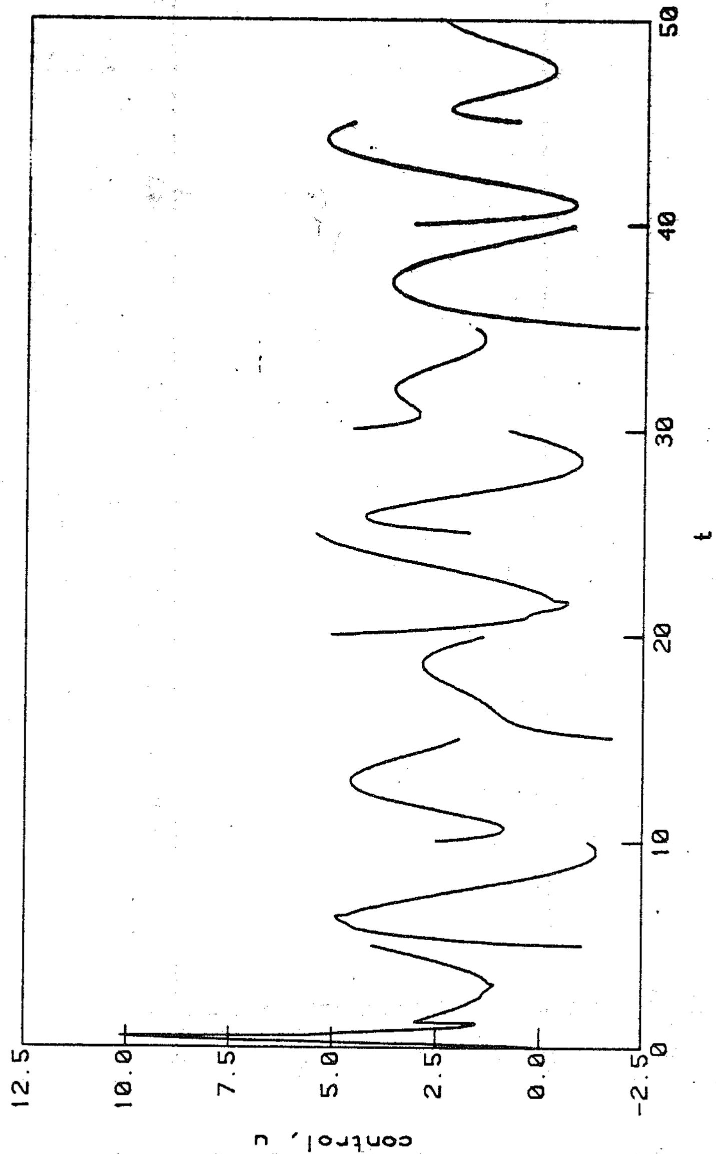


Figure 3. Control.

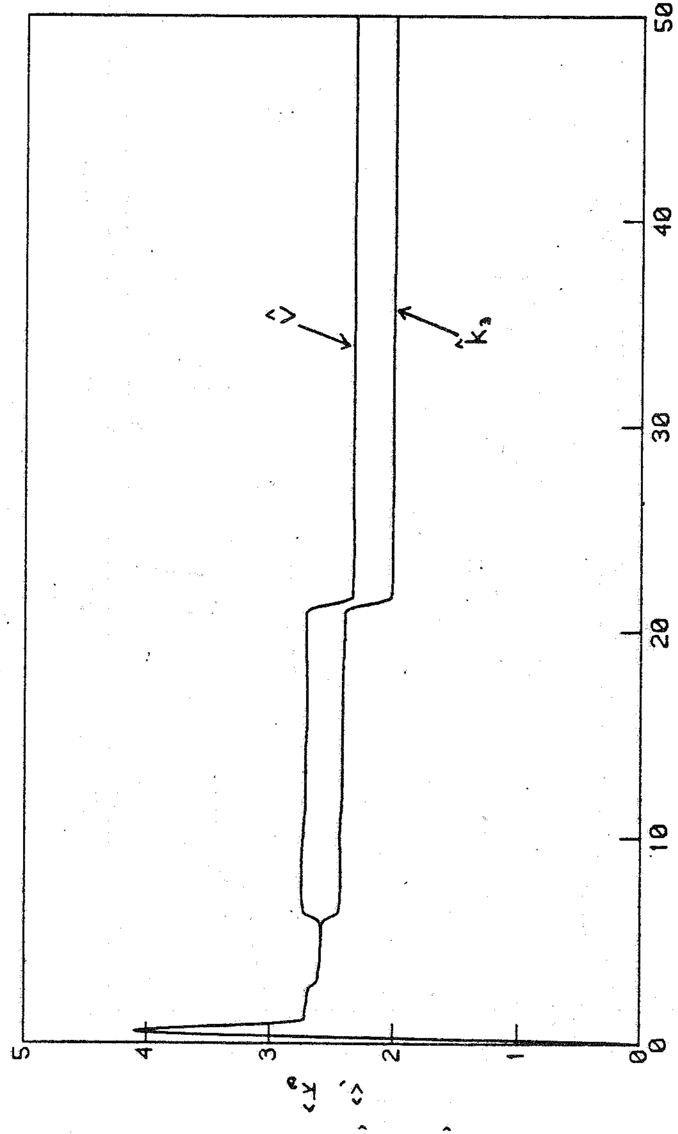


Figure 4. Adaptive parameters $\hat{v}$ and $\hat{k}_3$.

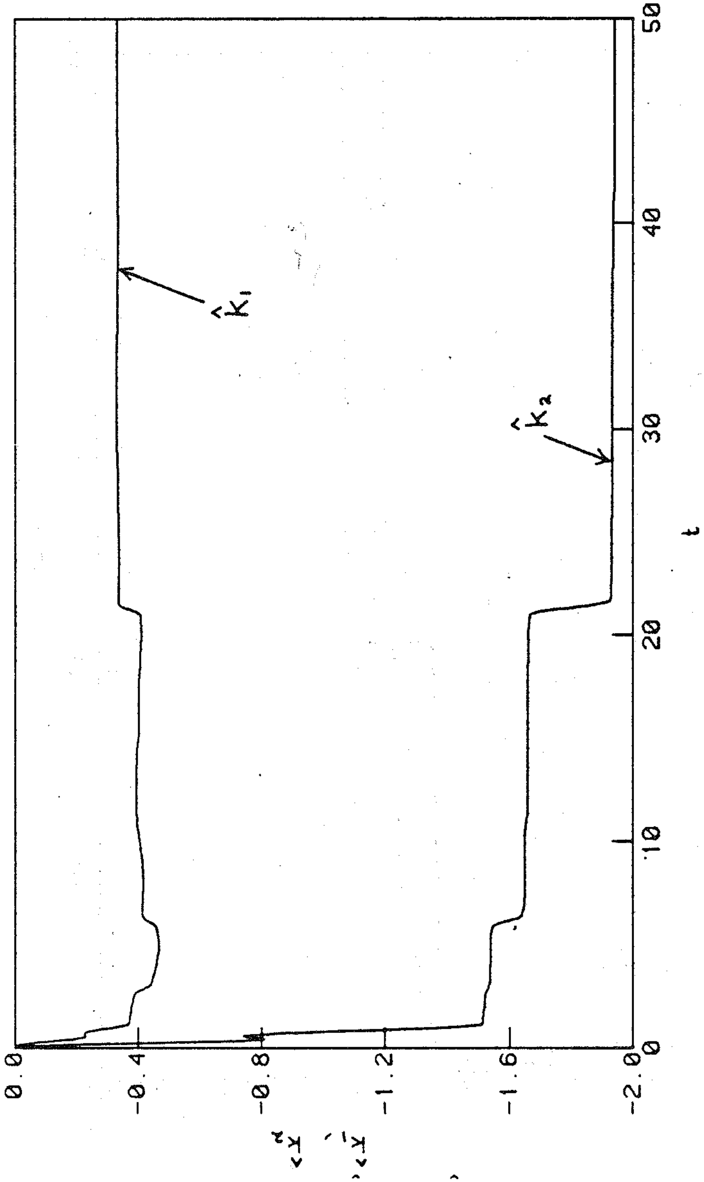


Figure 5. Adaptive parameters $\hat{k}_1$ and $\hat{k}_2$.

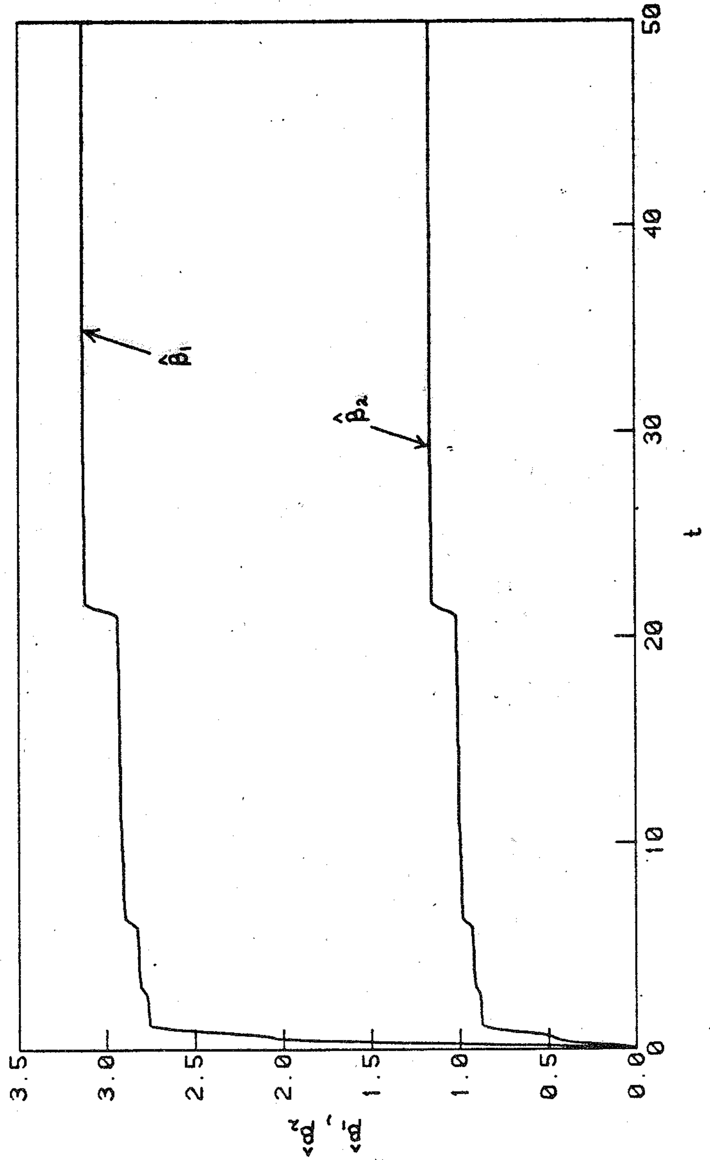


Figure 6. Adaptive parameters β_1 and β_2.