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Bloch walls, solitons, particles : an analogy

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Abstract : The analogy between Bloch walls, solitons and particles is studied in the framework of the sine-Gordon equation. The breather solution of this equation can be wiewed to be an extended oscillator moving as a whole with constant velocity. The fundamental insights of L. de Broglie (1924) leading to quantum mechanics were based on the study of extended moving oscillators. In this context we investigate the properties of the breather solution and show that a "de Broglie wavelength" can be attributed to the breather. In this simple, entirely classical case the momentum of the breather is proportional to its wave vector, and the total energy is proportional to the oscillator frequency. We distinguish between the periodic, but locally anharmonic oscillations of the breather and a harmonic plane wave of equal wave vector but constant amplitude, a distinction similar to that made by de Broglie between the classical, localised u-wave and the probabilistic ψ-wave of quantum mechanics. The present example opens the way to identify classical (three dimensional) solitons with particles.

Résumé : L'analogie entre les parois de Bloch, les solitons et les particules est étudiée dans le cadre de l'équation Sinus-Gordon. La solution oscillante (breather solution) de cette équation peut être considérée comme un oscillateur étendu se déplaçant en bloc à vitesse constante. Les vues fondamentales de L. de Broglie (1924) qui conduisirent à la mécanique quantique étaient basées sur l'étude d'oscillateurs

étendus en mouvement. Dans ce contexte, nous étudions les propriétés de la solution oscillante et montrons qu'une "longueur d'onde de de Broglie" peut être attribuée à la solution oscillante. Dans ce cas, entièrement classique, l'impulsion de la solution oscillante est proportionnelle à son vecteur d'onde et l'énergie totale est proportionnelle à la fréquence d'oscillateur. Nous faisons la distinction entre les oscillations de la solution oscillante, périodiques mais localement anharmoniques, et une onde plane harmonique de même vecteur d'onde mais avec une amplitude constante, une distinction similaire à celle faite par de Broglie entre l'onde classique u, localisée, et l'onde probabiliste ψ de la mécanique quantique. L'exemple présenté ici ouvre la voie à l'identification de solitons classiques (à 3 dimensions) avec des particules.

1. Introduction

The present paper deals with the analogy between the statics and dynamics of Bloch walls or solitons on the one hand and particles or masspoints on the other hand. This analogy was perceived early, but is still incompletely recognized and exploited [1]. The analogy has several aspects, including properties like inertia and kinetic energy, the existence of a limiting velocity, creation or annihilation of a pair of Bloch walls or particles, and, as pointed out recently [2], also wave aspects. The latter will receive special emphasis in this paper.

The common base for the description of the above phenomena is a scalar field governed by a non-linear field equation. Bloch walls are described, in the simplest case ($K/2\pi M^2 << 1$), by the sine-Gordon equation. We will restrict ourselves to this equation because it has solutions describable in closed form by elementary functions, which is very suitable to illustrate the essential aspects of the analogy. Among these solutions are the sine-Gordon soliton solution and the breather solution, which describes an oscillating field structure. In the following, we will first recall some properties of the sine-Gordon equation and its application to Bloch walls, and then treat the breather solution as a moving oscillator.

We then recall de Broglie's concept of "moving oscillator" which was the starting point of quantum mechanics, and his postulate of a localized function u(x,t). Finally, the breather solution is associated with de Broglie's u-wave, which leads to some rather remarkable results.

2. The sine-Gordon equation

We write the sine-Gordon equation in the dimensionless form

(1) $$u_{xx} - u_{tt} = \sin u.$$

The scalar u(x,t) is a function of a coordinate x and the time t. The well-known static soliton solution u(x) of eq.(1) is

(2) $$u = 2 \text{ arc } \sin[\cosh(x-x_0)]^{-1}.$$

The "site" of the soliton is the position x_0 on the x-axis.

Solution (2) describes two distinct field structures, a left-handed and a right-handed one. They are also called $\pm 2\pi$ kinks. Each single soliton of one kind is topologically stable, but a pair with opposite sign can annihilate.

Another solution of eq.(1) is the breather solution [3]

(3) $$u = 4 \text{ arc } \tan[\frac{s \sin rt}{r \cosh s(x-x_0)}],$$

which is governed by a single parameter q, with $s = \sin q$ and $r = \cos q$. The parameter q is considered to be a constant. While performing an internal oscillation with frequency r the field structure described by (3) remains localized around the position $x=x_0$. For small values of r eq.(3) can be viewed to describe a coupled, oscillating soliton-antisoliton pair. The oscillation is strongly anharmonic. Fig. 1 shows an example of this oscillation for $r = 0.02$ ($s \approx 1$). The maximum value of u(t) at $x=x_0$ is about 6.20, reached at the time t_m where $\sin rt_m = 1$. The soliton-antisoliton separation, as measured by the distance b of the two points $u = \pi$, reaches also a maximum at $t = t_m$.

Both field configurations discussed remain permanently localized at a certain position on the x-axis. To describe moving structures, we execute transformations according to

(4) $$x \longrightarrow \frac{x - \beta t}{\sqrt{1 - \beta^2}}$$

and

(5) $$t \longrightarrow \frac{t - \beta x}{\sqrt{1 - \beta^2}} .$$

The Lorentz-invariance of the sine-Gordon equation assures that the expressions obtained from (2) and (3) by the transformations (4) and (5) are solutions of (1).

3. Dynamics of Bloch walls

In this section we recall some aspects of domain wall dynamics, concentrating on infinite planar Bloch walls in magnetically "soft" uniaxial crystals. A complete treatment of domain wall and bubble dynamics has been given by Malozemoff and Slonczewski [4] and by de Leeuw et al [5]. We depart from a material having a uniaxial magnetic anisotropy energy density w_a given by

(6) $$w_a = K \sin^2\theta,$$

with $K > 0$. The (polar) angle θ between the c-axis of the crystal and the magnetization vector is assumed to depend on x and t only. The exchange energy w_e is then given, in the continuous, classical limit, by

(7) $$w_e = A\, \theta_x^2 ,$$

and the resulting total energy E is

(8) $$E = \int (w_a + w_e)dx.$$

The principle of minimum energy $\delta E = 0$ governs the static structure of the Bloch wall.

Instead of departing from a minimum energy principle we may assume a least action principle $\delta W = 0$, which yields, as a Euler equation, the sine-Gordon equation

(9) $$\theta_{xx} - \frac{1}{c_w^2}\,\theta_{tt} = \frac{K}{2A}\sin 2\theta,$$

identical with eq.(1) for $u = 2\theta$. Equation (9) can also be derived directly [6] from the Landau-Lifshitz equation, under the conditions that the damping constant vanishes and $K/2\pi M^2 \ll 1$. The limiting velocity c_w turns out to be

(10) $$c_w = 2\gamma\sqrt{2\pi A},$$

where γ is the gyromagnetic ratio. The velocity c_w is of the order of 100 m s^{-1} for practical materials. The relevant solution of eq.(9) describing a moving 180° Bloch wall is given by

(11) $$\theta = \arcsin\left[\cosh \frac{x - vt}{\sqrt{1 - \beta^2}}\right]^{-1} .$$

For static walls (v=0) the structure (11) is identical with the structure resulting from the minimum energy principle (8). The moving wall shows a contraction by a factor of $(1-\beta^2)^{-\frac{1}{2}}$. An integration analogous to eq.(8) delivers the energy of the moving wall :

(12) $$E(v) = 4\sqrt{AK}\,(1-\beta^2)^{-\frac{1}{2}}$$

The wall mass m_0 is given by [6]

(13) $$m_0 = 4\sqrt{AK}\, c_w^{-2},$$

which turns out to yield a value of the order of 10^{-10} g cm^{-2}. For low velocities, $\beta \ll 1$, eq. (12) takes the form

(14) $$E(v) \simeq 4\sqrt{AK} + \frac{1}{2} m_0 v^2 .$$

The second term of (14) can be interpreted as kinetic energy of the wall. Inertia and kinetic energy of single walls have been both observed [5].

Bloch walls behave like massive particles or mass points in classical, relativistic mechanics. For low velocities ($\beta \ll 1$) they represent Newtonian particles [7]. They exhibit inertia, kinetic energy and a limiting velocity c_w.

In addition to these particle properties Bloch walls have an internal structure which has two consequences : Firstly the structure represents a spatially extended, stable energy distribution which is free from singularities and has a finite total energy, the rest energy, and secondly the structure governs directly the interaction between two walls, without the introduction of an additional interaction constant. This built-in interaction, which is of the exponentially decaying type, has been described elsewhere [7,8] and is not further discussed here.

Summarizing, we have reviewed the structure of static as well as moving Bloch walls, and the energies associated with these structures. The energy (or rest mass) has a discrete character because it follows from a minimum principle of energy or action. Clearly the Bloch wall represents a physical realization of a sine-Gordon soliton.

4. The breather solution as a moving oscillator

In this section we treat the breather as a solution of a field theory describing particles. We rewrite the sine-Gordon equation in the form

$$u_{xx} - \frac{1}{c^2}\, u_{tt} = \frac{1}{d^2}\,\sin u. \tag{15}$$

The function $u(x,t)$ is now interpreted as a scalar physical field similar to the electric potential. Eq. (15) governs the evolution of this field in space and time. Two physical constants are introduced : the velocity of light c and a fundamental length d. We focus on the breather solution, which is now considered to be a physical object or particle moving with velocity v/c ($\beta = v/c$). The breather solution reads in general form

$$u = 4\,\mathrm{arc\,tan}\left(\frac{s\,\sin[(r/d)(c(t-t_0)-\beta x)(1-\beta^2)^{-\frac{1}{2}}]}{r\,\cosh[(s/d)(x-x_0-vt)(1-\beta^2)^{-\frac{1}{2}}]}\right), \tag{16}$$

again with a single parameter q ($s=\sin q$, $r=\cos q$). This solution follows directly from eq.(3) and the transformations (4) and (5). The parameter q determines both the oscillation frequency of the breather and its spatial extension. The constants x_0 and t_0 define the position of the breather and its phase at time $t=0$. Fig. 2 shows schematically the oscillation of the breather and its rectilinear motion during one period τ of the oscillation. As seen from eq. (16) the oscillation frequency ω_b is

$$\omega_b = \frac{rc}{d}(1-\beta^2)^{-\frac{1}{2}} \tag{17}$$

The oscillation is highly anharmonic near the center of the breather, but harmonic far from it.

The period τ is given by

$$\tau = 2\pi/\omega_b. \tag{18}$$

During the time τ the breather travels a distance equal to $v\tau$ along the x-axis. Eq. (16) also yields a wave vector k_b equal to

$$k_b = \frac{rv}{dc}(1-\beta^2)^{-\frac{1}{2}} \tag{19}$$

In the non-relativistic limit, this expression reduces to a wavelength

$$\lambda_b = \frac{2\pi cd}{r}\,\frac{1}{v}, \tag{20}$$

representing, at a fixed time, the distance between positions of equal phase on the x-axis.

If we define the "size" of the breather as the maximum separation b of the soliton-antisoliton pair, i.e. the maximum separation of the two points on the x-axis where $u = \pi$, we find from eq. (16)

$$r \simeq 2\,\exp\!\left(-\,\frac{b}{2d}\,(1-\beta)^{-\frac{1}{2}}\right). \tag{21}$$

Eq. (21) defines b in terms of r and d. The size b is small compared with the wavelength (20) ($d < b \ll \lambda_b$).

We now turn to the energy of the breather. The energy density [9] of the sine-Gordon system is given by

$$(22) \qquad w(x,t) = \frac{1}{2} G(d^2(u_x{}^2 + \frac{1}{c^2} u_t{}^2) + 4 \sin^2\frac{u}{2}).$$

A constant G with the dimension of a force is introduced in (22), so that the space-like integral, taken oven the entire x-axis, yields the total energy of the breather. For the (oscillating) breather at rest (β=0), the total energy resulting from the integration [9] is

$$(23) \qquad E_0 = 16\ sGd.$$

Most of this energy is localized in a space region of the order of the breather size b.

The energy (23) is equivalent to a rest mass $m_b = E_0/c^2$, the mass of the breather. The energy of the moving breather is

$$(24) \qquad E_b = E_0(1-\beta^2)^{-\frac{1}{2}}$$

corresponding to a momentum $p_b = E_b v/c^2$. In the non-relativistic limit ($0 < \beta \ll 1$) we therefore find a kinetic energy equal to $\frac{1}{2} m_b v^2$ for the moving breather.

We now return to the wave aspects of the breather. We have seen that the moving oscillator is governed by a frequency ω_b and a wave vector k_b. If we define a constant

$$(25) \qquad \hbar_b = \frac{16 sGd^2}{rc} = E_0 d/rc$$

with the aid of constants introduced earlier, we find, from eq. (17), (19) and (24) the following relations

$$(26) \qquad E_b = \hbar_b \omega_b$$

and

$$(27) \qquad p_b = \hbar_b k_b,$$

These relations constitute a perfect analogy to the corresponding quantum mechanical relations ; they emerge, however, from the entirely classical field solution u(x,t). The constant h_b introduced here can therefore be viewed as "Planck's constant of the breather".

5. De Broglie's u-wave

The results obtained up till now follow in a straightforward way from the exact solutions (2) and (3) of the sine-Gordon equation, which contains elementary functions only. The close analogy between the dynamics of a Bloch wall and a masspoint is evident. The breather solution reveals, apart from aspects as inertia and kinetic energy, also wave properties in striking analogy to quantum-mechanical properties. Beyond establishing a formal analogy only, we now ascribe physical significance to the results obtained by returning to the origins of quantum mechanics.

The point of departure leading to wave mechanics was de Broglie's idea of ascribing an internal oscillation to a particle [10]. De Broglie did not specify the nature of the oscillation, which he described, in the rest frame of the particle, simply by

$$(28) \qquad \psi = a \exp(i\omega t),$$

but assumed that ψ reflects some internal periodic element of the particle. Here we use ψ in this restricted sense. The frequency was assumed to be

$$(29) \qquad \omega = mc^2/\hbar,$$

where mc^2 is the rest energy of the particle, e.g. the electron. In the first instance, the amplitude a was chosen to be a constant independent of the coordinates x,y,z. De Broglie's original "Ansatz" leads, via transformations similar to eq. (4) and (5) directly to the basic wave properties of matter i.e. the wave

$$(30) \qquad \psi = a \exp \frac{i}{\hbar} (Et-px),$$

where E and p are the energy and the momentum of the electron.

In the non-relativistic limit, the wavelength of this wave reduces to $\lambda = h/mv$, the familiar de Broglie wavelength. It is important to realize that it is the phase only of the wave (30) that contains physical parameters, viz. energy and momentum, whereas the amplitude a is just an arbitrary constant.

Already in his original work, but also later [11], de Broglie postulated that to every ψ-wave of the type described there should correspond a u-wave having the same phase, but an amplitude varying in space. The amplitude should be large near the particle but small far away. De Broglie attributes a profound physical meaning to the u-wave, while considering the ψ-wave to be incomplete. Concerning the phase, both waves are equivalent, but the u-wave is thought to contain additional information on the constitution or structure of the particle. Moreover this idea of a "double solution" was expected to open the way to a realistic, classical (or causal) theory of particles on a deeper level. De Broglie also postulated that the u-wave should be the solution of some non-linear differential equation. However, he did not give an explicit example of a u-wave meeting his postulates, nor has, to the knowledge of the author, such an example emerged in the meantime. We think that the breather solution (16) represents the first explicite example of a u-wave in the sense of de Broglie.

6. Discussion

We may construct a plane wave ψ_b of constant amplitude of the form

$$\psi_b = a \exp \frac{i}{\hbar} (E_b t - p_b x), \tag{31}$$

with the aid of the energy (26) and the momentum (27) of the breather. The wave (31) can be seen as a solution of the corresponding linear "Schrödinger" equation (with "Compton wave" length d/r !). It is clear that the wave (31) describes only a partial aspect of the phenomenon (16), i.e. the wave aspect, whereas the structural and the particle properties are neglected. As in quantum mechanics, the particle aspect can be re-

covered by adopting a statistical interpretation for the wave (31). For a breather moving in a square potential well, to give an example, eigenvalues of the energy will result in a completely analogous way.

We come to the conclusion that the most fundamental description of the phenomenon "breather" or "particle" is one including the following elements : 1. a non-linear Lorentz-invariant equation. 2. extended, stable, oscillating solutions i.e. an internal structure. 3. no singularities, yet high energy concentration in a small space region. 4. discrete energy as a consequence of a least action principle. 5. a classical interpretation. (However, the incomplete "linearized" wave (or theory) leads necessarily to a statistical interpretation, as will be discussed elsewhere [12]).

We think that the above elements will turn out to be essential ingredients of a satisfactory theory of particles.

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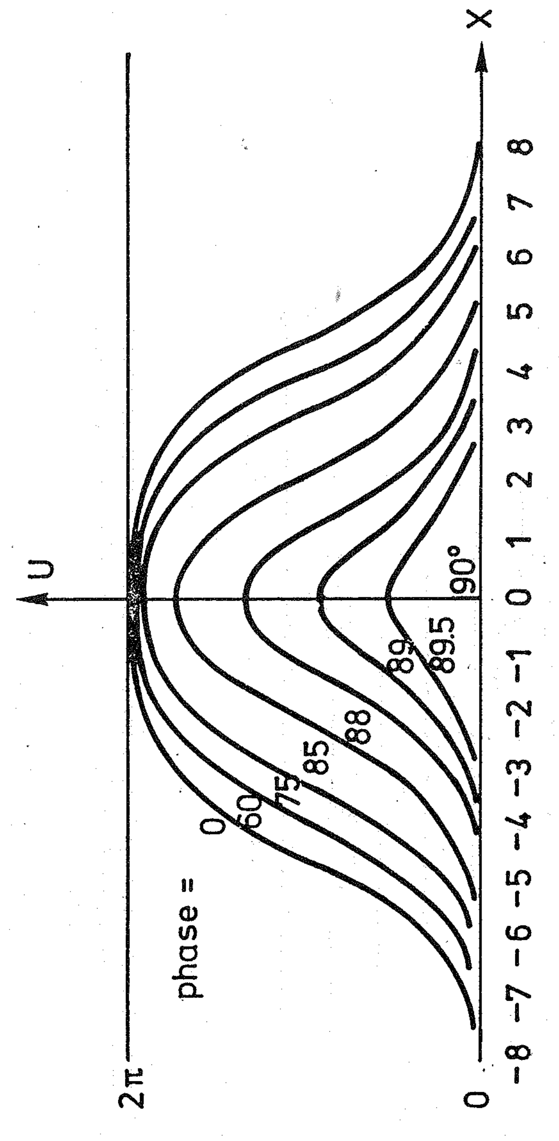

Figure 1 : The breather solution (eq. (3)) is represented by plotting u(x) for some fixed times (phases between 0 and 90°) during the first quarter of the oscillation (r=0.02).

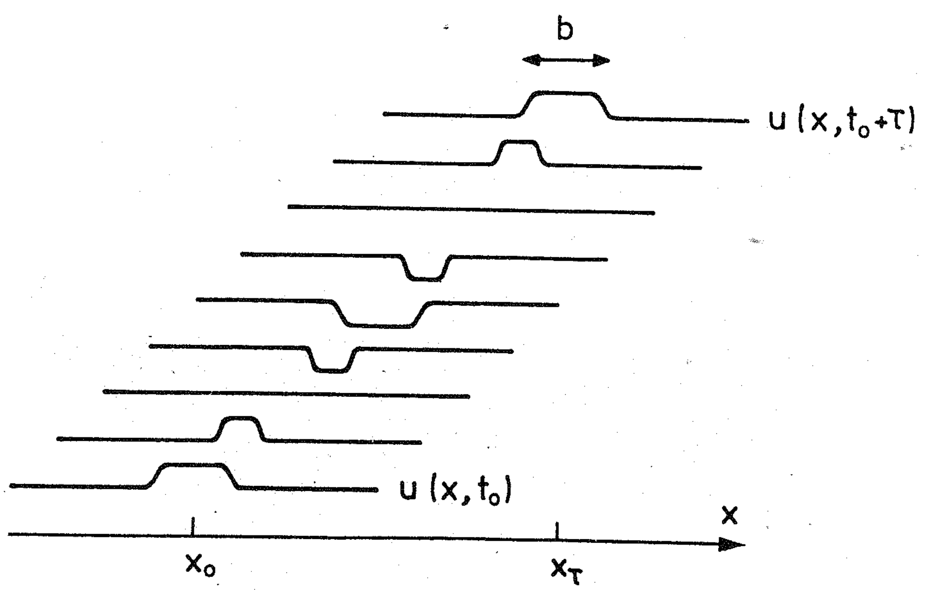

Figure 2 : Schematical representation of the moving breather described by eq. (16) for a full period of the oscillation.