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NON-LINEAR EQUATIONS OF TRANS-QUANTUM PHYSICS

by

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PART III

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. EQUATION FOR THE VECTOR FIELD OF A PARTICLE

i) Definitions

Let the wave function associated with a particle e represented by the four-vector $\underline{\varepsilon}(\underline{x})$ defined by

100) $$\underline{\varepsilon}(\underline{x}) = \underline{A}(\underline{x}) \exp i\, W(\underline{x})/\hbar$$

e would like to express the four-momentum density+ $\underline{\varepsilon}.\underline{\varepsilon}^*\underline{P}(\underline{x})$ s a function of the wave field $\underline{\varepsilon}(\underline{x})$, (and *not* by the gradient f its phase W).

onsequently, we define it, (analogous to eq.(11)), by

.01) $$\underline{\varepsilon}.\underline{\varepsilon}^*\underline{P} = \frac{\hbar}{2i}[(\underline{\varepsilon}^*.\underline{\nabla})\underline{\varepsilon} - (\underline{\varepsilon}.\underline{\nabla})\underline{\varepsilon}^*]$$

Ie shall see below that in order to avoid confusion with sym-ls used previously, we are to express the generalized 4-mo-ntum of the vector field by a new symbol $\underline{P}$, (instead of = $\underline{\nabla}W$ used for the scalar field, eq.(9)).

where

(102)+ $\underline{\varepsilon}^*(\underline{x}) = \underline{A}(\underline{x})\exp - iW(\underline{x})/\hbar$

Using the 4-dimensional vector calculus, one can write eq. (101) in the form

(103) $\underline{\varepsilon}.\underline{\varepsilon}^*\underline{P} = \frac{\hbar}{2i}[\underline{\varepsilon}^*(\underline{\nabla}.\underline{\varepsilon}) - \underline{\varepsilon}(\underline{\nabla}.\underline{\varepsilon}^*)]$,

(since $[\underline{\varepsilon\varepsilon}^*] = 0$,

(104) $= (\underline{A}.\underline{p})\underline{A}$

(105) $= \underline{A}^2.\underline{p} + [\underline{A}[\underline{Ap}]]$

(106) $= \underline{A}^2.\underline{p} + \underline{A}^2.\underline{\omega}$

or,

(107) $\underline{P} = \underline{p} + \underline{\omega}$

where

(108) $\underline{p} = \underline{\nabla}W$

and

(109) $\underline{\omega} = [\underline{s}_A[\underline{s}_A\underline{p}]]$

($\underline{s}_A$ is the unit vector along $\underline{A}$).

+It is to be noted that, (contrary to the scalar case), $\underline{\varepsilon}^*(\underline{x})$ defined in this way is not the mathematical complex conjugate of $\underline{\varepsilon}(\underline{x})$. Since the scalar product of two A-vectors must be Lorentz invariant we cannot write $\underline{\varepsilon}^*(\underline{x}) = \underline{A}^*(\underline{x})\exp - iW(\underline{x})/h$. In our notation 4-vectors have the imaginary time components and therefore $\underline{A}.\underline{A}^*$ is not Lorentz invariant where $\underline{A}(\underline{x}) = \vec{A} + i\,A_0$; $\underline{A}^* = \vec{A} - i\,A_0$ and $\vec{A}$, A_0 are real quantities.

From (104) we note that the vector $\underline{P}$ is parallel to $\underline{A}$. Since a general vector consists of an irrotational part and a rotational part, $\underline{\omega}$ must be the rotational part of $\underline{P}$. As before, we associate $\underline{p} = \underline{\nabla}W$ with the translatory 4-momentum of the "corpuscle" given by equations (2-5), but the scalar mass factor μ should now be expressed as

(110) $$\mu = \left(1 - \frac{(\underline{A}\,\square\,\underline{A})}{\underline{A}^2} \cdot \left[\frac{\hbar}{m_0 c}\right]^2\right)^{\frac{1}{2}}$$

We associate the corpuscular properties of the particle with those of the vortical domain of the wave field having the linear dimension of the order of the Compton wave length of the particle. The total energy of the wave field is practically concentrated in this domain. For a particle at rest the centre of the vortex also remains at rest, though the stationary wave field circulates around it. It will be seen later that the equation for Dirac "point electron" can be represented by averaging the equation valid for this vortical wave.

The function $\underline{\omega}(\underline{x})$ is then to be associated with the linear 4-momentum of the wave field due to its vortical motion so that $[\underline{r}_0\underline{\omega}]$ would represent the "intrinsic" 4-angular momentum of the particle where $\underline{r}_0$ is the 4-distance from the centre of the vortex to the world point of the vortex concerned.

(ii) Differential Equations

In order to obtain the relevant PDE we again postulate the continuity condition (111)

(111) $\underline{\nabla}.(\underline{\varepsilon}.\underline{\varepsilon}^*\underline{P}) = 0$

Substituting here the expression (103) for $\underline{\varepsilon\varepsilon^*P}$ and using the relevant formulae for 4-vectors, we get the most general differential equation (112).

(112) $\underline{\varepsilon}^*.\square\underline{\varepsilon} + \underline{\varepsilon}^*.[\underline{\nabla}[\underline{\nabla\varepsilon}]] - \underline{\varepsilon}.\square\underline{\varepsilon}^* - \underline{\varepsilon}.[\underline{\nabla}[\underline{\nabla\varepsilon}^*]] = 0$

Since the divergence of the vortical field must be zero, i.e.,

$$(113) \qquad \underline{\nabla}.(\underline{A}^2.\underline{\omega}) = 0$$

it follows that the continuity condition applies separately to the translational part also, so that

$$(114) \qquad \underline{\nabla}.(\underline{A}^2\underline{p}) = 0$$

That means, the "intrinsic" circulation motion of the wave field represented by $\underline{\omega}(x)$ is independent of the linear translational motion given by $\underline{p}(\underline{x})$. In the case of the rotational motion of the particle itself, (i.e. of the rotational motion of the centroid of the vortex and the accompanying wave field), we should expect that the "intrinsic" angular momentum would be coupled -with the "orbital" angular momentum of the particle.

The eq.(112) again does not contain explicitly any of the characteristic properties of the particle. In order to obtain a PDE containing the specific properties of a given particle, we verify by direct calculation, provided (114) holds,

$$(115) \qquad \underline{\varepsilon}^*.\ \underline{\varepsilon} = \underline{A}.\Box\underline{A} - \frac{1}{\hbar^2}\ \underline{A}^2.\underline{p}^2$$

Substituting here the relation, (cf. eqs 2-5)

$$\underline{p}_N^2 = (\underline{p} - \underline{p}_e)^2$$

or,
$$\underline{p}^2 = 2(\underline{p}_e.\underline{p}) - \underline{p}_e^2 - \mu^2\ m_0^2\ c^2$$

and the value of μ given by (110), we obtain

$$(116a) \qquad \underline{\varepsilon}^*.\Box\underline{\varepsilon} + \frac{1}{\hbar^2}[2(\underline{p}_e.\underline{p}) - \underline{p}_e^2 - m_0^2\ c^2]\underline{\varepsilon}.\underline{\varepsilon}^* = 0$$

Since for the existence of a particle neither $\underline{\varepsilon}$ nor $\underline{\varepsilon}^*$ can vanish, we have in general

$$(116) \qquad \Box\underline{\varepsilon} + \frac{1}{\hbar^2}[2(p_e.p) - p_e^2 - m_0^2\ c^2]\underline{\varepsilon} = 0$$

Finally, using the relation (107) we get

$$(117) \qquad \Box\underline{\varepsilon} + \frac{1}{\hbar^2}[2(\underline{p}_e.\underline{P}) - \underline{p}_e^2 - m_0^2c^2]\underline{\varepsilon} = \frac{2}{\hbar^2}(\underline{p}_e.\underline{\omega})\underline{\varepsilon}$$

Although the eq.(117) appears as an equation containing only $\underline{\varepsilon}$, in reality because of the presence of $\underline{P}$ given by (103), it contains both $\underline{\varepsilon}$ and $\underline{\varepsilon}^*$, just as in the case of the scalar eq. (19), (note the remarks following it).

We again assert that (117) is the general equation for any single particle containing its restmass and charge whose wave field can be represented by a four-vector. As in the scalar case, we justify this assertion by deducing well known relevant equations as special cases.

XI. IMPORTANT SPECIAL CASES OF EQ.(117)

(i) Wave Equation for Photons

For photons since $m_0 = e = 0$, eq.(117) reduces to eq.(118), an equation similar to the well known equation for the four-potential in the absence of sources, i.e. of free electromagnetic field.

$$(118) \qquad \Box\underline{\varepsilon} = 0$$

Nevertheless, at the present state of our knowledge it would be premature to identify the pilot wave function $\underline{\varepsilon}(x)$ *in general* as the "self four-potential" of the field due to the particle.

Note also that eq.(118) results from (116a) when $\underline{\varepsilon}(\underline{x})$ is a light-like vector.

(ii) Proca Equation

For a neutral particle and also in the absence of an external field, (i.e. $\underline{p}_e = 0$) we get an equation of Proca type

$$(119) \qquad \Box\underline{\varepsilon} - \left(\frac{m_0^2c^2}{\hbar^2}\right)\underline{\varepsilon} = 0$$

We note that ω does not enter into this equation explicitly. We, therefore, believe that the eq.(119) is valid also for *all* particles, including electrons[+], in the absence of an external field.

With reference to the long-standing controversies regarding the "spin" of free electrons (cf.[14]), let us note that neither a neutral particle nor a charged particle in the absence of an external field would show directly the effect of the "spin" of the particle. Since it is generally accepted that particles obeying eqs.(118 and 119) possess a "spin" equal to $\hbar$, it is reasonable to assume that the general solutions of these equations in which $\underline{\varepsilon}(x)$-field shows the vortical motion[++] must exhibit the "intrinsic" angular momentum of the particle. It would therefore be interesting to investigate mechanical properties of a collection of photons and of electrons which could measure the "intrinsic" angular momentum of each particle. It will also then settle the question whether the actual "intrinsic" angular momentum, (as a function of space and time), of electrons is $n\hbar$, (n = integer) as expected from the pilot wave theory. Regarding the difference between the space-time dependent "intrinsic" angular momentum of an electron and its space-time independent "spin" $\frac{\hbar}{2}$ obtained from the interaction of the electron in the presence of an external electromagnetic field, see the remarks below.

We now show that eq.(117) is also valid for electrons.

(iii) Iterated Dirac Equation

It is shown in the next subsection that the properties of Dirac "point" electron can be represented by the average vector wave field. For such an averaged field, eq.(117)

[+]It should be noted that a four-vector can be represented by a bispinor.

[++]It should be noted that plane wave solutions of these equations would not serve our purpose.

in which $\underline{P}$ is replaced by $\frac{\hbar}{i}\underline{\nabla}$ can be converted to Sommerfeld's *iterated* Dirac equation (120). It is interesting to note that in the early days of Dirac equation, many distinguished authors [15] tried to find the corresponding equivalent second order wave equation. Sommerfeld [11, p.217] seems to prefer the iterated second order equation to the original linearized first order Dirac equations.

$$(120)\quad \Box\underline{u} - \frac{2ie}{\hbar c}(\underline{\overline{\Phi}}\cdot\underline{\nabla u}) - \frac{e^2\underline{\overline{\Phi}}^2}{\hbar^2c^2}\underline{u} - \frac{m_0^2c^2}{\hbar^2}\underline{u}$$

$$= \frac{1}{\hbar}\overline{[\underline{\nabla p}_e]\cdot[s_{\alpha\beta}]}\underline{u}$$

$$= \frac{e}{\hbar c}\sum_{\alpha,\beta=1}^{3}(F_{\alpha\beta}\cdot s_{\alpha\beta})\underline{u} + \frac{e}{\hbar c}\sum_{\alpha=1}^{3}(F_{\alpha 0}\cdot s_{\alpha 0})\underline{u}$$

Note that the left-hand side of eq.(120) is the Schroedinger-Gordon equation (25) in the absence of the right-hand side, which can be considered as the correction term due to the "spin" of the particle.

Here, $\underline{u}$ represents the space-time average of $\underline{\varepsilon}$; $s_{\alpha\beta}$ the $\alpha\beta^{th}$ component of the unit antisymmetric six-vector corresponding to the four-rotation of the field, $F_{\alpha\beta}$ are the components of the field tensor given by $[\underline{\nabla\Phi}]$, namely,

$$F_{12} = H_z \;;\; F_{23} = H_x \;;\; F_{31} = H_y \;;$$

$$F_{10} = -iE_x \;;\; F_{20} = -iE_y \;;\; F_{30} = -iE_z.$$

Eq.(120) is completely equivalent to the iterated Dirac equation of Sommerfeld provided we replace the hypercomplex quantities $\gamma_\alpha\gamma_\beta$ by $s_{\alpha\beta}$. At least formally, $s_{\alpha\beta}$ has the same algebraic properties as Dirac matrices and hypercomplex quantities. Obviously, $s_{\alpha\beta}$ is physically more meaningful.

Sommerfeld has also shown that the iterated equation gives the same eigenvalues as the original linearized

Dirac equations. Eq.(120), however, being a second order PDE may in general cases provide solutions which need not be contained in the linearized first order equations, since in general a system of two first order PDE is not equivalent to a single second order PDE, (cf. ref[16] p.13). Nevertheless, it should offer all the solutions obtainable from linearized Dirac equations.

Finally, it should be noted that the actual equation for any vector field of a particle is given by eq.(117) or (112). Since $\underline{P}$ in the general case is given by eq.(113), the eq.(117) is nonlinear. Consequently, its nonanalytic and singular solutions might be of great physical interest.

(iv) Averaged Field

For simplicity, let us assume that the wave field shows a columnar vortex of radius r_0 around the centroid of the particle, (i.e., the singularity of the vortex field) at $\underline{x}_a$.

If we replace in the left hand side of eq.(117) $\underline{P}$ by $\frac{\hbar}{i}\underline{\nabla}$ and p_e^2 by $\overline{p_e^2}$ (or, if p_e remains practically constant in the vortical region) and the average field by $\underline{u}$, we have only to find the average value of the coefficient $\frac{2}{\hbar^2}(\underline{p}_e.\underline{\omega})$ on the right-hand side.

The average over the path of the vortex field at a distance $\underline{y}$ from the centre of the vortex and also over the entire vortex area is given by

$$(121)\quad \frac{2}{\hbar^2}\frac{1}{2\pi\underline{y}}\oint \underline{p}_e.\underline{\omega}\,d\underline{s}, \quad \text{(averaged over the path)}$$

$$= \frac{2}{\hbar^2}\frac{1}{2\pi\underline{y}}|\underline{\omega}|\oint \underline{p}_e.d\underline{s}, \quad \text{(since } \underline{\omega} \text{ is parallel to } d\underline{s})$$

$$= \frac{2}{\hbar^2}\frac{1}{2\pi\underline{y}}|\underline{\omega}|.\overline{[\underline{\nabla}p_e].[s_{\alpha\beta}]}.\pi\underline{y}^2, \quad \text{(averaged over the area)}$$

$$= \frac{1}{\hbar}\overline{[\underline{\nabla}p_e].[s_{\alpha\beta}]} \quad ,$$

since,

$$\oint \underline{\omega}.ds = h, \quad \text{(cf. the quantization rule for circulation)}$$

or,

$$(122)\qquad |\underline{\omega}| = \frac{\hbar}{\underline{y}}$$

It is obvious from (122) that the "intrinsic" angular momentum of a particle whose wave field can be represented by a 4-vector, is always h. This is also to be expected from the concept of the pilot wave. But thanks to the differential eq.(117) or eq.(120), this does not contradict the spectral data, since we get the correct energy eigenvalues of a Dirac electron in units of $\frac{\hbar}{2}$.

In quantum theory one usually recognizes (on the basis of correspondence principle) a canonical dynamical variable from its corresponding energy eigenvalues. Recalling this feature of quantum mechanical formalism, we can as well say without any contradiction whatsoever that whereas the "intrinsic" angular momentum of an electron as a function of space and time and in the sense of classical mechanics is nh, (n = integer), its space-time independent "spin" momentum, in the quantum mechanical sense, is n $\frac{\hbar}{2}$, (n = integer). Consequently, as noted before, a mechanical experiment to determine the actual "intrinsic" angular momentum of a free electron can be significant to avoid the confusion.

XII. CONCLUSIONS : CONJECTURES AND PROSPECTS

(i) Properties of Quantum Vortices

The detailed properties of a stable quantum particle can be studied only when all possible solutions of the nonlinear equations (112 and 117) are available. Plane wave solutions would not represent the wave field of a particle endowed with "intrinsic" angular momentum, i.e., a wave field showing vortical motion.

We also are compelled to study the effects of mutual interactions between different particles in order to understand the results of high energy collisions between "elementary" particles. Such interactions may generate unstable elementary particles. To answer these questions we have to know how the individual quantum vortices and the resulting composite wave field would behave when a vortex of the wave field belonging to one particle penetrates into the vortical domain of other particles. Superposition principle would not be applicable in this case since the equations are nonlinear and their nonanalytic solutions would be of physical interest. It is likely that such interpenetrating quantum vortices would produce a turbulent composite wave field which while coming to stable conditions would produce *unstable* elementary particles disguised as eddies. It is therefore obvious that in order to proceed further we are faced with unsolved problems of physics and of mathematics.

The properties of a vector field show that "point" particles are mathematical extrapolations without much actual physical significance in subatomic dimensions. Though as yet we do not know how to relate the charge of a particle with its wave field, it seems nevertheless entirely comprehensible why we must expect difficulties when charged particles treated as point charges penetrate the vortical domain of the given charged particle.

These are important physical problems, but at the present moment it is hardly possible to treat them quantitatively in a satisfactory fashion. Consequently, we have to fall back upon conjectures and extrapolations to explore the possible outcome of such interactions as well as some relevant properties of a quantum particle.

It is therefore interesting to know that on the basis of the vortical wave field of a particle many strange properties of a quantum particle can be made comprehensible even if we apply our existing knowledge of hydrodynamics and of quantum theory, (cf. ref[17]).

We have already remarked that the wave field of a

particle at rest would exhibit a circulatory motion and the elementary Huygen waves created out of the world aether move with the velocity c, (see ref[7]), and the "intrinsic" angular momentum of the particle is $\hbar$.

This could make the Schroedinger picture of the "trembling" motion of the stationary electron physically consistent. Also, the strange property of stationary Dirac electron at rest, namely, its velocity at rest is $\pm c$ would be physically meaningful.

It is also intriguing to note that if we assume that the binding energy of nucleons is mainly due to the binding energy of coupled parallel vortices, a simple calculation shows that an energy of 8.4 Mev is required to separate two neutral nucleons, (each of nuclear mass 1), bound in such a manner that the centre of each vortex rotates with a common radius of $r_0 = 10^{-13}$ cm. For atomic distance (10^{-8} cm) this energy already reduces to the order of 10^{-4} ev.

For some other physical properties of particles possessing vortical wave fields see ref[17]. It should however be emphasized that these plausible models have been constructed only to point out that it would be important to investigate in details the properties of such particles from the actual solutions of PDE (eqs. 112 and 117).

(ii) Universal Field Theory

The principal goal of theoretical physics is to formulate a physically meaningful and mathematically consistent field theory which would unify not only gravitational and electromagnetic fields but also fields of elementary particles. In spite of many efforts even gravitational and electromagnetic fields have not yet been satisfactorily unified. Quantum theory has not yet been successful to incorporate in its structure the theory of gravitation. In spite of notable predictive successes, present day quantum field theories suffer from the plague of infinities. It is very doubtful whether "subtraction" physics will ever be mathematically consistent and satisfactory even if we utilize the concept of physically observable functions as distinct from mathematical functions discussed in ref.[18]. It seems very unlikely that the situation will improve, at least mathematically, unless one gives up entirely the present day methodologies of field physics and starts anew from a different point of view. With this in mind I had suggested a certain approach to tackle these problems which unfortunately has not yet been formulated concretely in quantitative mathematical form (see ref[10]). Nevertheless, after much hesitation, I have ventured to put forward here such a *tentative* programme.

As shown before mass, energy-momentum and "spin" of a particle can be expressed entirely in terms of its associated wave field. But it has not yet been possible to relate its charge to its wave field. I had suggested already that just as one had to introduce a vector field to obtain the "spin" of a particle, in order to obtain the charge one has to study a non-symmetric tensor field whose torsion would be a measure of the charge.

Our first task therefore is to find an appropriate PDE for a nonsymmetric tensor field of a particle, (perhaps following a procedure similar to those adopted for scalar and vector fields). I am inclined to believe that it may lead to an equation which would reduce to the relevant equations for gravitational and electromagnetic fields and would reveal their interconnections as well.

But in order to formulate the Universal Field Theory quantitatively in a mathematically tractable but consistent way, one has to proceed in a completely different way with different hypotheses. First, we note that the wave field of a single *stable* particle should show properties which could be related to its mass, energy-momentum, spin and charge. We also note that unstable elementary particles do not occur in nature *ab initio*. They are produced when stable particles collide with one another at sufficiently high energies. Of course, unstable elementary particles once available give rise to other stable and unstable particles either by spontaneous disintegration or as a result of collisions with other particles.

We would therefore assume that only two stable "primary" particles, electrons and protons, produced all other stable and unstable "elementary" particles of our world and try to formulate a theory which would account for such productions. (We are here ignoring the extremely high average life of photons, $\sim 10^{31}$ yrs, compared to the age of our universe as well as the possibility that at the beginning of the universe only undifferentiated concentrated field of high energy density existed). Even stable particles like photons and neutrinos arise when electrons and protons are accelerated and the nucleons break down. Anti-particles of stable particles are presumably also stable, but they do not *naturally* occur in our universe. They are produced from nuclear reactions.

With these premises, a programme to formulate a Universal Field Theory would be based on the following steps.

1. To derive an appropriate PDE for a nonsymmetric second rank tensor wave field. We assume further that this would be a nonlinear hyperbolic PDE of the second order. This PDE is expected to yield equations for the gravitational field as well as the electromagnetic field of the given particle.

2. The space-time curvature of the wave field would be a measure of its mass, its circulation of its "spin", its torsion of the magnitude of the charge and its sense the sign of the charge.

3. The criteria of stability are given by the restriction that the wave field of a stable particle has a constant curvature, constant torsion and the same sense of torsion and of circulation throughout space and time.

The stability of the pertinent PDE is not expected to be affected by a linear transformation. But when this wave field is subjected to a suitable nonlinear coordinate transformation, (kinematic equivalent to the phenomenological forces of impact), the resulting equation governing the wave field becomes unstable and consequently, returns to the stable states in one or many intermediate steps.

The final stable states of the wave fields may represent the original "primary" particles and/or other stable particles like photons, neutrinos and anti-particles. All of these secondary stable particles are also characterized by the fact that their respective wave fields have constant curvature, constant torsion and the same sense of torsion and of circulation. They differ from one another in the magnitudes of curvatures and torsions and the senses of torsions and circulations.

Before returning to the final stable states the wave field may pass through intermediate states which are unstable and they would represent the wave fields of unstable elementary particles. The properties of these unstable particles would also be determined by these wave fields in a similar way to those of stable particles, only now these will change with space and time.

4. Discrete values of certain physical quantities -the so called quantized values- would be given by the eigenvalues of the corresponding nonlinear operators.

It is obvious that this programme avoids all the infinities of the present quantum field theories and offers a plausible mechanism for the production of elementary particles and of their characteristic properties. Whether it would succeed depends on the actual working out of this programme mathematically. For the moment this lies in the womb of future and I feel that it would be unwise to reject this approach and scheme as a speculation of no real physical consequence.

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It has been proved that the algebra of physically observable functions is mathematically equivalent to Schwartz-Temple approach of generalized functions see the appendix of the monograph. Nevertheless, the former is more useful for physics.