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Contributions to the theories of electromagnetism and gravitation II

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IV. Gravitational analogies and relationship between electromagnetism and gravitation

1. The two fields of gravitation and gravity waves.

The *second gravitational field* $\vec{\Omega}$ (*gravitational vortex*) marks the beginning of this writer research in gravitation. It aroused surprize. Ph. Le Corbeiller told Brillouin that Bridgman [22, pp. 159-160] wondered about the role of the inertial $\vec{H}$, assuming the inertial $\vec{E}$ to correspond to gravitation. After writing my two "Comptes Rendus" notes [23] on the subject, I discovered that Heaviside [6, pp. 115-118] suggests for gravitation a set of equations very similar to Maxwell's equations for electromagnetism and my formulas. L. de Broglie [16, p. 187] gives a set of Maxwellian equations for the particle *graviton* and establishes a *unitary theory* for the Light and for the Gravitation. A few scientists resented my simple and fecund ideas. They obviously did not read the books quoted above (and others) and their discussion of experimental facts was, as Brillouin put it, "disgraceful".

My equations for gravity waves are discussed by Brillouin [1, pp. 101-103]. To complete my analogy with the Maxwell equations, I shall write

$$\text{(137)}\quad \begin{aligned} &\mathrm{curl}\,\vec{\Omega} - \frac{1}{C^2}\frac{\partial\vec{G}}{\partial t} = -\frac{4\pi\gamma}{C^2}\vec{J}_g, && \mathrm{div}\,\vec{G} = -4\pi\gamma\rho_g, \\ &\mathrm{curl}\,\vec{G} + \frac{\partial\vec{\Omega}}{\partial t} = \frac{4\pi\gamma}{C^2}\vec{J}_\omega, && \mathrm{div}\,\vec{\Omega} = -\frac{4\pi\gamma}{C^2}\rho_\omega, \end{aligned}$$

where $\vec{G}$ is the Newtonian field, $\vec{\Omega}$ is the "gravitational vortex". $\vec{J}_g$ and $\vec{J}_\omega$ are densities of gravitational currents, ρ_g and ρ_ω are densities of matter (ρ_g, the only density of matter, defined so far, in physics, is usually denoted by ρ), and γ is the Newtonian inductive capacity of matter ; in vacuum : $\gamma = \gamma_0 \approx 6.17\times10^{-11}$ M.K.S.

The velocity of propagation $C_g = C$ for gravity waves is given by

$$\text{(138)}\qquad \varepsilon_g\mu_g C_g^2 = \varepsilon\mu C^2 = 1,$$

where

$$\text{(139)}\qquad \varepsilon_g = -\frac{1}{4\sigma\gamma}, \qquad \mu_g = -\frac{4\pi\gamma}{C^2}.$$

The relation for ε_g is due to Brillouin and Lucas [24]. In vacuum : $C = c$, the velocity of light.

We define the vectors $\vec{D}_g$ and $\vec{H}_g$ of the gravity by the relations

$$\text{(140)}\qquad \vec{D}_g = \varepsilon_g\vec{G} = -\frac{1}{4\pi\gamma}\vec{G}, \qquad \vec{H}_g = \frac{1}{\mu_g}\vec{\Omega} = -\frac{C^2}{4\pi\gamma}\vec{\Omega}.$$

The *gravitational energy density* is

$$\text{(141)}\qquad \frac{1}{2}(\vec{G}.\vec{D}_g + \vec{H}_g.\vec{\Omega}) = -\frac{1}{8\pi\gamma}(G^2 + C^2\Omega^2),$$

which, *characteristic to gravitation, is a negative quantity*. This yields new mass density, hence additional ρ_{add} term :

$$\text{(142)}\qquad \rho_{add} = -\frac{1}{8\pi\gamma C^2}(G^2 + C^2\Omega^2),$$

and we end up with nonlinear equations for gravity propagation.

Let us note that in vacuum, and omitting the term in Ω^2, we obtain *Brillouin's fundamental nonlinear law for gravistatics* :

$$\text{(143)}\qquad \mathrm{div}\,\vec{D}_g = -\frac{2\pi\gamma_0}{C^2}D_g^2.$$

The reader is refered to Brillouin's book [1, pp. 87-95] for a new approach to the famous Schwartzschild problem and a brilliant discussion of equation (143).

A somewhat similar equation is obtained if one takes $\rho_\omega = C\rho_g\tan\theta$ (*see*, for comparison equation (7)) and omitts the term in G^2 in (142) ; one has

$$\text{(144)}\qquad \mathrm{div}\,\vec{\Omega} = \frac{\tan\theta}{2C}\Omega^2,$$

a problem open for discussion.

Brillouin [1, pp. 102-103] notes that electromagnetic fields create an energy density, according to a classical formula

$$\text{(145)}\qquad E_{EM} = \frac{1}{2}(\varepsilon_0E^2 + \mu_0H^2)\rho_{EM,add}C^2.$$

This electromagnetic energy density E_{EM} represents a positive mass-density $\rho_{EM,add}$ to be added to our previous negative ρ_{add} of equation (142), and this mass-density distribution in any type of electromagnetic field must generate new gravitational fields. Thus, we have a very clear indication of a simple coupling between electromagnetism and gravitation.

We shall subject the densities of currents $\vec{J}_g$, $\vec{J}_\omega$ and densities of matter ρ_g, ρ_ω to the equations :

$$\text{(146)}\quad \begin{aligned} &\operatorname{curl}\vec{J}_g - \frac{1}{C^2}\frac{\partial \vec{J}_\omega}{\partial t} = \operatorname{grad}\rho_\omega, && \frac{\partial \rho_\omega}{\partial t} + \operatorname{div}\vec{J}_\omega = 0,\\ &\operatorname{curl}\vec{J}_\omega + \frac{\partial \vec{J}_g}{\partial t} = -C^2\operatorname{grad}\rho_g, && \frac{\partial \rho_g}{\partial t} + \operatorname{div}\vec{J}_g = 0, \end{aligned}$$

which, under initial and boundary conditions, determine the quantities, $\vec{J}_g$, $\vec{J}_\omega$, ρ_g and ρ_ω. The equations (137) determine then, $\vec{G}$, $\vec{\Omega}$. All these quantities propagate with the velocity C.

We now introduce the gravity-potentials ϕ_g, ϕ_ω, $\vec{A}_g$ and $\vec{A}_\omega$ by writing

$$\text{(147)}\quad \begin{aligned} &\vec{G} = -\frac{\partial \vec{A}_g}{\partial t} - \operatorname{grad}\phi_g - C^2\operatorname{curl}\vec{A}_\omega, && \frac{1}{C^2}\frac{\partial \phi_g}{\partial t} + \operatorname{div}\vec{A}_g = 0,\\ &\vec{\Omega} = -\frac{\partial \vec{A}_\omega}{\partial t} - \operatorname{grad}\phi_\omega + \operatorname{curl}\vec{A}_g, && \frac{1}{C^2}\frac{\partial \phi_\omega}{\partial t} + \operatorname{div}\vec{A}_\omega = 0. \end{aligned}$$

In the presence of *gravitons*, we replace the equations (146) by the following

$$\text{(148)}\quad \begin{aligned} &\operatorname{curl}\vec{J}_g - \frac{1}{C^2}\frac{\partial J_\omega}{\partial t} = \operatorname{grad}\rho_\omega + \frac{C^2}{4\pi\gamma}k^2\vec{\Omega},\\ &\operatorname{curl}\vec{J}_\omega + \frac{\partial \vec{J}_g}{\partial t} = -C^2\operatorname{grad}\rho_g - \frac{C^2}{4\pi\gamma}k^2\vec{G}, \end{aligned}$$

where $k = (1/\hbar)m_{g_0}C$ is L. de Broglie's constant and m_{g_0} is the proper mass of graviton : $m_{g_0} \approx 10^{-69}$ kg.

Comparison of equations (147) and (148) yield the *L. de Broglie* [16, p. 186] *equations for the particle graviton*. These are the equations (137), where

$$\text{(149)}\quad \begin{aligned} &\vec{J}_g = \frac{C^2}{4\pi\gamma}k^2\vec{A}_g, && \rho_g = \frac{1}{4\pi\gamma}k^2\phi_g,\\ &\vec{J}_\omega = \frac{C^4}{4\pi\gamma}k^2\vec{A}_\omega, && \rho_\omega = \frac{C^2}{4\pi\gamma}\phi_\omega. \end{aligned}$$

Finally, the singular gravitational field :

$$\text{(150)}\quad \vec{G} = C\vec{\Omega},$$

with

$$\text{(151)}\quad \rho_\omega = C\rho_g,$$

lead to a non-Maxwellian system of equations analog of equations (72)-(75) and (77). All the quantities involved obey the Klein-Gordon equation.

2. The analog of the Larmor precession in gravity and the advance of planetary perihelia. The Larmor precession in magnetism is well studied especially by Brillouin [25, p. 124 *passim*] and Bates [26, p. 11 *passim*]. We here replace the electron by a massive particle of mass m moving in a fixed frame of reference (x,y,z) in the two fields of gravitation : $\vec{G} = -(\gamma_0 M/r^3)\vec{r}$ and $\vec{\Omega}$ *which is a function of time only and is perpendicular to* $\vec{r}$. The equation of motion* is

$$\text{(152)}\quad \ddot{\vec{r}} = -\frac{\gamma_0 M}{r^3}\vec{r} + \frac{1}{2}\vec{r}\times\frac{\partial \vec{\Omega}}{\partial t} + \dot{\vec{r}}\times\vec{\Omega}.$$

We have

$$\text{(153)}\quad \frac{d}{dt}(\vec{r}\times\dot{\vec{r}}) = -\frac{1}{2}r^2\frac{\partial \vec{\Omega}}{\partial t} - r\dot{r}\vec{\Omega}\ , \quad (\vec{r}.\vec{\Omega} = 0),$$

that is

$$\text{(154)}\quad \frac{d}{dt}(\vec{r}\times\dot{\vec{r}} + \frac{1}{2}r^2\vec{\Omega}) = 0.$$

Hence

$$\text{(155)}\quad \vec{r}\times\dot{\vec{r}} + \frac{1}{2}r^2\vec{\Omega} = \text{const.} = \vec{L}.$$

*At the surface of the Earth : $(\gamma_0 M/r^3)\vec{r} = g\vec{k}$, $\vec{k}$ being the vertical upwards. Assuming $\vec{\Omega}$ = const. = $-(1/2)$ (Earth's angular velocity), equation (152) gives the motion of a free particle relative to the Earth's surface.

In polar coordinates (r,θ), we have

(156) $$r^2(\dot\theta + \frac{1}{2}\Omega) = L.$$

Thus, *the angular momentum per unit mass :* $r^2[\dot\theta+(1/2)\Omega]$ *is preserved during the motion* (in Kepler's second law : $r^2\theta$ = const.).

The trajectory described by equation (152) is in the plane

(157) $$\vec{L}.\vec{r} = 0,$$

as shown by the equation (155). In this plane, along and perpendicular to the radius vector $\vec{r}$, the equation (152) yields

(158) $$\ddot r - r\dot\theta^2 = -\frac{\gamma_0 M}{r^2} + r\dot\theta\Omega,$$
$$\frac{d}{dt}(r^2\dot\theta) = -(\frac{1}{2}\, r^2\, \frac{\partial\Omega}{\partial t} + r\dot r\Omega),$$

the latter equation being the equation (154). The first equation in (158) by virtue of (156), reduces to

(159) $$\ddot r = \frac{L^2}{r^3} - \frac{\gamma_0 M}{r^2}\ ,$$

if we neglect the term $-(1/4)\Omega^2 r$, assuming Ω very small. *This is the differential equation of the orbit and has the same form as if* Ω=0.

Let T, V, E be respectively the kinetic energy, potential energy and constant total energy, all per unit mass. Then

(160) $$T + V = E$$

Now, scalar multiplication of equation (152) with $\dot{\vec{r}}$ gives

(161) $$\frac{V^2}{2} - \frac{\gamma_0 M}{r} + \frac{1}{2}\, L\Omega = E,$$

where the relation (155) has been used and the term in Ω^2 has been neglected.

We have

(162) $$V^2 = \dot r^2 + r^2\dot\theta^2 = \dot r^2 + r^2(\frac{L}{r^2} - \frac{1}{2}\Omega)^2$$
$$= \dot r^2 + \frac{L^2}{r^2} - L\Omega.$$

Thus

(163) $$\dot r^2 = -\frac{L^2}{r^2} + 2\,\frac{\gamma_0 M}{r} + 2E,$$

which is precisely the integral of equation (159). We rewrite it as follows

(164) $$r^2\dot r^2 = -2E\Big[(\frac{\gamma_0 M}{2E})^2(1 + \frac{2EL^2}{(\gamma_0 M)^2}) - (r + \frac{\gamma_0 M}{2E})^2\Big].$$

Let us put

(165) $$a = -\frac{\gamma_0 M}{2E}, \qquad e^2 = 1 + \frac{2EL^2}{(\gamma_0 M)^2}.$$

Equation (164) becomes

(166) $$r^2\dot r^2 = -2E[a^2e^2 - (r-a)^2].$$

Writing

(167) $$r = a(1 - e\cos\phi),$$

equation (166) reduces to

(168) $$a^2(1 - e\cos\phi)^2\dot\phi^2 = -2E,$$

which implies E < 0; *the orbit described by the particle is the ellipse* (167). Equation (168) gives at once

$$\phi - e\sin\phi = \frac{\sqrt{-2E}}{a}\, t,$$

which is *Kepler's equation. Kepler's third law* follows at once :

$$T = 2\pi \frac{a}{\sqrt{-2E}} = 2\pi\sqrt{\frac{a^3}{\gamma_0 M}}. \tag{169}$$

Let us verify the relations (165). The maximum and minimum values of r are obtained from equation (163) where we put $\dot{r} = 0$. We then have

$$2Er^2 + 2\gamma_0 Mr - L^2 = 0. \tag{170}$$

Let r_1 and r_2 be the roots of the equation (170) ; we have

$$r_1 = a(1 - e), \qquad r_2 = a(1 + e), \tag{171}$$

where a is the semi-major axis of the ellipse and e is its eccentricity. Hence

$$r_1 + r_2 = 2a = -\frac{\gamma_0 M}{E}, \quad r_1 r_2 = a^2(1-e^2) = -\frac{L^2}{2E}, \tag{172}$$

$$L^2 = \gamma_0 Ma(1-e^2), \text{ C.Q.F.D.}$$

In search for the motion of the perihelion of a planet we found necessary to serutinize the equation (159) and its consequences, which are known in astronomy *except for the definition of* L. The independent variable we used was the time t. The motion in question hides in the equation (152). We, therefore, return to the equation (152) and rewrite it in a rotating frame (x',y') about the z-axis. We have

$$\begin{aligned} x &= x' \cos\alpha - y' \sin\alpha, \\ y &= x' \sin\alpha + y' \cos\alpha. \end{aligned} \tag{172}$$

Hence

$$\begin{aligned} \dot{x} &= (\dot{x}' - y'\dot{\alpha})\cos\alpha - (\dot{y}' + x'\dot{\alpha})\sin\alpha, \\ \dot{y} &= (\dot{x}' - y'\dot{\alpha})\sin\alpha + (\dot{y}' + x'\dot{\alpha})\cos\alpha, \end{aligned} \tag{173}$$

$$\begin{aligned} \ddot{x} &= (\ddot{x}' - y'\ddot{\alpha} - 2\dot{y}'\dot{\alpha} - x'\dot{\alpha}^2)\cos\alpha - (\ddot{y} + x'\ddot{\alpha} + 2\dot{x}'\dot{\alpha} - y'\dot{\alpha}^2)\sin\alpha, \\ \ddot{y} &= (\ddot{x}' - y'\ddot{\alpha} - 2\dot{y}'\dot{\alpha} - x'\dot{\alpha}^2)\sin\alpha + (\ddot{y} + x'\ddot{\alpha} + 2\dot{x}'\dot{\alpha} - y'\dot{\alpha}^2)\cos\alpha. \end{aligned} \tag{174}$$

Substitution of relations (172)-(174) in equation (152) yields

$$\begin{aligned} \ddot{x}' - y'\ddot{\alpha} - 2\dot{y}'\dot{\alpha} - x'\dot{\alpha}^2 &= -\frac{\gamma_0 M}{r'^3}x' + \frac{1}{2}y'\frac{\partial\Omega}{\partial t} + (\dot{y}' + x'\dot{\alpha})\Omega, \\ \ddot{y}' + x'\ddot{\alpha} + 2\dot{x}'\dot{\alpha} - y'\dot{\alpha}^2 &= -\frac{\gamma_0 M}{r'^3}y' - \frac{1}{2}x'\frac{\partial\Omega}{\partial t} - (\dot{x}' - y'\dot{\alpha})\Omega. \end{aligned} \tag{175}$$

We shall take

$$\dot{\alpha} = \frac{1}{2}\Omega. \tag{176}$$

Assuming Ω small enough as to neglect the terms in Ω^2, we obtain the equations

$$\begin{aligned} \ddot{x}' &= -\frac{\gamma_0 M}{r'^3}x', \\ \ddot{y}' &= -\frac{\gamma_0 M}{r'^3}y', \\ \ddot{z}' &= -\frac{\gamma_0 M}{r'^3}z', \end{aligned} \tag{177}$$

that is

$$\ddot{\vec{r}}' = -\frac{\gamma_0 M}{r'^3}\vec{r}' \tag{178}$$

which is the *Newtonian equation for planetary orbits written in the rotating frame of reference* (x',y',z'). An astronomer in the rotating frame knows the integrals of the equation (178) namely

$$\vec{r}' \times \dot{\vec{r}} = \vec{L}', \tag{179}$$

$$\ddot{\vec{r}} \times \vec{L}' = \gamma_0 M \frac{d}{dt}\left(\frac{\vec{r}'}{r'}\right), \tag{180}$$

which give

$$r'^2\dot{\theta}' = L',$$

(181)
$$\dot{\vec{r}}' \times \vec{L}' = \frac{\gamma_0 M}{r'}\vec{r}' + \vec{d}',$$

where θ' is the angle between $\vec{r}'$ and the x'-axis, and $\vec{d}'$ is a constant vector in the plane $\vec{L}'.\vec{r}' = 0$ of motion.

If we dot-multiply both members of the equation (181) by $\vec{r}'$, we find

(182)
$$L'^2 - \gamma_0 M\, r' = \vec{d}'.\vec{r}' = d'r' \cos\theta',$$

if we take the x'-axis in the direction of the vector $\vec{d}'$. Thus

(183)
$$r' = \frac{L'^2/(\gamma_0 M)}{1+e' \cos\theta'},$$

where $e' = d'/(\gamma_0 M)$ is the eccentricity and $L'^2/(\gamma_0 M)$ is the semi-latus rectum of our ellipse.

If we cross-multiply both members of the equation (181) by $\vec{L}'$, we find the velocity $\vec{v}'$:

(184)
$$\vec{v}' = \dot{\vec{r}}' = \frac{\gamma_0 M}{L'^2 r'}\vec{L}' \times \vec{r}' + \frac{1}{L'^2}\vec{L}' \times \vec{d}'.$$

We stop here to make a very important remark, which we do not find in the literature : *the planet experiences a rotation* ω' *about the* z'*-axis*, namely

(185)
$$\omega' = (\text{curl}'\vec{v}')_{z'} = \sqrt{\frac{\gamma_0 M}{a'}}\,\frac{1}{r'}\,\frac{1}{\sqrt{1-e'^2}},$$

which is maximum at the perihelion and is minimum at the aphelion. The time average of ω is

(186)
$$\begin{aligned}\bar{\omega}' &= \sqrt{\frac{\gamma_0 M}{a'}}\left(\overline{\frac{1}{r'}}\right)\frac{1}{\sqrt{1-e'^2}}\\ &= \sqrt{\frac{\gamma_0 M}{a'}}\,\frac{1}{a'}\,\frac{1}{\sqrt{1-e'^2}} = \frac{2\pi}{T'}\,\frac{1}{\sqrt{1-e'^2}}.\end{aligned}$$

$T' = T$ being *the periodic time* of our planet, given by (169). This result will please, we believe, our primed astronomer who, so far, ignored it ! In his position, he does not realize either the small advance of the perihelion of the planet, whose motion he observes. It does not appear in the equations above.

Let us now see the reactions and observations of his colleague in the fixed frame (x,y,z). He computes the angular momentum (per unit mass) $x'\dot{y}' - y'\dot{x}'$. Using the relations (172)-(173), one has

(187)
$$x'\dot{y}' - y'\dot{x}' = x\dot{y} - y\dot{x} - r^2\dot{\alpha} = x\dot{y} - y\dot{x} + \frac{1}{2}r^2\Omega.$$

Hence

(188)
$$L' = L.$$

Now

(189)
$$\begin{aligned} d'_{x'} &= d_x \cos\alpha + d_y \sin\alpha,\\ d'_{y'} &= 0 = -d_x \sin\alpha + d_y \cos\alpha.\end{aligned}$$

Thus $d'=d$ and one, therefore, has $e'=e$, and $a'=a$.

The mapping of the ellipse (183) onto the plane (x,y) reads

(190)
$$r = \frac{L^2/(\gamma_0 M)}{1+e \cos\left(\theta + \frac{1}{2}\int_0^t \Omega\, dt\right)},$$

which for Ω = const., reduces to

(191)
$$r = \frac{L^2/(\gamma_0 M)}{1+e \cos(\theta + \frac{1}{2}\Omega t)} = \frac{a(1-e^2)}{1+e \cos(\theta + \frac{1}{2}\Omega t)}.$$

The motion of the perihelion is described by the equations [*see*, formulas (172) where $\alpha = -(1/2)\Omega t$, $x'=a(1-e)$,

$y'=0$] :

$$\text{(192)}\qquad \begin{aligned} x_{per.} &= a(1-e)\cos(\tfrac{1}{2}\Omega t),\\ y_{per.} &= -a(1-e)\sin(\tfrac{1}{2}\Omega t), \end{aligned}$$

which represent a circle with center at the sun and radius $a(1-e)$. Perihelion's advance requires $\Omega<0$. It will return to its initial position at the time $t=4\pi/\Omega$. The planet, itself, finds then its old position.

The velocity $\vec{v}$ of the planet is

$$\text{(193)}\qquad \vec{v} = \frac{\gamma_0 M}{L^2 r}\,\vec{L}\times\vec{r} + \frac{1}{L^2}(\vec{L}\times\vec{d}) - \frac{1}{2}\,\vec{\Omega}\times\vec{r}.$$

Its rotation ω is

$$\text{(194)}\qquad \omega = \sqrt{\frac{\gamma_0 M}{a}}\,\frac{1}{r}\,\frac{1}{\sqrt{1-e^2}} - \Omega.$$

This is the manner in which we represent the motion of the perihelion of a planet, the so-called "advance of perihelion". It is caused by the gravitational vortex $\vec{\Omega}$. This is to be compared with the genial computation of Einstein. Ω, if constant, has to be determined by observation. If we take the time $t=T$, the period of revolution, we may write

$$\text{(195)}\qquad \frac{1}{2}|\Omega|\left(2\pi\sqrt{\frac{a^3}{\gamma_0 M}}\right) = \frac{6\pi\gamma_0 M}{ac^2(1-e^2)},$$

using Einstein's formula. Hence

$$\text{(196)}\qquad |\Omega| = \frac{6(\gamma_0 M)^{3/2}}{a^{5/2}c^2(1-e^2)}$$

a very small quantity, indeed. This, surely, is the new physical effect that Bridgman [22, p. 159] was looking for. He wrote : "*Something analogous to the electromagnetic field equations, but applicable to inertial matter, seems demanded here, but there is nothing in sight that meets the requirements. In fact, the need for something analogous appears so imperative*

that one is strongly tempted to believe that there must be some new physical effect so small as to have hither to escaped direction". Our equations (137) and relation (196) fill up his guest.

Much has been written about Einstein's formula, the second member of equation (195). We refer ther reader to the book by Misner, Thorne and Wheeler [3, p. 110-116] and references. To this, we add the elegant book by Tonnelat [27, *see* pp. 55-58 and 184-187] and the cautious books by Chazy [28] and Brillouin [1, p. 99]. We also note with considerable interest the new derivation of Einstein's result by Schwinger [11, pp. 83-85]. He is concerned with the interaction between two bodies : the "Sun" and a second test body : a "planet". Our "Comptes Rendus" Note [29] of December 3, 1969 : "*Sur l'énergie potentielle de deux particules électriques : une correction à la relativité classique*", appeared too late to be quoted in Schwinger's excellent book, published in 1970. It was our last "Comptes Rendus" note communicated by Henri Villat to the French Academy of Sciences.

3. On the Bode's law. The distances of the planets from the Sun do not seem to be arranged randomly. In 1776, J.D. Titius formulated empirically a relation which closely approximated these distances. This is known as Bode's law, named in honor of the famous astronomer who published it in 1772. The law states that, in astronomical units (1 astronomical unit = semi-major axis of the Earth = 149.4×10^6 km) the planetary semi-major axis are : 0.4 for Mercury, 0.4+0.3 for Venus ; thereafter always adding twice the previous value to 0.4, i.e., 0.6, 1.2, 2.4, etc. Clearly, one has

$$\text{(197)}\qquad a = 0.4 + 0.3\times 2^n,$$

where $n = 0$ for Venus, $n = 1$ for Earth, $n = 2$ for Mars, etc. *The planet Mercury corresponds to* $n = -\infty$.

At the time, only six planets were known. The discovery of Uranus in 1781 and of the first asteroid in 1801 corresponding closely to the missing numbers 19.6 and 2.8 lent considerable support to the law. However, the more recently discovered planets Neptune and Pluto deviate widely from the positions (38.8 and 77.2) predicted by Bode's law, and since there is no theoretical explanation for the law, astronomers now believe that the relationship may be incidental. Is it so ?

Without going into details, we replace Bode's law by the relation

$$a = 0.4 + 0.3 \times 2^n + s(20.1 - 28.9 \times 2^m), \tag{198}$$

where s is a *step function* : s = 0, for n = 0,1,2,...,6, and s = 1, for n = 7,8,...

For the planet Neptune, we take : n = 7, m = 0, and obtain a ≈ 30. For the planet Pluto, we take : n = 8, m = 1, and obtain a ≈ 39.5, value which fits the observation. This is not surprising. But, *what is surprising, indeed, is that if we take :* n=9, m=2, *our formula* (198) *gives :* a ≈ 58.5 A.U. *This is basically the same value as that predicted for the presumed missing planet based on the observed perturbations in the orbits of Uranus and Neptune.* This was confirmed to us by astronomer R.S. Harrington of the *U.S. Naval Observatory*, who finds here our thanks.

Our formula (198) restores, therefore, the power of prediction of Bode's law. It carries information which we are, unfortunately, unable as yet to read out. Kepler's laws led Newton to the formulation of universal gravitation. Who will now discover the physics behind our law (198) ? It is a challenging problem which involves the *theory of information* and *quantum mechanics*. It is of importance in a theory of the origin of the solar system, such as discussed by Jeffreys [30] but, where the Bode law and our formulation (198) are missing. Let us remember that Brillouin [31, pp. 162-183] identified Maxwell's demon, that "a being whose faculties are so sharpened that he can follow every molecule in his course, and would be able to do what is at present impossible to us", by application of the theory of information. We believe that the latter is the key of our problem ! But, there is research to be done about this.

The period of revolution of the 10th planet in our

solar system is given by (169) and the time average of the square of its speed is $\overline{v^2} = \gamma_0 M/a$; we do not insist.

4. Relationship between electromagnetism and gravitation

The ponderomotive electromagnetic forces acting on a fluid have been evaluated by Goldstein [32, pp. 50-51] when $\vec{J}_m = 0$, $\rho_m = 0$. See, also Penfield, Jr. and Hermann [33, p. 201 *passim*]. We write

$$\vec{f}_{em} = \rho_e\vec{E} + \rho_m\vec{B} + \vec{J}_e\times\vec{B} - \frac{1}{C^2}\vec{J}_m\times\vec{E}. \tag{199}$$

The ponderomotive gravitational forces, to be added to the former, are

$$\vec{f}_{g\omega} = \rho_g\vec{E} + \rho_\omega\vec{\Omega} + \vec{J}_g\times\vec{\Omega} - \frac{1}{C^2}\vec{J}_\omega\times\vec{E}. \tag{200}$$

Let us assume that

$$\rho_e\vec{E} = F_e\,\rho_g\vec{E}, \qquad \rho_m\vec{B} = F_m\,\rho_\omega\vec{\Omega}, \tag{201}$$

where F_e and F_g are dimensionless constants. To determine them, we use the divergence equations in (2) and (137). We obtain at once

$$F_e = -\frac{\rho_e^2}{\rho_g^2}\frac{1}{4\pi\varepsilon\gamma}, \qquad F_m = -\frac{\rho_m^2}{\rho_\omega^2}\frac{1}{4\pi\varepsilon\gamma}. \tag{202}$$

We anticipate the relation

$$\frac{\rho_m}{\rho_\omega} = \frac{\rho_e}{\rho_g} = \text{const.} \tag{203}$$

for a particle of fluid. Hence

$$F_e = F_m = F = -\frac{\rho_e^2}{\rho_g^2}\frac{1}{4\pi\varepsilon\gamma}. \tag{204}$$

We shall call the ratio F the *"Faraday number"*. This famous British physicist undertook numerous experiments to discover such a ratio, but these were fruitless. He concluded in his laboratory diary : "Here end my trials for the present. The results are negative. They do not shake my strong feeling of the existence of a relation between gravity and electricity, though they give no proof that such a relation exists".

The formulas (201)-(204) imply that

$$(205)\qquad \vec{E} = -\frac{\rho_e}{\rho_g}\frac{1}{4\pi\varepsilon\gamma}\vec{G}, \qquad \vec{B} = -\frac{\rho_e}{\rho_g}\frac{1}{4\pi\varepsilon\gamma}\vec{\Omega}.$$

For a massive particle of mass M and electric charge Q_e, we have

$$(206)\qquad \vec{E} = -\frac{Q_e}{M}\frac{1}{4\pi\varepsilon\gamma}\vec{G}, \qquad \vec{B} = -\frac{Q_e}{M}\frac{1}{4\pi\varepsilon\gamma}\vec{\Omega},$$

which, surely, constitute the relations that Faraday was seeking !

Comparing the Maxwell equations (2) and our equations (137) we obtain, by virtue of relations (206),

$$(207)\qquad \vec{J}_e = \frac{\rho_e}{\rho_g}\vec{J}_g, \qquad \vec{J}_m = \frac{\rho_e}{\rho_g}\vec{J}_\omega.$$

Finally, inserting the values (207) in the equations C(4) we have, in view of our equations (146),

$$(208)\qquad \frac{\rho_g}{\rho_e}\,\text{grad}\,\rho_m = \text{grad}\,\rho_\omega, \quad \frac{\rho_g}{\rho_e}\,\text{grad}\,\rho_e = \text{grad}\,\rho_g.$$

The first relation yields

$$(209)\qquad \frac{\rho_g}{\rho_e}\frac{\rho_m}{\rho_\omega} = 1,$$

which proves our anticipated relation (203). The second relation in (208) is an identity.

The relation (203) *is of fundamental importance. It gives the subtle intercoupling between electromagnetism and gravitation.* It can be rewritten as

$$(210)\qquad \frac{M_\omega}{M} = \frac{Q_m}{Q_e},$$

where $M_\omega = \int\rho_\omega d\tau$, $d\tau = dx\,dy\,dz$, is a "second mass" (we have no better word for it), which corresponds to the magnetic charge in electromagnetism ; its dimensions are : $[M_\omega] = MLT^{-1}$, i.e., those of a momentum (or quantity of motion). Many features of paragraphs 3-8 and 3-9 by Schwinger [11, pp. 227-254], including "charge quantization", apply *mutatis mutandis* to our theory, but much remains to be done. We wonder which would be the reaction to our suggestions of Harold, this alert reader of limitless dedication, who so many times critically interrupts Schwinger in the presentation of his work [11, *See* Index].

We now proceed to discuss *frozen-in fields*. The equation of linear momentum is

$$(211)\qquad \rho_g\frac{D\vec{v}}{Dt} = -\text{grad}\,p + (1+F)(\rho_g\vec{E}+\rho_\omega\vec{\Omega}+\vec{J}_g\times\vec{\Omega} - \frac{1}{C^2}\vec{J}_\omega\times\vec{E}),$$

where $\vec{v}$ is the velocity of a fluid particle, p is the pressure, and D/Dt is the mobile operator

$$(212)\qquad \frac{D}{Dt} = \frac{\partial}{\partial t} + (\vec{v}.\nabla).$$

We shall assume that

$$(213)\qquad \vec{J}_g = \rho_g\,\vec{v}, \qquad \vec{J}_\omega = \rho_\omega\,\vec{v}.$$

Equation (211) becomes

$$(214)\qquad \frac{\partial\vec{v}}{\partial t} = -\text{grad}(\int\frac{dp}{\rho_g} + \frac{v^2}{2})+2\vec{v}\times\vec{\omega}+(1+F)\left[\vec{G} + \frac{\rho_\omega}{\rho_g}\vec{\Omega}+\vec{v}\times(\vec{\Omega} - \frac{1}{C^2}\frac{\rho_\omega}{\rho_g}\vec{G})\right],$$

where $\vec{\omega} = (1/2)\text{curl}\,\vec{v}$ is the velocity and p is assumed to be a function of ρ_g only.

Taking the curl of both sides of the last equation, we obtain, by virtue of curl equations in (137) where we omit the right sides,

$$(215)\quad \frac{\partial}{\partial t}\left[2\vec{\omega}+(1+F)(\vec{\Omega} - \frac{1}{C^2}\frac{\rho_\omega}{\rho_g}\vec{G})\right] = \mathrm{curl}\{\vec{v}\times[2\vec{\omega}+(1+F)(\vec{\Omega}-\frac{1}{C^2}\frac{\rho_\omega}{\rho_g}\vec{G})]\}.$$

Observing that, in view of divergence equations (137),

$$(216)\qquad \mathrm{div}(\vec{\Omega} - \frac{1}{C^2}\frac{\rho_\omega}{\rho_g}\vec{G}) = 0,$$

and because of the hydrodynamic equation of continuity

$$(217)\qquad \frac{\partial \rho_g}{\partial t} + \mathrm{div}(\rho_g\,\vec{v}) = 0,$$

the equation (215) can be rewritten

$$(218)\quad \frac{D}{Dt}\{\frac{1}{\rho_g}[2\vec{\omega}+(1+F)(\vec{\Omega} - \frac{1}{C^2}\frac{\rho_\omega}{\rho_g}\vec{G})]\} = \{\frac{1}{\rho_g}[2\vec{\omega}+(1+F)(\vec{\Omega}- \frac{1}{C^2}\frac{\rho_\omega}{\rho_g}\vec{G})].\nabla\}\vec{v}.$$

The quantity

$$(219)\qquad \frac{1}{\rho_g}[2\vec{\omega} + (1+F)(\vec{\Omega} - \frac{1}{C^2}\frac{\rho_\omega}{\rho_g}\vec{G})]$$

is, therefore, preserved during the motion, and thereby is frozen into the material. If we call $\vec{A}$ the quantity $(1+F)[\vec{\Omega}-(1/C^2)(\rho_\omega/\rho_g)\vec{G}]$, then, *if a time* $t=0$, $2\vec{\omega}_0=-\vec{A}_0$ *for any particule of fluid, we shall have at all time, for that particle* : $2\vec{\omega} = -\vec{A}$. Applications follow, but we do not insist here.

Let us check our formulas (206) for our planet, the Earth.

We use the data

$$(220)\qquad \begin{aligned} &M = 5.977\times10^{24}\ \mathrm{kg}\ ;\ Q_e = 5\times10^{5}\ \text{coulombs, in fine weather},\\ &|\vec{G}| = g = 9.81\ \mathrm{m/sec^2},\ \varepsilon_0 = 8.854\times10^{-12}\ \mathrm{farad/m},\\ &\gamma_0 = 6.67\times10^{-11}\ \mathrm{M.K.S.} \end{aligned}$$

We shall assume that

$$(221)\qquad \frac{\gamma}{C^2} = \frac{\gamma_0}{C^2} = \frac{1}{8\pi}\,\chi,$$

where χ is Einstein's constant. Taking $\mu = \mu_0$, we have

$$(222)\qquad \varepsilon\gamma = \varepsilon_0\gamma_0 \simeq 5.9\times10^{-22}\ \mathrm{M.K.S.Q.}$$

We obtain

$$(223)\qquad |\vec{E}| \simeq 100\ \mathrm{volt/m}.$$

The field $\vec{E}$ *is directed downwards as it should :* Q_e *is found to be negative, hence* $\vec{E}$ *has the same direction as* $\vec{g}$. The result (223) is in excellent agreement with the fine-weather electric field observed in the air above the Earth. It was first found by Lemonnier in the year 1752.

The second formula in (206) does not admit an easy geophysical interpretation. Professor Dale M. Grimes wrote to this author, and I quote : "Now if your equation relating $\vec{B}$ and $\vec{\Omega}$ have substance on a geophysical scale then would one not expect that the reversal in direction of the magnetic field occurs together with a reversal in direction of rotation ? If so, the resultant drastic change in local climate during the reversing period should, it appears obvious, produce drastic change in both plant and animal life as is observed". We leave up the answer to reader's competence.

Conclusion. We come to an end of this mémoire. We enumerate our new contributions. The equations C which complement the Maxwell equations together with which they determine the quantities $\vec{E}$, $\vec{B}$, $\vec{J}_e$, $\vec{J}_m$, ρ_e and ρ_m of an electromagnetic theory. The equations for electromagnetic potentials and the modified equations C yield the L. de Broglie equations for the particle photon. Our retarded potentials lead to the definition of the two gravitational fields $\vec{G}$ and $\vec{\Omega}$ and show that these propagate in vacuum with the velocity c of light. Monochromatic waves contain the fundamental relations of special relativity to which we add the new relation $M = M_0\sqrt{1-v^2/c^2}$. A reformulation of the Doppler principle is given which takes into account the L. de Broglie wave length.

The magnetic potentials A_m and ϕ_m for a dipole lead to the oscillations of the Earth's magnetic field. The generalized L. de Broglie density-flux (which takes into account the potentials A_m and ϕ_m) gives the probability of presence of photons in the Earth's magnetic field and a new thoery of the aurora is contemplated. The non-Maxwellian equations of L. de Broglie appear as a singular case of the Maxwell equations and the modified equations C.

An entire chapter is devoted to a theory of rotating bodies, which is conspicuously missing in relativity and electromagnetism. The invariance of the Maxwell equations in a rotating frame of reference is subject to a new system of equations for the field vectors, which determine the latter. They are dipole fields. The Earth's magnetic field appear to be the same in both rotating and fixed frames of reference. If there are currents and charges, they will modify our magnetic and electric fields, but all quantities have the same representation in both frames. This is an introduction to planetary magnetism.

Brillouin [1, p. 71] emphasizes that the *Lorentz transformation* is a *mathematical, unobservable* tool, -very useful, but *definetely not physical*. We agree. Brillouin would be surprized to see its use in the *first principle of dynamics: An observer at rest* with respect to whom a massive particle moves with a constant velocity v in a fixed direction, *will sense the two fields of gravitation which, interesting enough, propagate in the direction of motion with the velocity* v. It is a simple, but major contribution of ours, which may serve as an introduction to the *theory of the double solution* of L. de Broglie [4]. We obtained this result by correcting an error of Sommerfeld [12, pp. 239-241] found in "the intrinsec field of an electron in uniform motion".

We conceive the gravitation in a simple fashion, not because gravitation is simple (look at the voluminous book by Misner, Thorne and Wheeler [3] and references on the subject) but, because our simple mind simplifies it. However, it is amazing how much information our simple ideas bring on the subject. A striking feature of our approach is the introduction of a "second mass density ρ_ω" in addition to the mass density ρ_g, and hence of a "second mass M_ω" corresponding to the magnetic charge Q_m. The ratio $\rho_\omega/\rho_m = \rho_g/\rho_e$ makes apparent an intercoupling between gravitation and electromagnetism and leads to simple relations between $\vec{E}$ and $\vec{G}$, and $\vec{B}$ and $\vec{\Omega}$, unsuccesfully sought by Faraday. We define a "Faraday number F". Einstein's *unitary theory* was never achieved since it was impossible to unite his geometric theory with electromagnetism.

The ratio $M_\omega/M_g = Q_m/Q_e$ could be a potential contributor to "second mass" quantization and mass normalization of "elementary particles", similar to charge quantization and mass normalization of Schwinger, *loc. cit.* [11].

The existence of our gravitational vortex $\vec{\Omega}$ is succesfully tested. The motion of planetary perihelia, the so-called "advance" of the latter, which is observed for Mercury, is caused by $\vec{\Omega}$.

By the way of digression, we reformulate Bode's law, and *our law*, so amended, *predicts the semi-major axis of the missing planet, the 10th planet in our solar system !*

We have said much, but much remains to be said, how much, we cannot tell. We may end and then revise a chapter of physics, but science will never be finished. The theory of the double solution of L. de Broglie finds an extraordinary application in lasers, and the lasers may bring more light to the double solution. In his last letter to me Brillouin said : "Et les quanta des lumière +hν, vont-ils s'annihiler avec les -hν des gravitons ?? Curieuse énigme ..." L. de Broglie [34] wrote a philosophical book "Matter and Light", and Newton in his "Opticks" in the year 1704, wonders "... And among such various and strange transmutations, why may not Nature change bodies into light, and light into bodies". A curious enigma, indeed !

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APPENDIX II - On planetary orbits

Here, we give a brief study on the advance of planetary perihelia. This advance, computed by Einstein and observed, so far, for the four inner planets, is exceedingly small. It represents an infinitesimally small fraction of a revolution of the planet. The trajectory of the perihelion is not known and will take centuries of observation to actually determine.

Subject to observation, our formulas (192) show that this trajectory is a circle with its center at the Sun and radius $a(1-e)$; the speed of the perihelion is $(\Omega/2)a(1-e)$. Clearly, the aphelion describes a concentric circle of radius $a(1+e)$; its speed is $(\Omega/2)a(1+e)$. The orbit rotates very slowly around the Sun between these two circles sweeping the entire area bounded by them. We desire to show that *the circles in question constitute the envelope of our family of orbits which depends on one parameter, the time* t.

In doing so, we shall give various formulas which are new and useful in astronomy.

We begin with the equation

$$\frac{(x'+ae)^2}{a^2} + \frac{y'^2}{b^2} = 1, \qquad \text{(AII.1)}$$

which represents the orbit in the rotating frame of reference (x',y') with origin O at the Sun.

We have

$$\begin{aligned} x' &= a(\cos\phi' - e), \\ y' &= b\sin\phi', \end{aligned} \qquad \text{(AII.2)}$$

where ϕ' is the eccentric anomaly of the planet. Hence :

$$r' = \sqrt{x'^2+y'^2} = a(1 - e\cos\phi'). \qquad \text{(AII.3)}$$

In the fixed frame of reference $(x,y,)$, we write

$$(AII.4)\quad \begin{aligned} x &= a_x(\cos\phi - e) + b_x \sin\phi, \\ y &= a_y(\cos\phi - e) + b_y \sin\phi, \end{aligned}$$

where

$$(AII.5)\quad \begin{aligned} a_x &= a\cos(\tfrac{1}{2}\Omega t), & b_x &= b\cos(\tfrac{1}{2}\Omega t), \\ a_y &= -a\sin(\tfrac{1}{2}\Omega t), & b_y &= b\sin(\tfrac{1}{2}\Omega t), \end{aligned}$$

are the components of semi-major and semi-minor axes a and b of the orbit along the Ox, Oy axes.

We have

$$(AII.6)\quad r = \sqrt{x^2+y^2} = a(1-e\cos\phi).$$

On comparing this with equation (AII.3) we see that $\phi' = \phi$, i.e., *the eccentric anomaly is the same in both frames of reference, Kepler's third law is, therefore, fulfilled regardless of rotation.* This confirms our result in the text.

We now map the ellipse (AII.1) onto the plane (x,y), using the formulas (172). We have

$$(AII.7)\quad \frac{(x\cos\alpha + y\sin\alpha + ae)^2}{a^2} + \frac{(-x\sin\alpha + y\cos\alpha)^2}{b^2} = 1,$$

where $\alpha = -(1/2)\Omega t$, Ω = const. *The advance of perihelia requires* $\Omega<0$.

To find the envelope of the family of ellipses (AII.7) we take the derivative with respect to α of this equation and put $\partial/\partial\alpha = 0$.

We have

$$(AII.8)\quad (-x\sin\alpha+y\cos\alpha)\left(\frac{x\cos\alpha+y\sin\alpha+ae}{a^2} - \frac{x\cos\alpha+y\sin\alpha}{b^2}\right) = 0.$$

We, first, study the solution (pencil of straight lines through the origin) :

$$(AII.9)\quad -x\sin\alpha + y\cos\alpha = 0.$$

By virtue of the latter, equation (AII.7) reduces to

$$(AII.10)\quad x\cos\alpha + y\sin\alpha = a(\pm 1-e),$$

which represents straight lines perpendicular to the former.

Solving the equations (AII.9) and (AII.10), we obtain

$$(AII.11)\quad \begin{aligned} x &= a(1-e)\cos\alpha = a(1-e)\cos(\tfrac{1}{2}\Omega t), \\ y &= a(1-e)\sin\alpha = -a(1-e\ )\sin(\tfrac{1}{2}\Omega t), \end{aligned}$$

and

$$(AII.12)\quad \begin{aligned} x &= -a(1+e)\cos\alpha = -a(1+e)\cos(\tfrac{1}{2}\Omega t), \\ y &= -a(1+e)\sin\alpha = a(1+e)\sin(\tfrac{1}{2}\Omega t). \end{aligned}$$

The solution

$$(AII.13)\quad \left(\frac{1}{a^2} - \frac{1}{b^2}\right)(x\cos\alpha + y\sin\alpha) = -\frac{e}{a},$$

is impossible, for it leads to imaginary values for x and y. Indeed, we have

$$(AII.14)\quad x\cos\alpha + y\sin\alpha = \frac{b^2}{ae}.$$

Equation (AII.7) becomes

$$(AII.15)\quad \frac{(-x\cos\alpha + y\sin\alpha)^2}{b^2} = 1 - \frac{1}{e^2},$$

which does not have a real solution because its right side is negative (e<1).

The formulas (AII.11) *and* (AII.12) *give the circles described respectively by the perihelion and the aphelion.* C.Q.F.D.

A few lines about the *hodograph* of the motion are in order. In the rotating frame, we have [*see*, formula (184)]:

(AII.16) $$v_x'^2 + (v_y' - \frac{d'_{x'}}{L'})^2 = (\frac{\gamma_0 M}{L'})^2 \quad (d'_{x'} = d' = d).$$

The *normalized form* of this equation is

(AII.17) $$x_H'^2 + (y_H' - ae)^2 = a^2,$$

where we put

(AII.18) $$x_H' = \tau v_x', \quad y_H' = \tau v_y', \quad \tau \frac{d'_{x'}}{L'} = ae, \quad \tau \frac{\gamma_0 M}{L'} = a,$$

(AII.19) $$\tau = \frac{T}{2\pi}\sqrt{1-e^2},$$

T being the periodic time of the planet.

Equation (AII.17) represents the hodograph of the motion, which is a circle having its center on the latus rectum at a distance ae from the focus and radius a.

Let us observe that if we put

(AII.20) $$x_V' = x' + x_H', \quad y_V' = y' + y_H',$$

the locus of the point (x_V', y_V') is the ellipse

(AII.21) $$\frac{(x_V' + ae)^2}{4a^2} + \frac{(y_V' - ae)^2}{(a+b)^2} = 1.$$

The area A_V of the latter is : $A_V = 2\pi a(a+b)$, and we have

(AII.22) $$A_V = 2(A_C + A_H),$$

where $A_C = \pi ab$ is the area of the ellipse (AII.1), and $A_H = \pi a^2$ is the area of the circle (AII.17). Our formula (AII.22) is reminiscent of the elegant expression given by Dimitrie Pompeiu (our "maître" in Bucharest) as a geometrical interpretation of the continuity equation for plane motion of an incompressible fluid. Pompeiu's formula is

(AII.23) $$A_V = A_C + A_H,$$

where A_C is the area bounded by a fluid curve C (area which is a constant), and A_H and A_V are the areas bounded by the curves traced by the velocity $\vec{v}$ and the vector $\vec{r}+\vec{v}$ put at the origin of the co-ordinates (the time τ, factor of $\vec{v}$ is assumed to be 1). Beautiful remembrance of our youth.

In the fixed frame, we have [*see*, formula (193)] :

(AII.24) $$(V_x + \frac{dy}{L})^2 + (V_y - \frac{dx}{L})^2 = (\frac{\gamma_0 M}{L})^2$$

where

(AII.25) $$V_x = v_x - \frac{1}{2}\Omega y, \quad V_y = v_y + \frac{1}{2}\Omega x,$$

are the components along the Ox, Oy axes of the total velocity $\vec{V} = \vec{v} + (1/2)(\vec{\Omega}\times\vec{r})$, and

(AII.26) $$d_x = d\cos\alpha, \quad d_y = d\sin\alpha.$$

Thus, the terminus of the velocity $\vec{V}$ laid off from the origin O lies on a circle with center at $(-dy/L, dx/L)$ and radius $(\gamma_0 M)/L$. This center, itself, describes a circle with center at the origin and radius a.

To find the envelope of the family of circles (AII.24) we follow the pattern discussed above. Using the relations (173), equation (AII.17) reads

(AII.27) $$(X_H\cos\alpha + Y_H\sin\alpha)^2 + (-X_H\sin\alpha + Y_H\cos\alpha - ae)^2 = a^2,$$

where $X_H = \tau V_x$, $Y_H = \tau V_y$.

Taking $\partial/\partial\alpha$ of this equation and putting $\partial/\partial\alpha = 0$, we obtain

(AII.28) $$X_H\cos\alpha + Y_H\sin\alpha = 0.$$

Hence, equation (AII.27) reduces to

(AII.29) $$-X_H \sin\alpha + Y_H \cos\alpha = a(e\pm 1).$$

The solution of the last two equations is

(AII.30) $$X_H = -a(1-e)\sin(\tfrac{1}{2}\Omega t),\quad Y_H = -a(1-e)\cos(\tfrac{1}{2}\Omega t),$$

and

(AII.31) $$X_H = a(1+e)\sin(\tfrac{1}{2}\Omega t),\quad Y_H = a(1+e)\cos(\tfrac{1}{2}\Omega t),$$

which describe the envelope in question. *Equations* (AII.30) *and* (AII.31)*represent the same circles* (AII.11) *and* (AII.12). We have, in each case,

(AII.32) $$x\,X_H + y\,Y_H = 0,$$

which shows that *the vectors* (x,y) *and* (X_H,Y_H) *are perpendicular.* This is not surprizing in view of the definitions of envelope and hodograph.

It is significant to note here that Newton's constant γ_0 (usually denoted by G) is really not a constant ; it does depend on the distribution of matter in the neighborhood of the Sun. This is particularly felt by the inner planets at their perihelion. We propose to replace γ_0 by γ given by

(AII.33) $$\frac{\gamma}{\gamma_0} = \frac{1 - A\rho_g}{1 + 2A\rho_g},$$

where A is a constant and ρ_g denotes the density of the solar atmosphere.

The formula (AII.33) *constitutes the gravitational analog of the Clausius-Mossotti law* (for the latter *see*, Stratton [8 p. 140]).

In vacuum, $\gamma=\gamma_0$. For a body of very high density, we have

(AII.34) $$\gamma \simeq -\frac{1}{2}\gamma_0.$$

The body will desintegrate due to the repulsive forces between its particles. This corresponds to *gravitational collapse.*

The singular relation (AII.34) can occur only locally inside the body and in a time-interval of the order of a second or less. We then have a *quake*, such as an earthquake. We do not go further. An editor might say : "I wish the author would take fewer bulls by the horns. One at a time is enough, because a lot of back-ground reading is required".

Astronomers may be able to tell to what extent our theory predicts the motion of the node of Venus. Einstein's theory throws no light on the latter. Neither does his theory explains the advance of the Moon's perigee and node.* (For the Earth, one may expect a slow precession of the seasons). -And seismologists may be intrigued by our new approach to a theory of earthquakes. Well, this is the drama of science : it never ends.

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*Our colleague Victor M. Waage informs us that he found in the quarterly journal of the "British-American Scientific Research Association", June 1986 (XI:2); p. 40, this statement :

> "...Just recently two scientists (E.F. Guinan and F.P. Maloney of Villanova University, Pa., U.S.A.) report in the August 1985 *Astronomical Journal* that the precession of the periastron of the binary star DI Herculis is about *seven* times smaller than it should be according to the special theory of relativity. Not a small discrepancy ! This suggests that both Einstein's theories are wrong..."

Brillouin saw clearly the need for a critical reappraisal of Einstein's relativity, based on actual experimental conditions.