Annales de la Fondation Louis de Broglie,
Vol. 10, n° 2, 1985

Non-linear equations of trans-quantum physics

by S.N. BAGCHI

5550 Bellerive, Brossard,
Quebec, JAZ 3C8, Canada

(manuscrit reçu le 1er Octobre 1984)

Abstract : This review paper is intended to convince the sceptics that it is possible to develop a deterministic continuum approach to micro-physics. It is written with the hope that it will encourage others to formulate a more comprehensive deterministic theory of trans-quantum physics which would eventually supersede the presentday quantum theory.

In this work it is shown that, based on the hypothesis that the wave field associated with a particle is a physical reality and on two a priorily non objectionable plausible postulates for this wave field. One can obtain non-linear partial differential equations (PDE) both for a scalar and for a vector wave field of a single particle which under well defined restricted conditions reduce to the fundamental relevant equations of classical and of quantum physics.

This long paper, for lack of available space in a single issue, has been conveniently separated into three parts :

Part I. Derivation of a non-linear equation for the scalar field of a single particle. Klein-Gordon equation, Schroedinger equations, (scalar) wave equation of optics as special cases.

Part II. Extension of point mechanics. Generalized Hamilton-Jacobi, Lagrangian and Hamiltonian equations of analytical mechanics. Diffraction forces. Ensemble description, Heisenberg's uncertainty relation, current density. The corpuscular properties of photons in terms of its wave field.

Part III. Derivation of the non-linear equation valid for a vector field. Proca equation, wave equation for photons, iterated Dirac equation as special cases.

Conclusions : Conjectures and a tentative programme for Universal field theory.

Résumé : Cet article de synthèse est destiné à convaincre les sceptiques qu'il est possible de développer une approche continue et déterministe de la microphysique. Il est écrit dans l'espoir d'encourager d'autres à formuler une théorie déterministe plus large de physique "trans-quantique" qui pourrait ultérieurement remplacer la théorie quantique d'aujourd'hui.

On montre dans cet article, en se basant sur l'hypothèse que le champ ondulatoire associé à une particule est une réalité physique et sur deux postulats plausibles, et ne soulevant pas d'objections a priori, qu'on peut obtenir des équations aux dérivées partielles non linéaires (PDE) portant sur deux champs, l'un scalaire, l'autre vectoriel, d'une même particule, et qui, dans certaines conditions particulières bien définies, se réduisent aux équations fondamentales correspondantes de la physique classique et quantique.

Ce long article, par faute de place dans un seul numéro a été divisé en trois parties :

- Première partie : déduction d'une équation non linéaire pour le champ scalaire d'une seule particule. Equations de Klein-Gordon, de Schrödinger, équation d'onde (scalaire) de l'optique comme cas particuliers.

- Deuxième partie : Extension de la mécanique du point. Equations généralisées de Hamilton-Jacobi, de Lagrange et de Hamilton de la mécanique analytique. Force de diffraction. Description en termes d'ensembles, relations d'incertitude de Heisenberg, densité de courant. Les propriétés corpusculaires des photons en termes de champ.

- Troisième partie : Déduction de l'équation non linéaire valable pour un champ vectoriel. Equation de Proca, équation d'onde des photons, équation de Dirac itérée comme cas particuliers.

- Conclusions : Conjectures et programme suggéré pour la théorie du champ universel.

Part I : Differential Equation for a Scalar Field

I. INTRODUCTION

This work, inspired mainly by Einstein-de Broglie concept of physics, has already vindicated *quantitatively* the view that it is possible to develop a deterministic theory for micro-physics, in particular to deduce Schroedinger and Dirac equations of quantum physics with deterministic physical interpretations. As a result the limitations of the usual quantum mechanical formalisms and recipes are also revealed.

Quantum mechanics had taught us that a particle possesses both corpuscular and wave properties and that all the observable properties of the particle could be obtained from its associated wave field. Our purpose here is neither to discuss the philosophy of science nor to assess the implications of Copenhagen interpretation of physics*, but to derive concrete and relevant equations of trans-quantum physics on certain (*a priorily* nonobjectionable) hypothesis and postulates. It suffices here to say that our work on Kepler problem *indicates* that if one uses Schroedinger's ψ-function, then one has to fall back upon statistical interpretation of quantum physics. But this is not necessary. This theory shows that in the presence of a slit the wave field of a particle

*I have briefly discussed elsewhere my point of view, (see ref. [1]).

suffers diffraction so that the "corpuscle", (represented here as the singular domain of finite dimension of the wave field), experiences certain force which, even in the absence of an external field, deviates its path according to deterministic laws.

Our fundamental hypothesis is that a particle possesses *simultaneously* both corpuscular and wave properties. Consequently, there must be a definite relation between the corpuscular properties and the wave properties of a given particle. The wave function $\varepsilon(\vec{x},t)$ belonging to a particle, (*not to be identified and interpreted as the usual quantum mechanical wave functions*), which leads to the observable properties of the particle, is a physical reality. This comes as an inevitable conclusion from Renninger's gedanken experiments [2]. This physical reality -in so far as it can be *guessed* from our presentday knowledge- is the energy density continuum whose space-time topological distortions and fluctuations give rise to observable phenomena. For brevity, this nonobservable continuum shall be designated as *World Aether*.

It must be emphasized here that this theory in its methodology and general outlook has nothing to do with the *hidden variable* theory. Further, although the theory described here is a causal deterministic theory and the formalism is closely related to Hamilton-Jacobi formalism of classical mechanics and to Hamilton-de Broglie pilot principle of quantum mechanics, it is nevertheless not a march back to classical mechanics. Rather, it is an extension of classical mechanics in which not only the phases but also the amplitude of the wave field of the particle is taken explicitly into consideration. As a result, the initial value problems of classical mechanics have been changed into boundary value problems of the wave field for the causal description of the behaviour of a particle. Point mechanics, (both relativistic and classical), as well as geometrical optics results from the restriction that the space-time curvature of the amplitude of the wave function is zero. We need not assume that $h = 0$ to get point mechanics, although we could arrive at it by making this non-permissible assumption. The fact remains that Planck's constant h has a finite value. It is proved that

point mechanics remain *strictly* valid as long as the wave field does not suffer diffraction. The relation between classical mechanics and the new mechanics of a particle derived from its associated wave field is exactly analogous to that between geometrical and physical optics. The Lagrangian and Hamiltonian formalisms of classical analytical mechanics remain applicable provided we *redefine* the dynamical mass of the particle which involves the space-time curvature of the amplitude of the wave field.

Now, before developing the theory on postulatory basis, let me first analyse Renninger's experiment to show that the wave field of a photon (or an electron) is a *physical reality*.

II. RENNINGER'S GEDANKEN EXPERIMENT

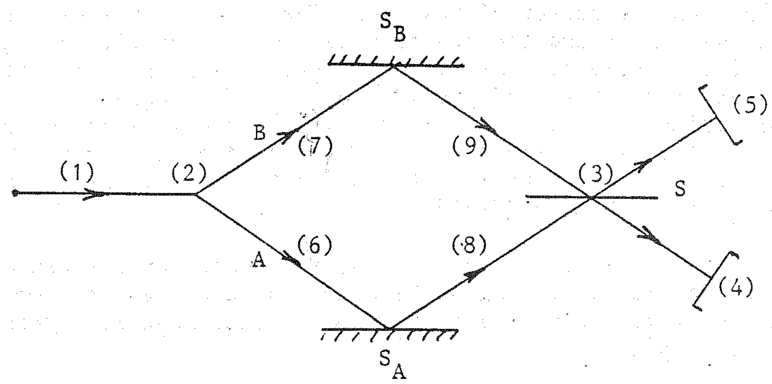
The ideal experiment designed by Renninger, (he asserted that such an experiment could be carried out in practice also both for photons and for electrons), showed that each light quantum is a "corpuscle" of energy guided causally by its wave field extending outside the domain of the "corpuscle". His conclusions are based on the following established results.

(i) All interference experiments yield the same result, (apart from the magnitude of the intensity distribution), whether a single photon or many photons are involved. That means, each photon interferes with itself.

(ii) Many partially coherent beams of light remain coherent when they travel in different and separate paths.

The figure below shows a schematic arrangement of Renninger's experiment.

A parallel beam of monochromatic light (1) is separated at (2) into two beams A and B and afterwards reunited at (3) where with the help of a half-silvered plate S they are allowed to interfere. The optical path difference between the two beams can be adjusted in such a way, (with the help of the mirrors S_A, S_B and/or phase plates), that either the field of



view (4) is bright and that of (5) dark or *vice-versa*. Further, the source of light is manipulated in the fashion that not more than one photon enters the path between (2) and (3) simultaneously. Also at different plates (6), (7), (8) and (9) one can insert detectors and/or λ/2-plates.

Let us assume that the experiment is initially so set up that the field (4) is bright and (5) is dark. Subsequently, we note the following :

(a) When nothing is inserted in the path, all the photons come to (4) and none to (5).

(b) If a detector is inserted in (6), we observe two kinds of phenomena, namely,

(i) If the photon is registered in (6), it vanishes and both (4) and (5) remain dark. No experiment can detect the presence of the photon (or its wave field) after it has been absorbed by the detector.

(ii) If the photon corpuscle does not pass through (6), obviously it travels through (7) and we get a different result. The photon comes either to (4) or to (5). Sometimes the field (4) is bright and (5) is dark and at other times the field (5) becomes bright whereas (4) remains dark. That means, by blocking the

path A we have changed the experimental outcome.

(c) Now, let us perform another experiment. In this case we replace the absorber by a perfectly transparent λ/2-plate. This time all the photons go to (5) and none to (4). Hence, it is possible through an experiment to guide photon corpuscles, whether then are located in the path A or in the path B, always to (5).

As a result of these experiments we can conclude that the "corpuscle" of energy belonging to the photon lying somewhere between (2) and (3) can be guided by tampering with its associated extended wave field without necessarily coming into direct contact with the "corpuscle of energy". Evidently, this extended wave field must be a physical reality, since one can direct the photons either to (4) or to (5), according to one's convenience.

Now, we have to answer the question : What happens to the wave field when the photon vanished in (6) through an absorber ? The presence of the wave field can never be detected after the photon had vanished.

There are two ready answers, but both of them are physically unacceptable. Either, one can suppose that the field vanishes instantaneously. The field then must contract with ultra photon velocity, contradictory the theory of relativity. Or, the wave field must be energyless which is absurd since causal connection revealed in this series of experiments must be due to interaction of energy.

Both these unsatisfactory explanations can be avoided if we assume that the physical reality is the continuum of energy density, (World Aether). The corpuscle and its associated wave field are space-time distortions of this continuum. Further, the entire energy of the particle is *practically* concentrated in the small domain (singularity) of the distorted wave field. After the photon is absorbed, the wave field which must carry some energy, however small, quickly returns back to the unperturbed state of the continuum whose properties cannot be measured. It also explains why elementary quantum of action plays such an important role in micro-physics.

It might be noted that this plausible explanation is perfectly consistent with de Broglie's idea of the double solution. Any way, whatever be the correct explanation of the experimental observations, there is no doubt that this series of experiments definitely proved that the wave field of a particle is a *physical reality*.

III. WORLD VECTORS

In this paper I shall restrict myself only to scalar and vector wave fields of a single particle interacting with an external field of electromagnetic type. All the equations will be expressed as world equations in Minkowski space. Corresponding nonrelativistic equations will be obtained as special cases. Consequently, for convenience of manipulations, I shall be using 4-dimensional vector analysis in the form introduced by Sommerfeld [3] and von Laue [4]. Four-vectors are underlined and the corresponding 3-vectors are printed with an arrow. In order to distinguish Minkowski spatial vectors from the corresponding usual 3-dimensional vectors, the latter, if necessary, will be denoted by a subscript N.

Our world is the Minkowski space with the coordinates x_1, x_2, x_3, x_0 (= ict) and the signature ++++.

For ready reference, some formulae for 4-dimensional vectors which will be used later are given below.

(i) Four-distance : $\underline{x} = \vec{x} + ict.\vec{S}_0 = \sum_{j=0}^{3} x_j.\vec{S}_j$

$\vec{S}_j$'s are unit vectors along the four mutually orthogonal axes, $\vec{S}_0$ lying along the time axis.

(ii) Four-velocity : $\underline{v} \equiv \frac{d\underline{x}}{d\tau} = k(\vec{v} + ic.\vec{S}_0)$

The proper time $d\tau$ is related to the local time dt by

$$dt = k d\tau, \quad \text{where} \quad k = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} .$$

It is to be noted that the scalar product of $\underline{v}$ with it-

self is $(\underline{v}.\underline{v}) = \underline{v}^2 = -c^2$.

(iii) Vector product of two four-vectors gives the so called six-vector

$$\mathbf{F} = [\underline{A}\ \underline{B}] ,$$

an antisymmetric tensor of the second rank. Its components are given by .

$$F_{k\ell} = \begin{vmatrix} A_k & A_\ell \\ B_k & B_\ell \end{vmatrix} = -F_{\ell k}, \quad (k,\ell = 0,1,2,3)$$

(iv) Vector multiplication of a six-vector with a four-vector gives again a four-vector.

$$\underline{D} = [\underline{A}[\underline{B}\ \underline{C}]] .$$

Its j^{th} component is given by

$$D_j = \sum_k A_k F_{jk} = \sum_{k=0}^{3} A_k(B_j C_k - B_k C_j)$$

(v) The four-gradient is symbolically represented by the four-dimensional "Nabla" operator, $\underline{\nabla}$.

$$\underline{\nabla} = \vec{\nabla} + \frac{\partial}{ic\partial t} . \vec{S}_0$$

(vi) D'Alembertian operator

$$\Box = \nabla^2 - \frac{\partial^2}{c^2 \partial t^2}$$

where $(\vec{\nabla}.\vec{\nabla}) = \nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$

is the Laplace operator.

(vii) For any arbitrary four-vector $\underline{A}(\vec{x},t)$, we have

$$\frac{d\underline{A}}{d\tau} = (\underline{u}.\underline{\nabla})\underline{A}$$

where $(\underline{u}.\underline{\nabla})\underline{A} \equiv (u_1\frac{\partial \underline{A}}{\partial x_1} + u_2\frac{\partial \underline{A}}{\partial x_2} + u_3\frac{\partial \underline{A}}{\partial x_3} + u_0\frac{\partial \underline{A}}{\partial x_0})$

and $\underline{u}$ is the velocity of propagation of the quantity $\underline{A}$.

(viii) Four-gradient of a four-scalar $\phi(\vec{x},t)$ is a four-vector $\underline{P}$

$$\underline{\nabla}\phi = \sum_{k=0}^{3} \frac{\partial \phi}{\partial x_k} = \underline{P}$$

(ix) Four-divergence of a four-vector $\underline{P}$ is a four-scalar ϕ

$$(\underline{\nabla}.\underline{P}) = \sum_{k=0}^{3} \frac{\partial P_k}{\partial x_k} = \phi$$

and

$$(\underline{\nabla}.\phi\underline{P}) = \phi(\underline{\nabla}.\underline{P}) + (\underline{P}.\underline{\nabla}\phi)$$

(x) Vector divergence of a six-vector $\mathbf{F}$ is a four-vector with its K^{th} component

$$\sum_{\ell=0}^{3} \frac{\partial F_{k\ell}}{\partial x_\ell}$$

It can be represented *symbolically* as the vector product of four-nabla $\underline{\nabla}$ with the six-vector $\mathbf{F}$; namely

$$[\underline{\nabla}\ \mathbf{F}]$$

This formal notation is also justified by the relation

$$(\underline{\nabla}.[\underline{\nabla}\mathbf{F}]) = 0$$

(xi) The curl of a four-vector gives a six-vector

$$[\underline{\nabla}\underline{P}] = \mathbf{F}$$

Its components are

$$F_{k\ell} = \frac{\partial P_\ell}{\partial x_k} - \frac{\partial P_k}{\partial x_\ell}, \quad (k,\ell = 0,1,2,3)$$

Note also

$$[\underline{\nabla}[\underline{\nabla}\underline{P}]^*] \equiv [\underline{\nabla}\mathbf{F}^*] = 0$$

where

$[\underline{\nabla}\underline{P}]^* \equiv \mathbf{F}^*$ is the dual of $\mathbf{F}$.

The dual is defined by

$$F^*_{k\ell} = \frac{1}{2}\,\epsilon_{k\ell mn}\,F_{mn}$$

where $\epsilon_{k\ell mn}$ is the completely antisymmetric Levi-Civita unit tensor.

(xii) We note also the following relations

$$[\underline{\nabla}(\underline{\nabla}\phi)] = 0$$

$$\underline{\nabla}.\underline{\nabla}\phi) = \Box\phi$$

$$[\underline{\nabla}[\underline{\nabla}\underline{P}]] = \underline{\nabla}(\underline{\nabla}.\underline{P}) - \Box\underline{P}$$

$$[\underline{\nabla}[\underline{U}\underline{V}]] = (\underline{V}.\underline{\nabla})\underline{U} - \underline{V}.(\underline{\nabla}.\underline{U}) - (\underline{U}.\underline{\nabla})\underline{V} + \underline{U}.(\underline{\nabla}.\underline{V})$$

corresponding to three-dimensional formulae

curl grad ϕ = 0 ; div grad ϕ = $\nabla^2\phi$;

curl curl $\vec{v}$ = grad.div $\vec{v}$ – div.grad $\vec{v}$

curl$[\vec{u}\vec{v}]$ = $(\vec{v}.\text{grad})\vec{u} - \vec{v}$ div $\vec{u} - (\vec{u}.\text{grad})\vec{v} + \vec{u}.\text{div}\ \vec{v}$

(xiii) Finally, we come to the operation of rotation of the six-vector which has no analogue in three-dimensional vector analysis. It is usually written with three indices, *all different*. It gives a four-vector defined by

$$\text{curl}_{jk\ell}\ \mathbf{F} \equiv \frac{\partial F_{jk}}{\partial x_\ell} + \frac{\partial F_{k\ell}}{\partial x_j} + \frac{\partial F_{\ell j}}{\partial x_k}$$

It can be easily verified that this represents the i^{th} component of the vector divergence of the dual of $\mathbf{F}$.

That is,

$$\text{Curl } \mathbf{F} = [\underline{\nabla}\mathbf{F}^*]$$

and its scalar divergence is zero

$$(\underline{\nabla}.(\text{Curl } \mathbf{F})) = 0$$

If **F** is a *special* six-vector obtained from the curl of a four-vector, then

$$\text{Curl } \mathbf{F} = \text{Curl}[\underline{\nabla P}] \equiv 0.$$

IV. DIFFERENTIAL EQUATIONS FOR THE SCALAR FIELD OF A PARTICLE*

(i) Definitions

Let the wave function associated with the particle be represented by

(1) $$\varepsilon(\underline{x}) = a(\underline{x}) \exp i\, W(\underline{x})/\hbar$$

a and W are real ; $\hbar = h/2\pi$; h is Planck's constant. Later it will be seen that $W(\underline{x})$ can be identified with Hamilton's principal function.

The generalized 4-momentum of the particle

(2) $$\underline{p} = \underline{p}_N + \underline{p}_e \equiv \vec{p} + \frac{i}{c} H.\vec{S}_0$$

where, (in Gaussian units),

(3) $$\underline{p}_e = \frac{e}{c}\,\underline{\Phi} = \frac{e}{c}(\vec{\Phi} + i\phi_0\vec{S}_0) \equiv \vec{p}_e + \frac{i}{c} U.\vec{S}_0$$

$\underline{p}_e$ is the field momentum due to an external field of electro-

*Most of the results on the scalar field were obtained in collaboration with R. Hosemann, (see refs[5-7]).

magnetic type with the 4-potential $\underline{\Phi}$. H is the total energy and U the potential energy. e is the invariant charge of the particle

(4) $$\underline{p}_N = M_0(\underline{x})\underline{v} \equiv \vec{p}_N + \frac{i}{c} E_N.\vec{S}_0 = \vec{p}_N + \frac{i}{c} Mc^2.\vec{S}_0$$

(5) $$M_0(\underline{x}) = \mu(\underline{x})m_0$$

(6)* $$\mu(\underline{x}) = [1 - \frac{\Box a}{a}(\frac{\hbar}{m_0 c})^2]^{\frac{1}{2}}$$

$\underline{p}_N$ represents the kinetic 4-momentum and E_N the kinetic energy of the particle whose rest mass is m_0. μ is called the mass factor which seems to be more appropriate [see VII(V)]. It depends on the amplitude of the wave field. The "effective rest-mass" M_0 is not generally constant but depends on $\mu(\underline{x})$. The dynamical mass M of the particle is given by

(7) $$M = \mu m_0 k \quad ; \quad [k = (1 - v^2/c^2)^{-\frac{1}{2}}]$$

Thus the amplitude of the wave field is also incorporated in the mechanics of the particle by this *redefinition* of mass.

It should be carefully noted here that the usual expressions of classical and relativistic point mechanics as well as those of geometrical optics are obtained from the condition

(8) $$\Box a = 0 \quad ; \quad \mu = 1$$

(ii) Postulates

According to our fundamental hypothesis the corpuscular and wave properties of the particle should be intimately

*This relation was obtained first by de Broglie[8] in 1927. In the language of the de Broglie school μ is designated as "quantum potential".

connected in a unique way and every property of the particle is capable of being expressed in terms of the wave functions ε and ε*. We postulate that this connection is given by Hamilton-de Broglie pilot principle so that the generalized 4-momentum of the particle is also given by its wave function

(9) $\underline{p} = \underline{\nabla} W = \frac{\hbar}{2i}\left(\frac{\nabla\varepsilon}{\varepsilon} - \frac{\nabla\varepsilon^*}{\varepsilon^*}\right)$, (Postulate I)

where $\varepsilon^*(\underline{x}) = a(\underline{x})\exp -i(W(\underline{x})/\hbar)$ (9a)

Comparing this with (2) we can also write

(10) $\vec{p} = \vec{\nabla} W$; $H = -\frac{\partial W}{\partial t}$

Thus W is Hamilton's principal function, confirmed also by eq(36). The energy-momentum density of the wave field therefore is given by

(11) $\varepsilon\varepsilon^*\underline{p} = \frac{\hbar}{2i}[\varepsilon^*\underline{\nabla}\varepsilon - \varepsilon\underline{\nabla}\varepsilon^*]$

The mass of the particle can also be expressed completely in terms of the wave field. From the definitions given by eqs(2-6) and eq(10), we can easily verify the expressions (12) and (13)

(12) $m_0 = \frac{1}{c}\left[\frac{1}{c^2}\left(\frac{\partial W}{\partial t} + U\right)^2 - (\vec{\nabla}W - \vec{p}_e)^2 + \frac{\Box a}{a}\hbar^2\right]^{\frac{1}{2}}$

(13) $M_0 = \frac{1}{c}\left[\frac{1}{c^2}\left(\frac{\partial W}{\partial t} + U\right)^2 - (\vec{\nabla}W - \vec{p}_e)^2\right]^{\frac{1}{2}}$

The expressions (12) and (13) also answer the pertinent criticism of Brillouin[9], namely, "one completely ignores any possibility of mass connected with the external potential energy, p.14".

It will be shown later that in order to obtain the "spin" of the particle we have to use a vector wave field*.

*It should be noted that a 4-vector can be expressed in terms of a bispinor.

It is related to the "intrinsic" angular momentum around the "singularity" of the vector field endowed with vorticity. Thus a scalar wave here also represents a particle without "spin".

As yet it is not known how the charge of a particle is related to its wave field. My *conjecture*, (cf.[10]), is that the sign and magnitude of the charge would depend on the torsion of the field. If this is correct, we have to study the properties of a non-symmetric tensor field in order to get all the properties of a charged particle, (see the conclusion in Part III). For the moment, as in classical and quantum physics, we have to take the charge of a particle as an empirically given quantity.

We postulate further that the energy-momentum of the field is conserved. That is,

(14) $\underline{\nabla}.(\varepsilon\varepsilon^*\underline{p}) = 0$, (Postulate II),

or,

(14a) $2\underline{\nabla}a.\underline{p} + a^2\underline{\nabla}.\underline{p} = 0$.

Eq(14) can be looked upon also as a continuity condition. Eq.(14a) shows the intimate connection between the amplitude and the 4-momentum.

(iii) <u>Differential Equations</u>

To obtain the differential equation, we simply substitute the expression (11) in (14). Thus, we get the symmetrical equation (15)

(15) $\varepsilon^*\Box\varepsilon - \varepsilon\Box\varepsilon^* = 0$

One can also get other equivalent equations for the wave field. For example, calculating $\nabla\varepsilon$ and $\nabla\varepsilon^*$ from (1) and (9a) and using equations (9 and 14) we obtain

(16) $\Box\varepsilon + \left[\frac{p^2}{\hbar^2} - \frac{\Box a}{a}\right]\varepsilon = 0$

or,

$$(17) \qquad \Box\varepsilon - [\frac{1}{4}(\frac{\nabla\varepsilon}{\varepsilon} - \frac{\nabla\varepsilon^*}{\varepsilon^*})^2 + \frac{\Box a}{a}]\varepsilon = 0$$

All these equations are valid for any scalar field of a particle. They are all nonlinear and imply that the most general solution is nonanalytic. The singular solutions may also be of great physical interest. Unfortunately, the general solutions of these nonlinear equations have not yet been obtained.

Since these equations do not contain *explicitly* characteristic properties of the particle, (e.g. rest mass and charge), in order to relate these equations to the fundamental equations of classical and of quantum physics, we convert eq(16) to the desired form. First, we note, (cf. eqs 2, 4 & 5),

$$(18) \qquad (\underline{p} - \underline{p}_e)^2 = \underline{p}_N^2 = -(\mu m_0 c)^2$$

Replacing the value of $\underline{p}^2$ obtained from (18) in eq(16) and remembering the value of μ given in eq(6), we finally get the desired equation (19)

$$(19) \qquad \Box\varepsilon + \frac{1}{\hbar^2}[2(\underline{p}_e.\underline{p}) - \underline{p}_e^2 - m_0^2c^2]\varepsilon = 0$$

It should be noted that eq(19) also contains ε^* through the definition of $\underline{p}$, eq(9).

It is doubtful whether a general equation can be derived containing only ε, (or ε^*). In this context it is worth noting the following.

(a) Eq(15), at least *formally*, looks like the difference of two adjoint systems where the divergence of current density automatically vanishes. But unlike the usual cases of linear D.E. we have not to postulate that the linear operators separately vanish, (cf. Sommerfeld[11] p. 724). In fact, $\Box\varepsilon = 0$ or $\Box\varepsilon^* = 0$ does not hold generally in our case.

(b) Sommerfeld[11] had also remarked that even in the wave mechanical case, although the linear wave equation can be expressed in terms of one wave function alone, "the wave mechanical problem is not determined by one wave equation but by a pair of equations, the original and its adjoint, p. 228". Note also this remark, "physically significant is not the individual function ε, but the pair of functions ε and ε^*, p. 49".

(c) Tolman[12] also emphasizes this point. He writes, "Just as a dual specification of coordinates and velocity is necessary to determine the state of a system in classical mechanics, so, too, a dual specification corresponding to the two numbers specifying our complex probability amplitude is necessary to determine the state of a system. p. 192".

We assert that eq(19) gives the most general differential equation governing the motion of *any* particle whose wave field can be represented by a scalar function. Further, it contains explicitly the rest mass and the invariant charge of the particle.

This assertion gets its full *a posteriori* justification from the fact that eq(19) reduces to the well known equations of classical and of quantum physics under appropriate restricted conditions.

It should also be noted that these pilot waves are running waves. This was pointed out to us first by Einstein in a private communication. Later it was confirmed in the case of the Kepler problem, (see Section VI).

V. <u>IMPORTANT SPECIAL CASES</u>

(i) Klein-Gordon Equation

In the absence of an external field, ($\underline{p}_e = 0$), as well as for a neutral particle eq(19) reduces to K-G equation (20)

(20) $\Box\varepsilon - (\frac{m_0 c}{\hbar})^2 \varepsilon = 0$

(ii) <u>The (Scalar) Wave Equation of Optics</u>

For photons, $m_0 = e = 0$. Its pilot wave therefore satisfies the eq(21)

(21) $\nabla^2\varepsilon - \frac{1}{c^2}\frac{\partial^2\varepsilon}{\partial t^2} = 0$

(iii) <u>Equations of Wave Mechanics</u>

In wave mechanics the differential equations are obtained from the Hamiltonian by using the prescriptions of operator formalisms. In this theory we do not need such *ad hoc* recipes. Quantum mechanical recipes come out in a straightforward way from the definition of generalized four-momentum, (eq9), under special restricted conditions which also reveal their limitations, (see below). In this theory $\underline{p}$ as a function of $\vec{x}$ and t is directly connected with the wave function which is also a function of $\vec{x}$ and t by the postulate I, eq(9). It is shown below that all the equations reduce to relevant wave mechanical equations if the wave function $\varepsilon(\vec{x},t)$ is such that it satisfies the particular relation

(22) $(\underline{p}_e\cdot\underline{\nabla}\varepsilon/\varepsilon) = -(\underline{p}_e\cdot\underline{\nabla}\varepsilon^*/\varepsilon^*)$

or, equivalently,

(23) $(\underline{p}_e\cdot\underline{\nabla}a) = 0$

We can then *formally* write

(24a) $\frac{\hbar}{i}\underline{\nabla} = \underline{p}_{op}$

But one must be careful in using this recipe, since although one can write

$$(\underline{p}_e\cdot\frac{\hbar}{i}\underline{\nabla}\varepsilon) = (\frac{\hbar}{i}\underline{\nabla}\varepsilon\cdot\underline{p}_e),$$

in operator formalism one must take the correct expression as

$(\underline{p}_e\cdot\underline{p}_{op})\varepsilon$ and *not* $(\underline{p}_{op}\cdot\underline{p}_e)\varepsilon$.

Yet another apparent inconsistency of quantum mechanical operator formalism should be noted. From eqs(24a), (9) & (10), we get

(24b) $\vec{p}_N + \vec{p}_e = \vec{p} = \frac{\hbar}{i}\vec{\nabla}\varepsilon$; (24c) $H = i\hbar\frac{\partial}{\partial t}\varepsilon$

Here the spatial part $\vec{p}$ of the generalized momentum, (the canonically conjugate momentum), is represented by the 3-dimensional gradient $\frac{\hbar}{i}\vec{\nabla}\varepsilon$ and the total energy H by $i\hbar\frac{\partial}{\partial t}\varepsilon$. In wave mechanics (24c) is retained, but only the kinetic momentum $\vec{p}_N$ is set equal to $\frac{\hbar}{i}\vec{\nabla}\varepsilon$. In practice, however, this apparent loss of symmetry is remedied by another prescription, namely, that of minimal coupling where $\vec{\nabla}\varepsilon$ is replaced by $(\vec{\nabla} - \frac{e}{c}\vec{\phi})\varepsilon$ which, as seen from (2), is $\vec{p}_N$. Further, as we shall see below, $\underline{p}_{op} = \frac{\hbar}{i}\underline{\nabla}$ does not represent the space-time dependent energy-momentum operator but its Fourier transform. Moreover, even when eq(24a) is valid the wave function $\varepsilon(\vec{x},t)$ should not be identified and physically interpreted as quantum mechanical wave functions.

Under the restriction (24a) we get from (19) the relativistic Schroedinger-Gordon equation, (cf.[11], p.212),

(25) $\Box\varepsilon + \frac{1}{\hbar^2}[2\frac{\hbar}{i}(\underline{p}_e\cdot\underline{\nabla}\varepsilon) - \underline{p}_e^2\varepsilon - (m_0 c)^2\varepsilon] = 0$

i.e.

(25a) $\Box\varepsilon + \frac{1}{\hbar^2}[2(\underline{p}_e\cdot\underline{p}_{op})\varepsilon - \underline{p}_e^2\varepsilon - m_0^2 c^2\varepsilon] = 0$

For the stationary state of the wave field, we have

(26) $\frac{\partial^2\varepsilon}{\partial t^2} = -(H/\hbar)^2\varepsilon$, provided $\frac{\partial a}{\partial t} = 0$ and $\frac{\partial H}{\partial t} = 0$

Consequently eq(19) reduces to, (cf. eqs(2-6)),

$$(27)\ \nabla^2\varepsilon + \frac{1}{\hbar^2}[(\frac{H-U}{c})^2 - \vec{p}_e^{\,2} - m_0^2c^2 - 2(\vec{p}_e.\vec{p})]\varepsilon = 0$$

For an electrostatic potential, ($\vec{p}_e = 0$), eq(27) is the relativistic Schroedinger equation used by Sommerfeld [11,p.215] in investigating the fine structures of hydrogen spectrum.

Finally, using (22 and 26) and the nonrelativistic approximation $\left|\frac{M - m_0}{m_0}\right| << 1$ as well as $|\vec{p}_e| << |\vec{p}|$ we get from (27) the time independent Schroedinger equation (28)

$$(28)\ \nabla^2\varepsilon + \frac{2m_0}{\hbar^2}(E - U)\varepsilon - \frac{2i}{\hbar}(\vec{p}_e\vec{\nabla}\varepsilon) = 0$$

where $E = H - m_0c^2$.

Although the wave function $\varepsilon(\vec{x},t)$ even in these cases should not be identified with wave mechanical ψ-function (see below), we have nevertheless come to the fundamental equations of quantum physics without the help of *any recipe*.

Hence, we repeat that Einstein's conviction, namely, it is possible to explain quantum phenomena deterministically, has been proved beyond doubt quantitatively at least for a single quantum particle whose wave field can be described by a scalar function.

VI. PILOT WAVES OF KEPLER ELLIPSES

Since the general solution of eq(19) is not known, it is instructive to study the role of pilot waves ε vis-a-vis the trajectory of the particle in a simple case.

Sommerfeld ([13], pp.110;611) had investigated the electronic orbits of the hydrogen atom from nonrelativistic Hamilton-Jacobi equation and discussed the results from the standpoint of old quantum theory. Of course, old quantum-theory did not take the amplitude of the wave field into consideration. Nevertheless, we can take the solutions of the H-J equation as the phases of the pilot waves which in this case of geometrical optics satisfy the time independent equation, (see ref.[6]),

$$(29)\qquad \nabla^2\varepsilon + \frac{2m_0(E - U)}{\hbar^2}\varepsilon = 0$$

But eq(29) also represents Schroedinger equation for stationary wave fields under the influence of static external field, (cf. eq(28) for $\vec{p}_e = 0$).

The object of this study of the Kepler problem, (ref[6]) was to compare the known solutions of the eigenvalue problem with the pilot wave functions for large quantum numbers.

The phases of the pilot waves were obtained from the H-J equation (30)

$$(30)\qquad \frac{1}{2m_0}[(\frac{\partial S}{\partial r})^2 + \frac{1}{r^2}(\frac{\partial S}{\partial \phi})^2] + U = E = \text{constant}$$

where

$$\frac{\partial S}{\partial \phi} = K_\phi = \text{constant}$$

$$f(r) = \int_{r_1}^{r} \frac{\partial S}{\partial r}\,dr \quad ; \quad \left|\frac{\partial S}{\partial r}\right| = \left|\sqrt{-A + \frac{2B}{r} - \frac{K_\phi^2}{r^2}}\right|$$

$$A = -2m_0E \quad ; \quad B = m_0c^2Z \quad ; \quad \oint \frac{\partial S}{\partial r}\,dr = 2\pi K_r = \text{constant}$$

For negative energy E, the square root is zero for $r = r_1$ (perihelion distance) and $r = r_2$ (apehelion distance). f(r) is positive and imaginary for $r > r_2$. It is negative and imaginary for $0 < r < r_1$. For $r_1 < r < r_2$ it branches in two real functions with branch points at r_1 and r_2 (see[13], p. 611).

Leaving aside a constant with no physical significance the solutions of eq(30) can be written as

(31a) $$S_1(r,\phi) = K_\phi \cdot \phi + |Re(f)| + i\,Im(f)$$

(31b) $$S_2(r,\phi) = K_\phi \cdot \phi - |Re(f)| + i\,Im(f)$$

In terms of the pilot wave theory, we can then say that the "corpuscle" for increasing r is guided by the wave

(32a) $$\epsilon_1(\vec{x},t) = a_1(\vec{x},t)\exp[\frac{i}{\hbar} S_1(\vec{x})].\exp[-\frac{i}{\hbar} Ht]$$

and with decreasing r by

(32b) $$\epsilon_2(\vec{x},t) = a_2(\vec{x},t)\exp[\frac{i}{\hbar} S_2(\vec{x})].\exp[-\frac{i}{\hbar} Ht]$$

The physical meaning of the complicated mathematical relations of the solutions of eq(30) interpreted in terms of the pilot wave concept is quite simple. The surfaces of constant phases S_1 = const are orthogonal to the trajectory of the particle in which the distance of the particle from the centre increases with time, that means, as the particle moves along its elliptic orbit from the perihelion to the apehelion, whereas the trajectory from the apehelion to the perihelion in which r decreases with time is determined by surfaces of constant phases S_2 = constant.

It should be noted that the wave function exists also outside the Kepler ellipse, so that $\epsilon\epsilon^*$ cannot be taken as the measure of the density of the particle which does not exist outside the orbit.

Outside the domain swept by the orbit, the wave functions are given by

(33a) $$\epsilon_{1,2} = a_{1,2}\exp[-Im(f)/\hbar].\exp[iK_\phi\phi/\hbar]\exp[-\frac{i}{\hbar} Ht],$$ for $r > r_2$

and

(33b) $$\epsilon_{1,2} = a_{1,2}\exp[-Im(f)/\hbar]\exp[\pm i\pi K_r/\hbar]\exp[iK_\phi\phi/\hbar] \times \exp[-\frac{i}{\hbar} Ht]$$ for $r < r_1$

In these domains the orbits are circular (concentric with r = 0) and the particles would have the momentum

$$m_0 v = \frac{1}{r}\frac{\partial S}{\partial\phi} = K_\phi/r$$

But classical mechanics teaches us that in these domains particles cannot occupy such orbits since S is not real.

Interestingly enough we also get a circular orbit in the domain $r_1 < r < r_2$ perpendicular to

$$S = \frac{1}{2}(S_1 + S_2) = K_\phi \cdot \phi$$

Though now S is real, such orbits have no physical significance, because actual orbits are perpendicular to S_1 and S_2 and not to the resultant of these two functions.

A discussion of the known solutions (eigenfunctions) of the Schroedinger equation (29) for large quantum numbers show, (for details see ref[6]), that Schroedinger ψ-functions can be represented as the sum of the two pilot waves, i.e.,

(34) $$\psi = \epsilon_1 + \epsilon_2$$

It should be noted that this does not contradict the quantum mechanical result that the ground state of the hydrogen atom is nondegenerated, since the pilot waves $\epsilon_{1,2}$, contrary to eigenfunctions, show branch points and as such do not fulfill the criteria of eigenfunctions. Moreover, similar results had been obtained for the ψ-function of the continuous hydrogen spectrum. Sommerfeld, ([11], p. 115) represented the Laguerre function R as the sum of two functions $Q_{1,2}$ which become infinite at r = 0 and $Q_1 = Q_2^*$.

Hence, if we discuss physical results with the help of ψ-function we could not find consistent results for the orbit of the particle and has to fall back upon statistical considerations for a consistent interpretation of physical results, since the actual path of the particle is governed by ε_1 and ε_2 and *not* by ψ.

Although the above results have been obtained by utilising the correspondence principle, it seems that our conclusions are valid for the motion of any quantum particle. Since ψ is a mathematical construct, we believe that in the general case also it can be replaced by a real physical wave of the particle and thus the probabilistic concept of quantum mechanics can be superseded by a deterministic one.

Moreover, this investigation shows that it would not be wise to neglect nonanalytic solutions of differential equations. Rather, the nonlinear differential equation involving both ε and ε^* suggest that such nonanalytic solutions are physically important and we would miss many interesting physical results if we insist that only single valued analytic solutions of differential equations are physically admissible. Consequently, it would be desirable to find all possible solutions of eq(19), particularly in view of the fact that general analytic solution may not even exist for such equations.

REFERENCES

[1] Bagchi, S.N. (a) Philosophy of Science. Some Problems Bull. Ramkrishna Mission Institute of Culture, Calcutta, India, June & July (1975)

(b) Crisis in Physics and a tentative approach to overcome it, Annal. de la Fondation Louis de Broglie, 8, 255 (1983)

[2] Renninger, M., Zum Wellen-Korpuskel Dualismus, Z. Physik 136, 261 (1953)

[3] Sommerfeld, A., I. Vierdimensionale Vektoralgebra Ann. der Physik, 32, 749 (1910)

II. Vierdimensionale Vektoranalysis, Ibid. 33, 649 (1910)

[4] Von Laue, Max, Die Spezielle Relativitätstheorie, Fried. Vieweg & Sohn, Braunschweig (1955)

[5] Hosemann, R. & Bagchi, S.N., Quanten mechanik als Beugungsphänomen von Führungswellen
(a) Part I Z. Phys. 142, 334 (1955)
(b) Part II Ibid. 142, 347 (1955)
(c) Part III Ibid. 142, 363 (1955)

[6] Hosemann, R. & Bagchi, S.N., Korrespondenz des Kepler-Problems der Klassischen Mechanik und der Schrödingerschen Eigenfunktionen des Wasserstoffatoms, Z. Phys. 145, 65, (1956)

[7] Hosemann, R. & Bagchi, S.N., Die Berechnung der Lorentz-invarianten Struktur der Greenschen Grundlösung der Sog. allgemeinen Wellen-gleichung, Z. Phys. 139, (1954)

[8] de Broglie, Louis, Une Tentative d'Interprétation Causale et non Linéaire de la Mécanique Ondulatoire, Gauthier-Villars, Paris (1956).

[9] Brillouin, L., Relativity Re-examined, Academic Press (1970)

[10] Bagchi, S.N., Fundamental Equations of Classical and of Quantum Physics a special cases of Nonlinear Equations, Lecture Notes in Physics, Vol. 120;338 (1980), Springer Verlag

[11] Sommerfeld, A., Atombau und Sperktrallinien, Fried. Vieweg & Sohn, Braunschweig (1951)

[12] Tolman, R., The Principles of Statistical Mechanics, Oxford University Press (1955)

[13] Sommerfeld, A., Atomic Structure & Spectral Lines, Methuen & Co Ltd, London (1934)

[14] de Broglie, Louis, La Théorie des Particles de spin $\frac{1}{2}$, Gauthier-Villars, Paris (1952)

[15] Darwin, C.G., Proc. Roy. Soc. London <u>118A</u>, 654 (1928)
Eddington, A.S., Ibid. <u>121A</u>, 524 (1928)
Sauter, F., Z. Phys. <u>61</u>, 803 ; <u>64</u>, 295 (1930)

[16] Courant, R. & Hilbert, D., Methods of mathematical Physics, Vol. II, Interscience Publishers(1965)

[17] Bagchi, S.N., Hydrodynamical Approach to Quantum Physics, Proc. Ind. Association for the Cultivation of Science, Calcutta, <u>58</u>, 21 (1975)

[18] Hosemann, R. & Bagchi, S.N., Direct Analysis of Diffraction by matter, Ch.V. & Appendix, North Holland, Amsterdam (1962). See also Z. Physik <u>135</u>, 50 (1953) and <u>137</u>, 1 (1954).
It has been proved that the algebra of physically observable functions is mathematically equivalent to Schwartz-Temple approach of generalized functions see the appendix of the monograph. Nevertheless, the former is more useful for physics.