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# ON SPECIAL RELATIVITY'S SECOND POSTULATE

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*Résumé : Nous insistons sur le fait que l'hypothèse que la vitesse de la lumière est la même dans tous les systèmes de référence inertiels (Postulat (B)), n'est pas une conséquence immédiate du principe de relativité et de l'hypothèse que la lumière se propage avec une et seulement une vitesse dans chaque système inertiel (Postulat (B')). Nous poussons les conséquences du principe de relativité aussi loin que possible, nous déterminons complètement la transformation de Lorentz et nous démontrons que la vitesse de la lumière est une constante universelle.*

*Abstract : It is emphasized that the assumption that the speed of light is the same in all inertial systems (postulate (B)) does not follow trivially from the relativity principle and the assumption that in each inertial system light propagates with one and only one speed (postulate (B')).*

*We follow the consequences of the relativity principle as far as possible, we use (B') to specify completely the Lorentz transformation and we demonstrate that the speed of light is a universal constant.*

## 1. Introduction

We describe below the two postulates on which Special Relativity is conventionally based :

(A) The principle of Relativity : There exists an infinity of coordinate systems (called inertial systems) moving relative to each other with constant velocity, which are equivalent as far as the physical laws are concerned i.e. the physical laws have the same *form* in all of them.

If we assume that the law of inertia holds, then, as we shall see later the transformation between two inertial systems $\Sigma$, $\Sigma'$ is linear. That is, if $x_\mu$, $x'_\mu$, $\mu=1,2,3,4$ are the coordinates of an event in $\Sigma$, $\Sigma'$ respectively[1] then

$$x'_\mu = \sum_{\nu=1}^{4} a_{\mu\nu} x_\nu + b_\mu \qquad (1)$$

where the matrix $A = (a_{\mu\nu})$ is a function of the velocity $\vec{v}$ of $\Sigma'$ relative to $\Sigma$ and $b_\mu$ are constants. A moment's reflection will persuade one that A has the following properties :

(i) A is invertible and

$$A\vec{U} = (0,0,0,\frac{1}{\gamma(\vec{v})}), \qquad A^{-1}\vec{V} = (0,0,0,\gamma(\vec{v})) \qquad (2)$$

where $\vec{U}$, $\vec{V}$ are the 4-vectors $(\vec{v},1)$, $(-\vec{v},1)$ ; (consider a stationary point in $\Sigma'$ or $\Sigma$ and differentiate the transformation -equations with respect to $x'_4$ or $x_4$- we reject the case $\gamma(\vec{v}) = \frac{dx_4}{dx'_4} = 0$ as physically impossible).

[1] We put $t = x_4$.



Because of postulate (A) :

(ii) $$A^{-1}(-\vec{v}) = A(\vec{v}) \qquad (3)$$

that is interchanging $\vec{v}$ with $-\vec{v}$ and $x'_\mu$ with $x_\mu$ in the homogeneous transformation $\Sigma \to \Sigma'$ we must obtain the inverse transformation $\Sigma' \to \Sigma$.

(iii) if $\Sigma''$ is a third inertial system moving with velocity $\vec{v}'$ relative to $\Sigma'$ and if $A'$, $A''$ are the matrices of the transformations $\Sigma' \to \Sigma''$, $\Sigma \to \Sigma''$ then

$$A'' = AA' \qquad (4)$$

i.e. the set of transformations between inertial systems is a group.

We will show in paragraph 2 that (i), (ii) together with homogeneity of time and isotropy of space suffice to express the 16 unknown $a_{\mu\nu}$ in terms of a single parameter. In order to determine this one *completely* a second postulate is necessary ; traditionally it is taken to be the following:

(B) The speed of light in vacuum is a universal constant, i.e. it is the same in all inertial systems.

Actually (B) together with eqs(1), the isotropy of space and homogeneity of time, completely specify the transformation (cf. [6],[7]). The disadvantage of this derivation, (common also to Einstein's original one) is that it obscures the fact, that as we shall see, essentially postulate (A) alone, suffices to specify the transformation to within a constant. In addition checking (B) experimentally is much harder than it is for the following weaker statement.

(B') In each inertial system light propagates in vacuum with only one speed irrespective of direction and state of motion of the source, i.e. each inertial system has its own vacuum light speed.

Indeed on considering the earth as a typical inertial system[2] and using the principle of relativity together with the results of classical experiments such as the Michelson-Morley experiment[3] or the absence of the de Sitter effect in double stars, [1], one can very safely assert that in each inertial system light propagates in vacuum with only one speed, (postulate (B')). On the other hand, the direct experimental verification of (B) requires the measurement of the speed of light at different times of the year[4], evidently a more difficult task to accomplish.

Therefore it would be instructive to show that on the basis of (A) and the *a priori* weaker postulate (B') the Lorentz transformation is completely specified and the vacuum light speed *comes* to be a universal constant. In the majority of literature (e.g. [2],[3],[5]) it is usually taken for granted or at least considered as self evident that the Principle of Relativity, (A), together with the fact that each inertial system has its own light-speed, (B'), imply *immediately* that the speed of light in vacuum is a universal constant, (B)[5]. But what (A) says is that the laws of physics have the same form in all inertial systems ; this does not *a priori* mean that the speed of light (or any other parameter) characterising each of these systems has the same *numerical* value in all of them.

It is the purpose of this paper to show that the two sets of postulates {(A),(B')}, {(A),(B)} are indeed equivalent but the proof is mathematical rather, than verbal (hence more doubtful).

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[2] for a time interval small compared with one year.

[3] where a comparison of the light speed in two different directions is made.

[4] Each time the earth being identified with a different inertial system.

[5] This is also true for Einstein's original paper ([4], p.45)



## 2. Consequences of the relativity principle

We make the fundamental assumption that events in the physical world form a space-time continuum which is a 4-dimensional manifold, an assumption which according to H. Weyl, [8], is the most certain fact of our empirical knowledge. Assuming that the law of inertia holds globally on this manifold we see that the transformation between inertial systems maps straight lines to straight lines so that it is projective ; however rejecting the possibility of mapping infinite points to finite ones and vice versa we are left with a linear transformation (cf. eq(1)). Assuming that space is isotropic about every point -hence homogeneous- and that time is homogeneous as well we may say that coordinate systems with arbitrary axes-orientations and/or space-time locations are equivalent in the sense of postulate (A). Therefore without loss of generality we may restrict ourselves to the study of the transformation between systems $\Sigma$, $\Sigma'$, such that the velocity $\vec{v}$ of $\Sigma'$ relative to $\Sigma$ is along the $x_1$-axis while the other two axes of $\Sigma'$ remain parallel to those of $\Sigma$, and the origins 0, 0' of $\Sigma$, $\Sigma'$ at $t = 0$ coincide[6]. These imply that in (1) $b_\mu = 0$, $\mu = 1,2,3,4$ and $a_{21} = a_{24} = a_{31} = a_{34} = a_{23} = a_{32} = a_{12} = a_{13} = 0$ ; (we simply notice : i) for any t the $x_1\, x_2, x_3\, x_4$-planes and for $t = 0$ the $x_2x_3$-plane of $\Sigma$ coincide with those of $\Sigma'$ ii) we consider the transformation of the events $(x_1,0,0,t)$, $(0,x_2,0,t)$, $(0,0,x_3,t)$, $(0,x_2,x_3,0)$). Therefore the transformation takes the form

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \\ t' \end{pmatrix} = \begin{pmatrix} \alpha_{11} & 0 & 0 & \alpha_{14} \\ 0 & \alpha_{22} & 0 & 0 \\ 0 & 0 & \alpha_{33} & 0 \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ t \end{pmatrix} \qquad (5)$$

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[6] From this special case the general one is recovered by including space-time translations, space-rotations, space-inversions, and time reversal.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ t \end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix} \alpha_{22}\alpha_{33}\alpha_{44} & \alpha_{14}\alpha_{33}\alpha_{42} & \alpha_{14}\alpha_{22}\alpha_{43} & -\alpha_{14}\alpha_{22}\alpha_{33} \\ 0 & \Delta\alpha_{22}^{-1} & 0 & 0 \\ 0 & 0 & \Delta\alpha_{33}^{-1} & 0 \\ -\alpha_{22}\alpha_{33}\alpha_{41} & -\alpha_{11}\alpha_{33}\alpha_{42} & -\alpha_{11}\alpha_{22}\alpha_{43} & \alpha_{11}\alpha_{22}\alpha_{33} \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ t' \end{pmatrix} \tag{6}$$

with $\Delta = \det(a_{\mu\nu}) = a_{22}a_{33}(a_{11}a_{44}-a_{41}a_{14}) \neq 0$. But by (A), (5), (6) must have the same form so that $a_{42} = a_{43} = 0$ ; furthermore from (2) we easily see that $\frac{\alpha_{14}}{\alpha_{11}} = -v$, $\frac{\alpha_{14}}{\alpha_{44}} = -v$ so that

$$a_{14} = -va_{11} \quad , \quad a_{44} = a_{11} = \gamma \tag{7}$$

Space isotropy can now be used to simplify (5), (6) ; we immediately have

$$a_{22} = a_{33} = b \tag{8}$$

In addition upon considering a rod of length 1 moving first with velocity v (along the $x_1$-axis) and then with $-v$, space isotropy implies that[7]

$$\gamma(v) = \gamma(-v) \tag{9}$$

Similarly

$$b(v) = b(-v) \tag{10}$$

Finally (3) together with (7), (8) provide further relations among the coefficients :

$$\gamma(-v) = \frac{1}{\gamma(v) + v\,\alpha_{41}(v)} \tag{11}$$

[7] i.e. the length of the rod doesn't depend on the direction of its motion.



$$b(-v) = \frac{1}{b(v)} \tag{12}$$

In view of (9), (10) we have[8]

$$b(v) = \pm 1, \quad a_{41} = \frac{1-\gamma^2(v)}{v\,\gamma(v)} \tag{13}$$

We have now expressed all the $a_{\mu\nu}$ in terms of one parameter (compare with eq(2)) so that (5), (6) take the form

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ t' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\gamma v \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1-\gamma^2}{v\gamma} & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ t \end{pmatrix}, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ t \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & \gamma v \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{\gamma^2-1}{v\gamma} & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ t' \end{pmatrix} \tag{14}$$

One may proceed further by using (4) together with (A) (A" must have the same *form* as A, A') which easily imply (cf. [2]) that

$$\left[1 - \frac{1}{\gamma^2(v)}\right]\frac{1}{v^2} = \left[1 - \frac{1}{\gamma^2(v')}\right]\frac{1}{v'^2} = \frac{1}{k}$$

where k is a universal constant with dimensions of a square velocity.

Therefore

$$\gamma = \left[1 - \frac{v^2}{k}\right]^{-\frac{1}{2}} \tag{15}$$

Eq.(15) shows that the Relativity Principle together with space-time isotropy determine the transformation uniquely to within a constant. But in order to interpret k physically we

[8] We need n't consider the case $b = -1$ since it can be included in the general case as a space inversion ; this is also tacitly done in what follows.

need a second postulate[9].

## 3. Complete specification of the transformation

The procedure leading to (13), (14) is given in a slightly different way in [2] ; however instead of following the argument leading to (14) we proceed directly as follows : we accept postulate (B') and suppose that in $\Sigma$ the light-speed is c. Two signals of light are sent at the same time from $\Sigma$, one along the $x_1$-axis, the other along the $x_2$-axis ; in $\Sigma$ the signals are the world-points $(ct,0,0,t)$, $(0,ct,0,t)$ and in $\Sigma'$ they are $(x_1',x_2',0,t')$, $(x_1'',x_2'',0,t'')$, where by (14) we have

$$x_1' = \gamma ct - \gamma vt \qquad x_1'' = -\gamma vt$$
$$x_2' = 0 \qquad x_2'' = ct \qquad (16)$$
$$t' = \frac{1-\gamma^2}{v\gamma} ct + \gamma t \qquad t'' = \gamma t$$

By (B') the two signals have the same speed in $\Sigma'$ :

$$\left(\frac{x_1'}{t'}\right)^2 = \left(\frac{x_1''}{t''}\right)^2 + \left(\frac{x_2''}{t''}\right)^2 \Rightarrow \frac{\gamma^2(c-v)^2}{\left[\frac{(1-\gamma^2)c}{v\gamma} + \gamma\right]^2} = v^2 + \frac{c^2}{\gamma^2}$$

After a straightforward calculation we obtain :

$$(2\beta^3-\beta^2-2\beta+1)\gamma^4 + (\beta^2+2\beta-2)\gamma^2+1 = 0 \qquad (17)$$

[9] The ad hoc identification of k with the square of the light-speed, [2], can be done only because we already know what the Lorentz transformation is, but is not otherwise justified ; in fact we cannot even say if k is positive or negative.



where $\beta = \frac{v}{c}$. Then :

$$\gamma^2 = \frac{-\beta^2-2\beta+2\pm|\beta(\beta-2)|}{2(\beta^2-1)(2\beta-1)} \qquad (18)$$

which implies

a) $\beta(\beta-2) > 0 \Rightarrow \gamma^2 = \frac{1}{1-\beta^2} \geqslant 0 \qquad -1 \leqslant \beta \leqslant 0$

$\gamma^2 = \frac{1}{1-2\beta} \geqslant 0 \qquad \beta \leqslant 0$

b) $\beta(\beta-2) < 0 \Rightarrow \gamma^2 = \frac{1}{1-\beta^2} \geqslant 0 \qquad 0 \leqslant \beta \leqslant 1$

$\gamma^2 = \frac{1}{1-2\beta} \geqslant 0 \qquad 0 \leqslant \beta \leqslant \frac{1}{2}$

The solution $\gamma^2 = \frac{1}{1-2\beta}$ is rejected since it is not an even function of v, so we are left with

$$\gamma = \frac{1}{\left[1 - \frac{v^2}{c^2}\right]^{\frac{1}{2}}} \qquad |v| \leqslant c \qquad (19)$$

Now we assume that the speed of light in $\Sigma'$ is c'. Then in the same way we obtain $\gamma' = \left(1 - \frac{v^2}{c'^2}\right)^{-\frac{1}{2}}$. But from (14) $\gamma = \gamma'$ since the transformation $\Sigma' \to \Sigma$ is the inverse of $\Sigma \to \Sigma'$. Then it follows that c = c'. That is the speed of light in vacuum is a universal constant and by (19) an upper limit to the speed of any object. Going back to (15) we see that k must be necessarily identified with $c^2$.

## 4. Conclusion

The above derivation although less geometrical than the usual one, shows clearly the strength of the Relativity Principle, rests on postulate (B') which is easier to verify experimentally than (B) and proves that the sets of postulates {(A),(B')}, {(A),(B)} are equivalent instead of justifying it verbally as is usually the case.

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