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ON THE EQUATION OF THE UNITARY QUANTUM THEORY

by Victor A. BOICHENKO and Lev G. SAPOGIN

Institute of Control Sciences,
Academy of Sciences of the USSR,
65 Profsoyuznaya St., Moscow, USSR

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Abstract : *The properties of the equation of the unitary quantum theory are studied. The relativistic invariance of the equation is proved and transformation laws of the wave function and its bilinear combinations are derived.*

Résumé : *On étudie les propriétés de l'équation de la théorie quantique unitaire. On démontre l'invariance relativiste de l'équation et on déduit les lois de transformation de la fonction d'onde et de ses combinaisons bilinéaires.*

References [1-3] have suggested a model of the unitary field theory where a particle of mass m is described by the equation

$$i\lambda^{\mu} \frac{\partial \Phi}{\partial x^{\mu}} - m\Phi = 0 \qquad (1)$$

and each component of the wave function satisfies the second order equation

$$u^{\mu}u^{\nu} \frac{\partial^2 \Phi}{\partial x^{\mu} \partial x^{\nu}} + m^2\Phi = 0 \qquad (2)$$

so that the commutation relations for matrices λ^{μ} have the form

$$\lambda^{\mu}\lambda^{\nu} + \lambda^{\nu}\lambda^{\mu} = 2g^{\mu\nu}\, I \qquad (3)$$

where $x^{\mu} = (t,\vec{x})$; $u^{\mu} = (\frac{1}{\gamma},\frac{\vec{v}}{\gamma})$ is the particle velocity ; $\mu,\nu = 0,\ 1,\ 2,\ 3$; a metrics with a signature $(+,-,-,-)$ is used ; c and h are made equal 1 and repeated indices are understood to be summed.

For equation (1) to be starting point of the theory, the equation should, first, result in the correct energy-momentum relation for a free particle and, second, be Lorentz covariant Eq. (2) insures the former condition in the form $(p^{\mu}u_{\mu})^2 = m^2$. Matrices are functions of the particle velocity and so the commutation relations (3) alone are insufficient for proving invariance of eq. (1) under the Lorentz transformations ; therefore let us first specify the functional dependence of the matrices on the velocity. Since the trivial solution $\lambda^{\mu} = u^{\mu}I$ is totally uninteresting let us consider the case of linear dependence on the velocity

$$\lambda^{\mu} = \lambda^{\mu\sigma}u_{\sigma} + \lambda^{\mu 4} \qquad (4)$$

where $\lambda^{\mu\sigma}$ and $\lambda^{\mu 4}$ are numerical matrices. The condition (3) holds identically if

$$\lambda^{\mu\sigma}\lambda^{\nu\tau} + \lambda^{\nu\tau}\lambda^{\mu\sigma} = 2(g^{\mu\tau}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\tau})I$$

$$\lambda^{\mu 4}\lambda^{\nu 4} + \lambda^{\nu 4}\lambda^{\mu 4} = 2g^{\mu\nu}\, I \qquad (5)$$

$$\lambda^{\mu 4}\lambda^{\nu\tau} + \lambda^{\nu\tau}\lambda^{\mu 4} = 0$$

Because of the antisymmetry of $\lambda^{\mu\sigma} = -\lambda^{\sigma\mu}$ only ten out of the twenty matrices are independent quantities. These matrices mutually anticommute ; the square of four of them is equal to unity and of six, to minus unity. To put it differently, eq. (5) is specified by ten generatrices of the alternion algebra ${}^{4}A_{11}$ which is isomorphous with the algebra of sixteenth order quaternion matrices [4]. Since they are not handy let us replace the quaternion matrices with ten complex, irreducible, unitary 32 nd order matrices

$$(\lambda^{\mu\nu})^{+} = (\lambda^{\mu\nu})^{-1} \ ; \ (\lambda^{\mu 4})^{+} = (\lambda^{\mu 4})^{-1} \qquad (6)$$

This situation arises in construction of Dirac matrices which are usually chosen as complex fourth order matrices even though the equation $\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu}I$ is satisfied by four second-order quaternion matrices.

From eqs. (5) and (6) it follows that four matrices are Hermitian and six, anti-Hermitian

$$(\lambda^{oa})^{+} = \lambda^{oa} \ ; \ (\lambda^{ab})^{+} = -\lambda^{ab} \ ; \ a,b = 1,2,3,4 \qquad (7)$$

If a matrix Λ is introduced

$$\Lambda = \lambda^{12}\lambda^{13}\lambda^{14}\lambda^{23}\lambda^{24}\lambda^{34} \ ; \ \Lambda^{+} = \Lambda^{-1} = -\Lambda \qquad (8)$$

then the Hermitian conjugation conditions (7) can be rearranged into

$$(\lambda^{\alpha\beta})^{+} = \Lambda\, \lambda^{\alpha\beta}\, \Lambda^{-1} \qquad (9)$$

Represented in the form (5) the commutation relations are unwieldy and unhandy in proving the relativistic invariance ; however, they can be represented in a simpler form. Let us define a symmetrical tensor $g_{\alpha\beta}$

$$g_{00} = -g_{11} = -g_{22} = -g_{33} = -g_{44} = 1,$$

$$g_{\alpha\beta} = 0 \ \text{ if } \ \alpha \neq \beta \qquad (10)$$

henceforth subscripts of initial letters of the Greek alphabet α, β, γ, δ take on values from 0 to 4 while those of the middle of the alphabet from 0 to 3. The inverse tensor $g^{\alpha\beta}$ provides a compact restatement of commutation relations (5)

$$\lambda^{\alpha\beta}\lambda^{\gamma\delta} + \lambda^{\gamma\delta}\lambda^{\alpha\beta} = 2(g^{\alpha\delta}g^{\beta\gamma} - g^{\alpha\gamma}g^{\beta\delta})I \qquad (11)$$

Eqs. (4), (10), and (11) make it possible to prove the relativistic invariance of eq. (1) by using a five-dimensional group of transformations of co-ordinates 0 (4,1). For this purpose extend eq. (1) to the case of a five-dimensional pseudo-Euclidian space with a metric tensor (10)

$$i\lambda^{\alpha\beta}\, u_\alpha \frac{\partial\Phi}{\partial x^\beta} - m\,\Phi = 0 \qquad (12)$$

(where u^α is the 5-velocity, $u^\alpha u_\alpha = 0$) and then prove invariance of this equation under the group of five-dimensional transformations 0 (4,1) which contains the Lorentz group as a subgroup. Under reduction of 0 (4,1) to the Lorentz group, assume that x^4 = const, $u^4 = 1$ and $\frac{\partial}{\partial x^4} \equiv 1$ we have eq. (1) ; in other words one can assume that eq. (1) is invariant under five-dimensional transformations but physical solutions do not depend on the fifth co-ordinate. Incidentally, eq. (12) can be interpreted differently but we will not discuss these possibilities because using the five dimensions is merely a convenient tool which enables making full use of simplicity of the commutation relations (11).

To prove invariance of the equation, it is sufficient to show [5] that for any transformation of co-ordinates

$$(x^\alpha)' = a^\alpha_\beta\, x^\beta \;;\quad (x^\alpha)' x'_\alpha = \text{inv} \qquad (13)$$

there is a linear transformation S(a) of wave functions in the primed and unprimed reference frame

$$\Phi'(x') = S(a)\Phi(x) \;;\; \Phi(x) = S^{-1}(a)\Phi'(x') \qquad (14)$$

and $\Phi'(x')$ is a solution of the equation which has the form of eq. (12) in the primed reference frame

$$\left\{i\,\lambda^{\gamma\delta}\,u'_\gamma \frac{\partial}{\partial(x^\delta)'} - m\right\}\Phi'(x') = 0 \qquad (15)$$

Substitute (14) into (12) ; multiply the left-hand side by S(a) ; and use the definition (13) to have

$$\left\{iS\lambda^{\alpha\beta}S^{-1}a^\gamma_\alpha a^\delta_\beta u'_\gamma \frac{\partial}{\partial(x^\delta)'} - m\right\}\Phi'(x') = 0$$

This equation coincides with (15) if the matrix has the property

$$a^\gamma_\alpha a^\delta_\beta S\lambda^{\alpha\beta}S^{-1} = \lambda^{\gamma\delta} \qquad (16)$$

Construct S for the infinitesimal proper transformation of the group 0 (4,1)

$$a^\beta_\alpha = \delta^\beta_\alpha + \varepsilon^\beta_\alpha \;;\quad a_{\alpha\beta} = g_{\alpha\beta} + \varepsilon_{\alpha\beta} \qquad (17)$$

with

$$\varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha} \qquad (18)$$

Expand S in powers of ε and keep only linear terms

$$S = 1 - \frac{1}{4}\,\sigma^{\alpha\beta}\,\varepsilon_{\alpha\beta} \qquad (19)$$

where $\sigma^{\alpha\beta} = -\sigma^{\beta\alpha}$ by eq. (18). Substitute eqs. (17) - (19) into eq. (16), keep first-order terms in ε, use the notation [B,C] = BC - CB for the commutator brackets and have

$$2[\sigma^{\alpha\beta},\lambda^{\gamma\delta}] = g^{\alpha\delta}\lambda^{\beta\gamma} - g^{\alpha\gamma}\lambda^{\beta\delta} + g^{\beta\gamma}\lambda^{\alpha\delta} - g^{\beta\delta}\lambda^{\alpha\gamma}$$

The antisymmetric solution of this equation

$$\sigma^{\alpha\beta} = \frac{1}{2}\,g_{\gamma\delta}[\lambda^{\beta\gamma},\lambda^{\alpha\delta}] \qquad (20)$$

is, by virtue of diagonality of the metric tensor and antisymmetry of $\lambda^{\alpha\beta}$, a sum of mutually commutating terms ; in particular, σ^{12} has the form

$$\sigma^{12} = \lambda^{20}\lambda^{10} - \lambda^{23}\lambda^{13} - \lambda^{24}\lambda^{14}$$

According to eq. (19), S for an infinitesimal transformation is given by

$$S = 1 - \frac{1}{8}\, g_{\gamma\delta}\, \varepsilon_{\alpha\beta}[\lambda^{\beta\gamma},\lambda^{\alpha\delta}]$$

Hence, for rotation through a finite angle ω about an axis in the direction labeled n is represented as

$$S = \exp\left\{-\frac{1}{4}\,\omega\sigma^{\alpha\beta}\,\mathcal{I}^{n}_{\alpha\beta}\right\} \qquad (21)$$

where $\mathcal{I}^{n}_{\alpha\beta}$ is the generator of rotation about this axis. The matrix S is not, generally speaking, unitary but formula (9) easily shows that $\Lambda^{-1}\sigma^{+}\Lambda = -\sigma$; consequently, for proper transformations

$$\Lambda^{-1}S^{+}\Lambda = S^{-1} \qquad (22)$$

Let us consider unproper transformations of space reflection and time reversal. For space reflection the matrix a is diagonal, $a^0_0 = a^4_4 = -a^1_1 = -a^2_2 = -a^3_3 = 1$, then eq. (16) for the space reflection operator P is satisfied by

$$P = \lambda^{01}\lambda^{02}\lambda^{03}\lambda^{14}\lambda^{24}\lambda^{34} = P^{+} = P^{-1} \qquad (23)$$

which insures invariance of both eq. (1) and eq. (12).

Construct a transformation of the time inversion; for this purpose introduce an interaction of a particle whose charge is e with an external electromagnetic field $A^{\mu} = (\phi, A^{k})$ by means of the gauge invariant substitution $i\frac{\partial}{\partial x^{\mu}} \to i\frac{\partial}{\partial x^{\mu}} - eA_{\mu}$ and rewrite eq. (1) in the form [5]

$$i\lambda^{0}\frac{\partial\Phi}{\partial t} = [\lambda^{k}(-i\frac{\partial}{\partial x^{k}} + eA_{k}) + m + e\phi\lambda^{0}]\Phi = H\Phi$$

Determine the transformation $\mathcal{T}$ such that if $t' = -t$, $\Phi'_{T} = \Phi'(t') = \mathcal{T}\Phi(t)$; then the latter equation becomes

$$-(\mathcal{T}i\lambda^{0}\mathcal{T}^{-1})\,\frac{\partial\Phi'(t')}{\partial t'} = (\mathcal{T}H\mathcal{T}^{-1})\Phi'(t')$$

When the sense of time is reversed, $u'_0 = u_0$, $u'_k = -u_k$; $\phi' = \phi$, $A'^{k} = -A^{k}$ and, before all, it is necessary to change sign between two terms $i\frac{\partial}{\partial x^{k}}$ and eA_k ; therefore the transformation is sought as a complex conjugation operator multiplied by the matrix T :

$$\Phi'_{T} = \mathcal{T}\Phi(t) = T\Phi^{*}(t) \qquad (24)$$

This gives

$$i(T\lambda^{*0}T^{-1})\,\frac{\partial\Phi'(t')}{\partial t'} = \left\{-(T\lambda^{*k}T^{-1})\left[-i\frac{\partial}{\partial (x^{k})'} + eA'_{k}\right] + m + e\phi(T\lambda^{*0}T^{-1})\right\}\Phi'(t')$$

and for invariance of the equation it is necessary that

$$T\lambda^{*0k}T^{-1} = -\lambda^{0k} \qquad T\lambda^{*k\ell}T^{-1} = \lambda^{k\ell}$$
$$T\lambda^{*k4}T^{-1} = -\lambda^{k4} \qquad T\lambda^{*04}T^{-1} = \lambda^{04} \qquad (25)$$

Thence it immediately follows that $T^{*} = T^{-1} = \bar{T}$ though the explicit form of the matrix T depends on the particular representation of the matrices $\lambda^{\alpha\beta}$. Note that there is just one matrix $\lambda = \prod_{\alpha<\beta}^{4}\lambda^{\alpha\beta}$ which commutes with both all generators $\sigma^{\alpha\beta}$ for the representation of the group O (4,1) and with operators of discrete transformations P and $\mathcal{T}$. Under reduction of O (4,1) to the Lorentz group two more matrices

$$\Lambda_1 = \lambda^{04}\lambda^{14}\lambda^{24}\lambda^{34} \; ; \; \Lambda_2 = \lambda\Lambda_1$$

are generated which commute with the generators $\sigma^{\mu\nu}$ of the representation of the Lorentz group and anticommute with P and $\mathcal{T}$. Consequently, formulae (21), (23) - (25) specify the reducible representation of the Lorentz group and this representation is double-valued. Indeed, consider a particular case, rotation through and angle ω about the Z-axis. In this case $\mathcal{J}^Z_{12} = -\mathcal{J}^Z_{21} = 1$; using the explicit form of σ^{12} we have

$$S = e^{-\frac{\omega}{2}\sigma^{12}} = \cos^3\frac{\omega}{2} + \sigma^{12}\cos^2\frac{\omega}{2}\sin\frac{\omega}{2} +$$

$$+ \frac{3+(\sigma^{12})^2}{2}\cos\frac{\omega}{2}\sin^2\frac{\omega}{2} + \lambda^{20}\lambda^{10}\lambda^{23}\lambda^{13}\lambda^{24}\lambda^{14}\sin^3\frac{\omega}{2}$$

The half-angle is an expression of the doublevaluedness of the wave function transformation. Therefore the observables in the theory should be bilinear in $\Phi(x)$. The matrix $\underline{\Lambda}$ makes it possible to determine the adjoint wave function $\bar{\Phi} = \Phi^+\Lambda$ which is a solution of the adjoint equation

$$i\frac{\partial\bar{\Phi}}{\partial x^\mu}\lambda^\mu + m\bar{\Phi} = 0$$

An adjoint wave function under an arbitrary transformation of the co-ordinates should be transformed by the equation $\bar{\Phi}' = \bar{\Phi}\Lambda^{-1}S^+\Lambda$ which for proper rotations (22) leads to $\bar{\Phi}' = \bar{\Phi}S^{-1}$; for space and time inversions $\bar{\Phi}'_P = -\bar{\Phi}P$ and $\bar{\Phi}' = -\bar{\Phi}*T^{-1}$, respectively. The adjoint wave function and the matrices λ ; Λ_1 and Λ_2 make it possible to construct four independent scalar functions $\bar{\Phi}\Phi$; $\bar{\Phi}\lambda\Phi$; $\bar{\Phi}\Lambda_1\Phi$; and $\bar{\Phi}\Lambda_2\Phi$ which under space and time inversions are transformed as

$$\bar{\Phi}'_P\Phi'_P = -\bar{\Phi}\Phi \quad ; \quad \bar{\Phi}'_T\Phi'_T = \bar{\Phi}\,\Phi \qquad (26a)$$

$$\bar{\Phi}'_P\lambda\Phi'_P = -\bar{\Phi}\lambda\Phi \quad ; \quad \bar{\Phi}'_T\lambda\Phi'_T = -\bar{\Phi}\lambda\Phi \qquad (26b)$$

$$\bar{\Phi}'_P\Lambda_1\Phi'_P = \bar{\Phi}\Lambda_1\Phi \quad ; \quad \bar{\Phi}'_T\Lambda_1\Phi'_T = -\bar{\Phi}\Lambda_1\Phi \qquad (26c)$$

$$\bar{\Phi}'_P\Lambda_2\Phi'_P = \bar{\Phi}\Lambda_2\Phi \quad ; \quad \bar{\Phi}'_T\Lambda_2\Phi'_T = \bar{\Phi}\Lambda_2\Phi \qquad (26d)$$

Following the classification of Ref.[4], the quantities (26a - d) are a singular and a simple pseudo-scalar and a singular and a simple scalar, respectively, each of these functions being a unique scalar function of the associated type, quadratic in $\Phi(x)$. To obtain a numerical scalar let us use a representation of the function $\Phi(x)$ as a four-dimensional Fourier integral. Since each component of $\Phi(x)$ satisfies the second order equation (2), the general solution represented entirely in relativistic terms has the form

$$\Phi(x) = \frac{1}{(2\pi)^{3/2}}\int d^4k\, e^{ik_\mu x^\mu}\,\delta\{(k_\mu u^\mu)^2 - m^2\}\Phi(k) \qquad (27)$$

where $\delta\{(k_\mu u^\mu)^2-m^2\} = \frac{1}{2m}\{\delta(k_\mu u^\mu-m) + \delta(k_\mu u^\mu+m)\}$ is the relativistic δ-function and the amplitude $\Phi(k) = \Phi(k^0;\vec{k})$ satisfies the equation

$$(\lambda^\mu k_\mu + m)\Phi(k)\Big|_{(ku)^2=m^2} = 0$$

Because the integrand includes a δ-function, the integration is performed over just two Lorentz-invariant hypersurfaces $k_\mu u^\mu = \pm m$, rather than the entire four-dimensional k-space. This enables decomposing the integral (27) into two summands

$$\Phi(x) = \overset{+}{\Phi}(x) + \overset{-}{\Phi}(x) \; ; \quad \overset{\pm}{\Phi}(x) = \frac{1}{(2\pi)^{3/2}}\int d^4k\,\frac{\delta(k_\mu u^\mu \mp m)}{2m}\,\Phi(k) \quad (28)$$

Using this representation and integrating over the three-dimensional volume we have

$$\int \bar{\Phi}^{\pm} u^{\mu} \frac{\partial \Phi}{\partial x^{\mu}} \frac{dV}{\gamma} = -\int u^{\mu} \frac{\partial \bar{\Phi}^{\pm}}{\partial x^{\mu}} \Phi^{\pm} \frac{dV}{\gamma} = \pm \frac{i}{2m} \int d^4k \; \delta(k_{\mu} u^{\mu} \mp m) \bar{\Phi}(k) \Phi(k)$$

$$\int \bar{\Phi}^{\pm} u^{\mu} \frac{\partial \Phi^{\mp}}{\partial x^{\mu}} \frac{dV}{\gamma} = \int u^{\mu} \frac{\partial \bar{\Phi}^{\pm}}{\partial x^{\mu}} \Phi^{\mp} \frac{dV}{\gamma} = \frac{\mp i}{2m} \int d\vec{k} \; e^{\mp \frac{2ix^0 m}{u^0}} \bar{\Phi}\left(\frac{\vec{k}\vec{u} \pm m}{u^0}; \vec{k}\right) \Phi\left(\frac{\vec{k}\vec{u} \mp m}{u^0}; \vec{k}\right)$$

Combining these relations and using the equality $\delta(k_{\mu}u^{\mu}-m) - \delta(k_{\mu}u^{\mu}+m) = \theta(k_{\mu}u^{\mu})\delta\{(k_{\mu}u^{\mu})^2-m^2\}$ we find that

$$\int \left(\bar{\Phi} u^{\mu} \frac{\partial \Phi}{\partial x^{\mu}} - u^{\mu} \frac{\partial \bar{\Phi}}{\partial x^{\mu}} \Phi\right) \frac{dV}{\gamma} = i \int d^4k \; \theta(k_{\mu}u^{\mu}) \delta\{(k_{\mu}u^{\mu})^2 - m^2\} \bar{\Phi}(k) \Phi(k) \quad (29)$$

where

$$\theta(ku) = \begin{cases} 1, & \text{if} \quad ku > 0 \\ -1, & \text{if} \quad ku < 0 \end{cases}$$

. The right-hand side of eq. (29) is explicitly represented in covariant form which facilitates a study of properties which can be traced to the space and time inversions. More specifically eq. (29) is a simple pseudo-scalar because $\int \ldots d^4k$ and $\delta\{(k_{\mu}u^{\mu})^2 - m^2\}$ are simple scalars, $\theta(k_{\mu}u^{\mu})$ is a singular scalar (θ is an odd function and k^{μ} and u^{μ} are a simple and a singular vector, respectively), and $\bar{\Phi}(k)\Phi(k)$ is a singular pseudo-scalar, according to the definition (27) and eq. (26a). It is easy to construct a simple scalar

$$\int \left(\bar{\Phi} \Lambda_1 u^{\mu} \frac{\partial \Phi}{\partial x^{\mu}} - u^{\mu} \frac{\partial \bar{\Phi}}{\partial x^{\mu}} \Lambda_1 \Phi\right) \frac{dV}{\gamma}$$

which can, following Refs[2,3] be interpreted as the particle mass while the nonlinear equation of Refs[2,3] is represented as

$$i\lambda^{\mu} \frac{\partial \Phi}{\partial x^{\mu}} - \Phi \int \left(\bar{\Phi} \Lambda_1 u^{\mu} \frac{\partial \Phi}{\partial x^{\mu}} - u^{\mu} \frac{\partial \bar{\Phi}}{\partial x^{\mu}} \Lambda_1 \Phi\right) \frac{dV}{\gamma} = 0$$

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