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Non-linear equations of trans-quantum physics

by S.N. BAGCHI

5550 Bellerive, Brossard,
Quebec, JAZ 3C8, Canada

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Part II

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VII. EXTENSION OF POINT MECHANICS

(i) Generalized Hamilton-Jacobi Equation

From our postulate I, eq(9), we note that the generalized four-momentum $\underline{p}$ is connected with the phase of the wave function which obeys the generalized Hamilton-Jacobi (H.J.) equation (35)

$$(35) \qquad (\underline{\nabla} W - \underline{p}_e)^2 + m_0^2 c^2 = \hbar^2 \frac{\Box a}{a}$$

Here we have used the relations (18) and (6). Eq.(35) in the case of point-mechanics, i.e. for $\Box a = 0$ (8), reduces to the well known relativistic H.J. eq. (36), [cf. eqs.(2,3,10)],

$$(36) \quad (\vec{\nabla} W - \vec{p}_e)^2 - \left(\frac{H - U}{c}\right)^2 + m_0^2 c^2 = 0$$

It is to be specially noted here that we could have derived eq.(36) from (35) by putting h = 0. But contrary to the connection between quantum mechanics and classical mechanics, this theory does not need this inadmissible condition to arrive at classical mechanics. As stated before, classical mechanics and geometrical optics can be derived from this theory by the requirement of eq.(8).

Eq.(36) shows that $W(\vec{x},t)$ is Hamilton's principal function, which can be expressed as

(37) $$W(\vec{x},t) = S(\vec{x}) - Ht$$

if the Hamiltonian $H = E + m_0c^2$ is a system constant.

Using this condition, we get the familiar non-relativistic Hamilton-Jacobi equation (38)

(38) $$\frac{(\vec{\nabla}S - \vec{p}_e)^2}{2m_0} + U = E$$

for the particular case

$$\Box a = 0 \ ; \ E = \text{constant} \ ; \ \left|\frac{m - m_0}{m_0}\right| << 1$$

It should be noted carefully that contrary to the case of classical mechanics, in order to obtain the trajectory of the quantum particle from the generalized H.J. eq.(35) one must determine the amplitude of the wave as a function of space and time from the relevant partial differential equation. Consequently, the initial value problems of classical mechanics has been changed to the boundary value problems of PDE of the microphysics.

It is obvious that the eq.(35 or 36) is a differential equation of the first order and cannot represent a wave equation. But in the hands of Schroedinger this was changed into a second order PDE by using the *recipes* of operator formalisms. Consequently, it will be worth while to investigate the physical meaning of the quantum mechanical operator formalisms.

As we have noted before, in order to be consistent, we have to define operator formalism by

(24a) $$\underline{p}_{op} = \frac{\hbar}{i}\,\underline{\nabla}$$

or, $\vec{p}_{op} = \frac{\hbar}{i}\vec{\nabla}$; $H_{op} = i\hbar\frac{\partial}{\partial t}$

Consequently, one can write

(39) $$\underline{p}^2_{op} = -\hbar^2(\underline{\nabla}.\underline{\nabla}) = -\hbar^2\Box$$

But the physical momentum, for the subsidiary condition (22) or (23), is given by

$$\underline{p} = \frac{\hbar}{i}\,\underline{\nabla}\,\varepsilon$$

and consequently

(40) $$\underline{p}^2 = (\underline{p}.\underline{p}) = -\hbar^2(\underline{\nabla}\varepsilon)^2 \neq -\hbar^2\Box\varepsilon$$

Relations (39) and (40) can be reconciled mathematically only if the quantum mechanical momentum in the coordinate language is identified as $h\underline{b}$ by Einstein-de Broglie relations. ($\underline{b}$ is the vector reciprocal to $\underline{x}$ in Fourier space). Thus, if

$$\underline{P}_{op}(\vec{x},t) = \frac{\hbar}{i}\,\underline{\nabla}(\vec{x},t) \ ;$$

$$(\underline{p}_{op}\cdot\underline{p}_{op}) = -\hbar^2(\underline{\nabla}.\underline{\nabla}) = -\hbar^2\Box \ ;$$

their Fourier transforms are given by

(41) $$\frac{\hbar}{i}(2\pi i\underline{b}) = h\underline{b}$$

and $$-\hbar^2.4\pi^2\underline{b}^2 = -h^2\,\underline{b}^2$$

Thus we clearly see that quantum mechanical expression for the momentum in the coordinate language is not the physical generalized momentum of the particle $\underline{p}(\vec{x},t)$ but its mathematical representation in the Fourier space in terms of its associated wave field.

Hence, Heisenberg's Uncertainty relation is *not* a Principle of Nature, but is the consequence of wave mechanical formalism. It is the price of representation of the physical quantity $\underline{p}(\vec{x},t)$ by its Fourier transform, as shown in the next sub-section.

Although this fact about the physical nature of quantum mechanical momentum has never been explicitly recognized and discussed, it has been implicit is the fundamental physical postulates of quantum mechanics. For example,

$$W(\vec{q},t) = \psi^*(\vec{q},t).\psi(\vec{q},t)$$

and

$$W(\vec{p},t) = \phi^*(\vec{p},t).\phi(\vec{p},t)$$

give the probability densities for finding particular values of q and p respectively at time t. But the amplitudes ψ and ϕ are related by Fourier integrals (see Tolman [12] p. 187-193).

(ii) Heisenberg's Uncertainty Relation

In actual experiments involving a collection of particles, we get the Uncertainty Relation connecting the integral widths of the centroid of the collection and its average momentum, the latter being expressed as a function of Fourier space. Obviously, this does not mean that the position and momentum, both considered as functions of space and time, cannot be determined simultaneously without errors, at least conceptually and in principle.

First, let us define the mean values and the integral widths in conventional ways, namely,

$$\bar{\underline{p}} = \frac{\int \varepsilon\varepsilon^* \underline{p} dv_{\underline{x}}}{\int \varepsilon\varepsilon^* dv_{\underline{x}}} \; ; \; \bar{\underline{x}} = \frac{\int \varepsilon\varepsilon^* \underline{x} dv_{\underline{x}}}{\int \varepsilon\varepsilon^* dv_{\underline{x}}} \; ; \; \bar{\underline{p}}^2 = \frac{\int \varepsilon\varepsilon^* \underline{p}^2 dv_{\underline{x}}}{\int \varepsilon\varepsilon^* dv_{\underline{x}}}$$

$$\bar{\underline{b}} = \frac{\int EE^* \underline{b} dv_{\underline{b}}}{\int EE^* dv_{\underline{b}}} \; ; \; \bar{\underline{b}}^2 = \frac{\int EE^* \underline{b}^2 dv_{\underline{b}}}{\int EE^* dv_{\underline{b}}} \; ; \; \overline{(\underline{b}')^2} = \frac{\int A^2 \underline{b}^2 dv_{\underline{b}}}{\int A^2 dv_{\underline{b}}}$$

$$(\delta x_j)^2 = \frac{\int \varepsilon\varepsilon^* (x_j^2 - \bar{x}_j^2) dv_{\underline{x}}}{\int \varepsilon\varepsilon^* dv_{\underline{x}}}$$

$$(\delta p_j)^2 = \frac{\int \varepsilon\varepsilon^* (p_j^2 - \bar{p}_j^2) dv_{\underline{x}}}{\int \varepsilon\varepsilon^* dv_{\underline{x}}}$$

$$(\delta b_j)^2 = \frac{\int EE^* (b_j^2 - \bar{b}_j^2) dv_{\underline{b}}}{\int EE^* dv_{\underline{b}}}$$

Here, $dv_{\underline{x}}$, $dv_{\underline{b}}$ are 4-dimensional volume elements in physical and Fourier spaces respectively. $E(\underline{b})$ and $A(\underline{b})$ are 4-dimensional Fourier transforms of $\varepsilon(\underline{x})$ and $\bar{a}(\underline{x})$.

Using the mathematical relations

$$E(\underline{b}) = F\varepsilon(\underline{x}) \equiv \int \varepsilon(\underline{x})\exp - 2\pi i(\underline{b}.\underline{x})dv_{\underline{x}}$$

$$A(b) = Fa(\underline{x}) \equiv \int a(\underline{x})\exp - 2\pi i(\underline{b}.\underline{x})dv_{\underline{x}}$$

$$F(\underline{\nabla}) = 2\pi i \, \underline{b} \quad , \quad F(\Box) = -4\pi^2 \underline{b}^2$$

$$\int \varepsilon\varepsilon^* dv_{\underline{x}} = \int EE^* dv_{\underline{b}}$$

$$\int (\varepsilon\varepsilon^* \underline{p}) dv_{\underline{x}} = h \int (EE^* \underline{b}) dv_{\underline{b}}$$

one can derive the following relations

$$\delta x_j \; \delta b_j = \beta \; ;$$

(for Gaussian distributions $\beta = 1$; for other reasonable distributions $\beta \simeq 1$)

$$\bar{\underline{p}} = h \, \bar{\underline{b}}$$

$$\delta x_j \; \delta p_j = \beta h \sqrt{1 - \overline{(b_j')^2}/(\delta b_j)^2}$$

$$\delta x_j \ \delta p_{Nj} = \beta h \sqrt{1 - \frac{b_j'^2 + [(2\delta^2 p_j \cdot p_{ej} - \delta p_{ej})^2]/\hbar^2}{(\delta b_j)^2}}$$

$$\simeq \beta h, \text{ if } |\mu - 1| << 1 \ ; \ |p_{ej}| << |p_j|$$

for detailed calculations, see ref[5b].

(iii) Lagrangian Mechanics

By virtue of the eq.(9), the generalized 4-momentum $\underline{p}(\vec{x},t)$ of the particle is a field quantity and consequently, its rate of change is given by

(42) $$\frac{d\underline{p}}{d\tau} = \underline{\nabla}(\underline{v}_c \cdot \underline{p})$$

or,

(42a) $$\frac{d\vec{p}}{dt} = \text{grad}[(\underline{v}_c \cdot \underline{p})/k_c]$$

(42b) $$\frac{dH}{dt} = - \frac{\partial}{\partial t}[(\underline{v}_c \cdot \underline{p})/k_c]$$

(The index c implies that in the differentiation the velocity of the particle $\underline{v}$ is to remain constant).

Now, regarding the Lagrangian as a function of x_j, v_j, t, let us *define* it by (43)

(43) $$L(x_j, v_j, t) = (\frac{\underline{v} \cdot \underline{p}}{k}) = - \frac{\mu m_0 c^2}{k} + (\vec{v} \cdot \vec{p}_e) - U$$

For $\mu = 1$, eq.(43) represents Schwartschild's Lagrangian.

Noting

(44) $$\frac{\partial}{\partial v_j}(\frac{1}{k}) = - \frac{kv_j}{c^2}$$

and the quantities $\vec{p}_e$, U and μ are functions of $\vec{x}$ and t, we get from (42a,b, 44 and 2)

(45) $$\frac{\partial L}{\partial \dot{x}_j} = \mu m_0 k v_j + P_{ej} = p_j \ ; \ (\dot{x}_j = v_j, \ j = 1,2,3)$$

(46) $$\frac{dp_j}{dt} = \frac{\partial L}{\partial x_j} = \frac{\partial}{\partial x_j}(\frac{\underline{v} \cdot \underline{p}_e}{k}) - \frac{m_0 c^2}{k} \frac{\partial \mu}{\partial x_j}, \quad (j = 1,2,3)$$

Differenting (45) with respect to time along the world line of the particle we get the Lagrangian equations of motion

(47) $$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}_j} - \frac{\partial L}{\partial x_j} = 0 \ ; \qquad (j = 1,2,3)$$

(iv) Hamiltonian Mechanics

Using eqs.(2-7,43) we get

(48) $$H = (\vec{p} \cdot \vec{v}) - L = U + c[(\mu m_0 c)^2 + (\vec{\nabla} W - \vec{p}_e)^2]^{\frac{1}{2}}$$

It can be directly proved that H given by (48) is a function of x_j, p_j, t, i.e.,

(49) $$\frac{\partial H}{\partial \dot{x}_j} = 0, \qquad (j = 1,2,3)$$

and it also satisfies the usual canonical equations of Hamilton, namely,

(50) $$\frac{\partial H}{\partial x_j} = - \frac{\partial p_j}{\partial t} \ ; \ \frac{\partial H}{\partial p_j} = \frac{dx_j}{dt} \ ; \ (j = 1,2,3)$$

Finally, it should be noted that both the Lagrangian, eq.(43), and the Hamiltonian, eq.(48), contain μ, i.e. the amplitude of the wave function and for $\mu = 1$ reduce to the well known expressions for point mechanics.

(v) Diffraction Force and the Total Force Acting on the Particle

From the preceeding sections it would be obvious that if we redefine the kinetic momentum of the particle $\underline{p}_N$ by eqs.(4-6) we can study the trajectory of a quantum particle deterministically following the methodology of analytical point mechanics. Of course, we have to introduce the mass factor μ in relevant expressions. So really speaking, we have to determine the amplitude of the wave function, i.e., solve the relevant PDE, to obtain the trajectory of the particle. Now, it is well known that classical and relativistic point mechanics cannot explain diffraction phenomena. So it is pertinent to enquire how the generalized analytical mechanics can account for the diffraction phenomena of a quantum particle. The answer is that contrary to usual point mechanics which does not take the amplitude of the particle into consideration, in this new analytical mechanics the amplitude of the wave field is incorporated in the formalism through the mass factor μ. We shall now show that this factor μ produces a new type of force, the diffraction force, which deviates the trajectory of the particle, even in the absence of an external field, when it passes through a slit (and also in nonstationary processes) always deterministically according to the pilot principle. Also in the absence of diffraction force $\underline{F}_D$, the force acting on the particle depends on μ, even if μ is a constant different from unity.

The four-force $\underline{F}$ is defined by

(51) $$\underline{F} = \frac{d}{d\tau}(M_0\underline{v})$$

Note that this definition is the customary one, except for the fact that the rest mass m_0 is replaced by the effective rest mass $M_0 = \mu m_0$.

Since the kinetic 4-momentum $\underline{p}_N$ is given by the difference of two field quantities $\underline{p}$ and $\underline{p}_e$, (cf. eq.2) it can itself be looked upon as a field quantity. Consequently, one can express (51) in the form

(52) $$\underline{F} = (\underline{v}\nabla)\underline{p}_N$$

From this, using the formalism of 4-dimensional vector analysis, we get

(53) $$\underline{F} = \underline{\nabla}(\underline{v}_e\cdot\underline{p}_N) - [\underline{v}[\underline{\nabla p}_N]]$$

Further, from eqs.(9&2), it follows

$$[\underline{\nabla p}] = 0$$

or,

(54) $$[\underline{\nabla p}_N] = -[\underline{\nabla p}_e]$$

So we can express the 4-force given by (51) as the sum of two types of forces, (cf. eq.53), namely,

(55) $$\underline{F} = \underline{F}_D + \underline{F}_e$$

where

$$\underline{F}_e = [\underline{v}[\underline{\nabla p}_e]]$$

is the well known Lorentz force due to the external electromagnetic field. $\underline{F}_D$ is called the diffraction 4-force where

(56) $$\underline{F}_D = \underline{\nabla}(\underline{v}_c\cdot\underline{p}_N) = -m_0c^2\,\underline{\nabla}\,\mu$$

Let us first note that in relativistic physics the spatial component of Minkowski force is k times the Newtonian force and $(\underline{v}.\underline{F}) = 0$, unless the rest mass changes. In the latter case,

$$(\underline{v}.\underline{F}) = -c^2\frac{dm_0}{d\tau}$$

According to this theory, however, the total four-force is not pseudo-orthogonal to the four-velocity even if the restmass m_0 remains unchanged, but μ changes. It can be easily verified from (51) that

$$(\underline{v}.\underline{F}) = -\frac{d}{d\tau}(M_0c^2) \tag{57}$$

Writing the diffraction 4-force $\underline{F}_D$ in an analogous fashion of the Minkowski 4-force, namely,

$$\underline{F}_D = k(\vec{F}_D + \frac{i}{c} W_D \vec{S}_0) \tag{58}$$

we get from (56)

$$\vec{F}_D = -\frac{m_0c^2}{k} \vec{\nabla}.\mu \tag{59}$$

and

$$W_D = \frac{m_0c^2}{k}\frac{\partial\mu}{\partial t} = \frac{m_0c^2}{k}\left(\frac{d\mu}{dt} - (\vec{v}.\vec{\nabla}\mu)\right) \tag{60}$$

$$= [\vec{v}.(-\frac{m_0c^2}{k}\vec{\nabla}\mu)] + \frac{m_0c^2}{k}\frac{d\mu}{dt}$$

$$= (\vec{v}.\vec{F}_D) + \frac{1}{k}\frac{dE_0'}{dt}\ ; \quad (E_0' \equiv \mu m_0c^2 = M_0c^2).$$

From eq.(60) we see that the work done per unit of time on the corpuscle by the diffraction force $\underline{F}_D$ consists of two parts :

(i) Work done per unit of time by the spatial change of μ, proportional to the gradient of μ and

(ii) due to the rate of change of its "effective internal energy" redefined by $E_0' = M_0c^2$.

Thus there is an interchange of energy between the extended wave field of the particle and its "corpuscular" domain. The total energy-momentum, however, remains conserved [cf. (14)]. The total force experienced by the particle, however, is affected by μ even if it remains constant but differs from unity. In order to see this let us consider the total four-force $\underline{F}$ given by (52)

$$\underline{F} = (v.\underline{\nabla})\underline{p}_N = k[\{(\vec{v}.\vec{\nabla})(\mu k m_0\vec{v}) + \frac{\partial}{\partial t}(\mu k m_0\vec{v})\} \tag{52}$$

$$+ i\vec{S}_0\{(\vec{v}.\vec{\nabla})(\mu k m_0 c) + \frac{\partial}{\partial t}(\mu k\ m_0 c)\}$$

$$= k[\frac{d}{dt}(\mu m_0 k\vec{v}) + i\vec{S}_0\frac{d}{dt}(\mu m_0 kc)]$$

$$= k[\mu\frac{d}{dt}(km_0\vec{v}) + km_0\vec{v}\frac{d\mu}{dt}] + \frac{i}{c}\vec{S}_0 k[\mu\frac{dE}{dt} + E\frac{d\mu}{dt}]$$

$$\underline{F} = k[\mu\vec{F}_N + km_0\vec{v}\frac{d\mu}{dt}] + \frac{i}{c}\vec{S}_0 k[\mu\frac{dE}{dt} + E\frac{d\mu}{dt}] \tag{61}$$

where $E = mc^2$, $m = km_0$.

We thus see that not only the Newtonian force $\vec{F}_N$ acting on the particle is multiplied by the factor μ, the effective force on the particle depends on $\frac{d\mu}{dt}$ as well. Further, even if μ remains constant but differs from unity so that its gradient vanishes, the force experienced by the particle depends on μ. Consequently, in spite of the relation (59) it would be misleading to characterise μ as the "quantum potential". In view of (61) it would be more appropriate to designate μ as the mass factor.

Also, the work per unit of time done on the particle by the total force depends on both μ and $\frac{d\mu}{dt}$.

As expected, everything returns to the familiar expressions of point mechanics for $\mu = 1$.

Finally, from the above discussions of the section VII, we can conclude that the limitation of the methodology of the conventional point mechanics is *solely* due to the fact that it did not take into account the amplitude of the wave field associated with a particle although its phase became a powerful tool in the hands of Hamilton and Jacobi. Previously, the phase was only a mathematical artifice for calculations without any physical significance. But microphysics changed this situation. Post quantum physics must take into account both the amplitude and the phase of the wave field associated with a particle as real physically meaningful quantities.

VIII. ENSEMBLE DESCRIPTION, NORMALIZATION, CURRENT DENSITY

(i) Ensemble of Particles and the Normalization of the Wave function

In classical physics usually we deal with a collection of particles or photons. Since we have set up a theory only for a single particle, in order to discuss the properties of the collective we have to consider only incoherent collection of similar particles. The pilot wave associated with each particle must therefore be the same except for statistically independent phase differences. In order to follow the trajectory of each corpuscle we can utilise the physical interpretation of classical H.J. formalism. Knowing the phase $W(\vec{x},t)$, the trajectory of each (classical) particle is determined as soon as one fixes its initial position and draw its worldline perpendicular to the surfaces of constant phases at different times. The different particles follow their world lines starting from their different initial positions. Here also we follow the world tube containing the corpuscle, i.e., the "singular domain" of its wave function. The observed properties of such an incoherent cloud of particles can be determined more conveniently if we normalize the wave function in the appropriate way. That is, instead of a single particle we deal with an initially fixed density distribution $\rho(\vec{x})$ proportional to the square of the amplitude of the wave function within a 3-dimensional time-like hypersurface, the physical volume V. Thus, we set at $t = t_0$

(62) $\rho(\vec{x}) = \gamma^2 \varepsilon\varepsilon^*$, ($\gamma^2$ = normalization constant)

In order to choose γ^2 and ρ for different cases of interest, we have first to study the energy-momentum current density both for the generalized 4-momentum $\underline{p}$ and for the kinetic 4-momentum $\underline{p}_N$ of the "corpuscle".

From postulate II, eq.(14) and eq.(2), we have

(63) $\underline{\nabla}.(\varepsilon\varepsilon^* \underline{p}) = \underline{\nabla}.(\varepsilon\varepsilon^* \underline{p}_N) + a^2(\underline{\nabla}.\underline{p}_e) + 2a(\underline{\nabla}a.\underline{p}_e) = 0$

Obviously, the current density of the energy-momentum of the "corpuscle" $\underline{p}_N$ is conserved *only* when the quantum mechanical operator formalism, eq.(23), is applicable, (provided the external field potential is chosen to satisfy the Lorentz gauge condition). Otherwise, eq.(64)

(64) $\underline{\nabla}.(\varepsilon\varepsilon^* \underline{p}_N) = 0$

remains valid either when the external field is zero, (i.e. $\underline{p}_e = 0$) or when $\frac{\partial a}{\partial t} = 0$ and $\vec{p}_e = 0$. That means, in the presence of an external field the energy-momentum of the "corpuscle" is conserved only in stationary states and in the particular rest system in which the sources generating the field lie at rest.

Hence, in physically interpreting the usual quantum mechanical equations and the corresponding ψ-functions one should be very careful if one desires to avoid confusions, (see below, section VIII(ii)).

It should be noted that in the general case the invalidity of eq.(64) does not mean that the energy-momentum of the "corpuscle" is lost. It only shows that there is an interchange of energy-momentum between the "singular domain" of the wave field and its extended part lying outside it. One should not forget that in investigating the properties of a quantum particle, the physical reality is the entire wave field and, as postulated, the generalized energy-momentum is always conserved, (see eq.(14)).

Before investigating the properties of an ensemble, we first note that the general continuity condition, eq.(14), can be transformed, by Gauss theorem, into an integral over a closed 3-dimensional hypersurface,

$$\int \varepsilon\varepsilon^* \underline{p}\, dS = 0$$

Now, let us choose this 3-dimensional hypersurface S as a 3-dimensional physical volume V and a world tube parallel to $\underline{p}$. Then, it follows

(65) $\int_{t_0} \varepsilon\varepsilon^* H\, dv = \int_{t_1} \varepsilon\varepsilon^* H\, dv = C_0$, (a constant for all time).

This means that the same amount of energy entering the volume V at time t_0 is also leaving it at time t_1.

For a collection of N particles we can choose the normalization constant $\gamma^2 = \frac{N}{C_0}$ (66) so that from eq. (65) we get

(67) $$\int \gamma^2 \varepsilon\varepsilon^* H\, dv = N$$

which means that the total energy of the system as well as the number of particles remains constant for all time.

If the function H remains constant for all time, we can choose

(68a) $$\gamma^2 = \frac{N}{C_0 H}$$

and define

(68b) $$\rho = N\, \varepsilon\varepsilon^*$$

so that

(68c) $$\int \rho\, dv = N$$

ρ is the density of the collection of N particles. For a single particle, one can choose

(69a) $$\gamma^2 = \frac{1}{C_0 H}$$

and

(69b) $$\rho = \varepsilon\varepsilon^*$$

so that

(69c) $$\int \rho\, dv = 1$$

It should be particularly noted that we have used the condition H = constant in deriving the eqs.(68 and 69). Thus in general $\varepsilon\varepsilon^*$ cannot be taken as a measure of density of particles.

Consequently, this not only points out the limitations of wave mechanical formalism, but also shows, as we have noted before, that ε cannot be identified with wave mechanical ψ-function. The correct interpretation of ψ for a single particle has been *inferred* from the study of the Kepler problem as the resultant of the pilot waves guiding the particle (cf.VI).

(ii) Wave Mechanical Current Density

Consider a collection of particles, ($m_0 \neq 0$), with the normalization constant (68a). [For a single particle we have to choose (69a)]. Set

(70) $$\gamma\varepsilon = \chi$$

and

(71) $$\frac{\gamma^2 \varepsilon\varepsilon^* \underline{p}_N}{m_0} = \underline{J}$$

The expression (71) obviously represents the current density of the corpuscles flowing through a world tube lying parallel to $\underline{p}_N$ field.

From eqs.(2,3 and 9), we then get

(72) $$\underline{J} = \frac{h}{4\pi i m_0}(\chi^* \underline{\nabla}\chi - \chi\underline{\nabla}\chi^*) - \frac{e}{m_0 c}\chi\chi^* \underline{\phi}$$

This expression is nothing but the usual quantum mechanical expression for four-current density provided we identify χ with wave mechanical ψ.

Using eqs.(2,9 and 14) as well as the Lorentz condition for $\underline{\phi}$, we obtain

(73) $$\underline{\nabla}.\underline{J} = -\frac{2|\chi|e}{m_0 c}(\underline{\phi}.\underline{\nabla}|\chi|)$$

Consequently, we see that it vanishes only if any one of the following conditions is satisfied :

a) There is no external field i.e. $\underline{\phi} = 0$

b) There exists an inertial system for which $\partial a/\partial t = 0$; $\vec{\phi}$.grad a = 0 for all $\vec{x}$ and t. Note that this is the condition (23) for the validity of the wave mechanical operator formalisms.

c) There exists an inertial system for which either $\phi_0 = 0$ or |grad χ| = 0 for all $\vec{x}$ and t.

In the presence of an external field, only when the condition (b) is fulfilled we can express the intensity I registered by the beam of particles as, (cf. eq.86)

(74) $I = \text{constant}\ \chi.\chi^* |\vec{v}| E_N = \text{constant}\ \chi.\chi^* |\text{grad } W - e\vec{\phi}|$

In the general case for ensemble normalization we have $\int \chi\chi^* dv = N = \text{constant}$ but $\chi\chi^*$ represents neither the density of the particles nor the intensity registered by them on a measuring instrument. The intensity as seen from (74) depends not merely on the square of the amplitude but also on the phase of the wave field.

The nonvanishing of (73) in the general case does not mean that some of the particles are lost. It means, as we have noted before, that in such cases there is an exchange of energy and momentum between the "corpuscle" and its extended wave outside the singular domain. The current density of the generalized energy-momentum and the number of particles in a world tube lying parallel to $\underline{p}$ field remain constant since

(75) $\underline{\nabla}.J' = \underline{\nabla} . \frac{\gamma^2}{m_0} \varepsilon\varepsilon^* \underline{p} = 0$, (cf. eq(14))

IX. PHOTONS AND THEIR PILOT WAVES

(i) Corpuscular Properties of Photon in Terms of its Pilot Wave

For $m_0 = e = 0$, we have already deduced from the generally valid equation for a scalar field, eq.(19), the equation valid for the pilot wave of the photon, namely, the (scalar) equation of wave optics (21) $\Box\varepsilon = 0$.

But as yet we have not investigated the corpuscular properties of the photon and their relations with its pilot wave.

First, we note that the momentum and energy of the photon are given by

(76) $\vec{p}_N = \mu\, m_0\, k\vec{v} \neq 0$

$E_N = \mu\, m_0\, kc^2 \neq 0$

although $m_0 = 0$.

From the definition of M_0, eqs.(5 and 6), it is easily shown that its "effective rest mass"

(77) $M_0 = \frac{\hbar}{c}\sqrt{\frac{\Box a}{a}}$

Hamilton-Jacobi equation (35), (because e = 0), becomes

(78) $(\vec{\nabla}W)^2 - \frac{1}{c^2}(\frac{\partial W}{\partial t})^2 = \hbar^2 \frac{\Box a}{a}$

As expected, for $\Box a = 0$ it gives the classical "eikonal equation" used to study geometrical optics.

Using (78), we can also verify from eqs.(12 & 13) that

$m_0 = 0$ and $M_0 = \frac{\hbar}{c}\sqrt{\frac{\Box a}{a}}$

From eqs.(43 & 6) we get the Lagrangian for a photon as

(79) $L = -\frac{\hbar}{c}\sqrt{\frac{\Box a}{a}}$

Its gradient gives the diffraction $\vec{F}_D$ acting on the photon when it passes through the slit

(80) $\vec{F}_D = -\frac{\hbar c}{k}\vec{\nabla}(\sqrt{-\frac{\Box a}{a}})$

The same expression (80) is also obtained from $\vec{F}_D$ given by eq.(59).

The Hamiltonian is given by (cf. eqs.(48 & 4)

(81) $$H = c\sqrt{(\vec{\nabla}W)^2 - \hbar^2 \frac{\Box a}{a}} = Mc^2$$

(ii) Einstein-de Broglie Relations

De Broglie had shown that the phase velocity of "matter wave" is a space-like vector and the particle velocity $\underline{v}$ is the group velocity.

The phase velocity $\underline{u}$ is determined by

(82) $$\frac{dW}{d\tau} = (\underline{u}\nabla)W = 0$$

or,

(83) $$(\vec{u}.\vec{p}) - H = 0, \text{ (cf. eq.(10))}$$

Since $\vec{p}$ is orthogonal to momentary surfaces of constant phase, we have

(84) $$u_p = \frac{H}{|\vec{p}|} = \frac{\mu m_0 kc^2 + U}{|\mu m_0 k\vec{v} + \vec{p}_e|}$$

Eq.(84) is known as de Broglie's reciprocal relation.

For photons eq.(84) gives

(85) $$u_p = \frac{c^2}{|\vec{v}|}$$

just as in the case of a free particle of non vanishing mass. $\vec{v}$ is the velocity of the photon which as we shall se below is not always equal to c.

Remembering eqs.(2-6 & 10) we can express the velocity $\vec{v}$ of any particle by the expression

(86) $$|\vec{v}| = \frac{c^2\vec{p}_N}{E_N} = -c^2 \frac{\vec{\nabla}W - e\vec{\phi}}{\frac{\partial W}{\partial t} + e\phi_0}$$

which for the case of photons reduces to

(87) $$|\vec{v}| = -c^2 \frac{\vec{\nabla}W}{\partial W/\partial t}$$

Eq.(86) is known as the "formule du guidage" cf. de Broglie.

For any particle, expanding the phase W of its associated wave at any world point $\underline{x}_a$ up to the first order of approximation, its wave function ε in the neighbourhood $\underline{x}_a$ can be represented by

$$\varepsilon(\underline{x}_a + d\underline{x}) \approx a(\underline{x}_a)\exp \frac{i}{\hbar}[W(\underline{x}_a) + (d\underline{x}.\underline{p}(x_a))]$$

This means that we can express the wave field of the particle around the point $\underline{x}_a$ approximately as a plane wave with the wave lentgh λ_a and frequency ν_a given by the famous Einstein-de Broglie relations, namely

(88) $$\nu_a \approx \frac{H(\underline{x}_a)}{h} \quad ; \quad \lambda_a \approx \frac{h}{|\vec{p}(\underline{x}_a)|}$$

Therefore, strictly speaking, these relations hold exactly for a monochromatic plane wave. Of course, these relations hold also for the pilot wave of a photon.

(iii) Diffraction of the Pilot Wave of a Photon

It is well known that in Fraunhofer diffraction, in contrast to Fresnel diffraction, the relative intensity distribution depends only on the scattering angle and not on the distance of the screen from the scatterer. Consequently, photons after being diffracted by the slit and after crossing the Fresnel zone completely travel in straight lines in the Fraunhofer domain. It is therefore obvious that in this domain photons do not experience diffraction force, (cf. eq.80). Consequently, we shall characterize the Fraunhofer zone, as in geometrical optics, by the relation

(8) $$\Box a = 0$$

As a result, M_0 also vanishes (cf. eq.77) and the velocity of the photon corpuscle $|\vec{v}| = c$; otherwise $\vec{p}_N$ and E_N would also vanish for photons (cf. eq. 76) contrary to experimental results.

Hence, from eqs.(9 & 13) we get

(89) $$M_0 = \frac{1}{c}\sqrt{-(\underline{\nabla W})^2} = 0$$

Consequently, $\underline{\nabla W}$ is a light-like vector.

From eq.(12) we can also corroborate that the rest mass of the photon corpuscle expressed in terms of its pilot wave is zero :

(90) $$m_0 = \frac{1}{c}\sqrt{\frac{\Box a}{a}\hbar^2 - (\underline{\nabla W})^2} = 0$$

Evidently, in the Fresnel zone $\Box a$ cannot be zero. From (77) we see that in this domain $\Box a$ must be negative, i.e.

(91) $$\Box a < 0,$$

otherewise M_0 would be imaginary. Therefore for a real positive M_0, as seen from eq.(89) $\underline{\nabla W}$ must be a time-like vector. Consequently, the velocity of the photon "corpuscle" in the Fresnel zone must be less than c and the phase velocity of the pilot wave is greater than c, just like the case of the particle of rest mass $m_0 \neq 0$, (cf. eq.85).

(92) $$|\vec{v}| < c \quad ; \quad |\vec{u}| > c, \text{ in the Fresnel zone.}$$

It will be shown below that in the Fresnel zone the intensity of the light beam is not proportional to the square of the amplitude of the wave function but is given by

(93) $$I = \text{constant } \varepsilon\varepsilon^* |\vec{\nabla}W|$$

It would be worth while to verify these relations, (92 & 93) experimentally.

(iv) <u>Photons and Classical Electromagnetic Radiation</u>

For photons the generalized 4-momentum $\underline{p}$ is the same as the kinetic 4-momentum $\underline{p}_N$, (cf. eqs.(2-6)). Consequently, the postulate II, eq.(14) becomes

(94) $$\vec{\nabla}.(\varepsilon\varepsilon^*\vec{p}_N) + \frac{1}{c^2}\frac{\partial(\varepsilon\varepsilon^* E_n)}{\partial t} = 0$$

Since classical electromagnetic radiation is now supposed to consist of a collection of photons, let us first utilize the ensemble normalization eq.(66) which in this case becomes, (cf. eq.(65)),

(95) $$\gamma^2 = \frac{N}{C_0}, \text{ where } C_0 = \int \varepsilon\varepsilon^* E_N dv$$

The continuity condition for a collection of photons given by eq.(94) can then be expressed as, (cf. eqs.(76)),

(96) $$\vec{\nabla}.(\rho\vec{v}) + \frac{\partial\rho}{\partial t} = 0,$$

where

(97) $$\rho = k\rho_0 \; ; \; \rho_0 = \gamma^2\varepsilon\varepsilon^*\mu m_0 c^2$$

ρ denotes the number density of photons in the volume element dv ; ρ_0 the proper density and $\int \rho dv = N$, the total number of photons in the ensemble lying within a world tube parallel to $\underline{p}_N$-field.

If we use instead the normalization constant γ^2 and the density ρ given by eqs.(68a,b) we can reformulate the continuity condition (94) as

(98) $$\vec{\nabla}.\vec{S} + \frac{1}{c^2}\frac{dE}{dt} = 0$$

where

(99) $$\vec{S} = \rho\,\vec{p}_N \quad ; \quad E = \rho E_N \quad ; \quad \rho = N\varepsilon\varepsilon^*$$

Eq.(98) can then be interpreted as the continuity condition for the electromagnetic radiation where $\vec{S}$ represents the Poynting's vector.

However, it must be noted that since eq.(98) is obtained by using the normalization constant given by (68a), it is valid only in the inertial system in which E is a constant for all $\vec{x}$ and t, contrary to the continuity condition (94) which is valid generally. The restriction E = constant means that we are dealing either with a monochromatic radiation or with a polychromatic radiation in which the number of photons of each frequency remains constant. Hence this restriction is practically of no significance for the validity of eq.(98). The real restriction for its general validity, contrary to eq.(94), is due to the subsidiary condition $\rho = N\varepsilon\varepsilon^*$ (cf. 99), which signifies that the intensity of radiation is proportional to the square of the amplitude of the pilot wave of photons. But we have remarked before that this is not quite true in the Fresnel zone, (cf. eq.(93)), which we now prove.

Evidently, $\gamma^2\varepsilon\varepsilon^* \underline{p}_N$ is a measure of the energy-momentum density of a collection of photons registered by the instrument placed normal to the world tube lying parallel to $\underline{p}_N$. The intensity I of a beam of photons is given by (cf. eq. (87))

$$(93)\qquad I = \text{constant } \varepsilon\varepsilon^* |\vec{v}| E_N = \text{constant } \varepsilon\varepsilon^* |\vec{\nabla} W|$$

Thus, contrary to the Fraunhofer zone*, in the Fresnel zone, the intensity of radiation is not proportional to the square of the amplitude of the wave field but also depends on the gradient of the phase of the pilot wave function of the photons.

*Note that in Fraunhofer zone, $|\vec{v}| = c$; $\vec{\nabla} W = -\frac{1}{c}\frac{\partial W}{\partial t} = \frac{E_N}{c}$ (cf. eq.(89), eq.(93) becomes $I = \text{constant } \varepsilon\varepsilon^*$ (93a).

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It has been proved that the algebra of physically observable functions is mathematically equivalent to Schwartz-Temple approach of generalized functions see the appendix of the monograph. Nevertheless, the former is more useful for physics.