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Qualitative Research on Soliton of Nonlinear Schrödinger Equation with External Field

by Guan Ke-ying

Beijing Institute of Aeronautics and Astronautics

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Abstract : For the nonlinear Schrödinger equation with external field

$$i\frac{\partial\psi}{\partial t}+\frac{\partial^2\psi}{\partial x^2}+f(|\psi|^2)\psi=U(x,t)\psi$$

the conditions of the existence of soliton solution, the nonlinear eigenvalue problem arising from seeking soliton and stationary solutions, the stability of soliton, and a method for seeking soliton solution are discussed. The obtained results are applied to the NLS equation for harmonic oscillator

$$i\frac{\partial\psi}{\partial t}+\frac{\partial^2\psi}{\partial x^2}+2|\psi|^2\psi=Kx^2\psi$$

and some facts similar to the linear Schrödinger equation are shown. A numerical result on the re-emergence of two solitons after a collision with each other is introduced.

Résumé : Sur l'équation de Schrödinger non linéaire avec champ externe

$$i\frac{\partial\psi}{\partial t}+\frac{\partial^2\psi}{\partial x^2}+f(|\psi|^2)\psi=U(x,t)\psi$$

on discute les conditions de l'existence de solutions soliton, le problème non linéaire des valeurs propres suscité par la *recherche de solutions soliton ou stationnaires, la stabilité des solitons, et une méthode de recherche des solutions soliton. Les résultats obtenus sont appliqués à l'équation NLS pour l'oscillateur harmonique*

$$i\frac{\partial\psi}{\partial t}+\frac{\partial^2\psi}{\partial x^2}+2|\psi|^2\psi=kx^2\psi$$

et on montre quelques propriétés similaires à l'équation de Schrödinger linéaire. Un résultat numérique sur la ré-émergence de deux solitons après une collision est présenté.

I. Introduction

In his famous book "Nonlinear Wave Mechanics-a causal interpretation" [1], Louis de Broglie, one of the discovers of the wave-particle dualism, proposed using nonlinear wave equations for giving a clear and causal interpretation to the wave-particle dualism -the fundamental myth in the microphysics. According to his idea, the elementary particle is represented as a small region of large amplitude of a nonlinear wave.

Stimulated by de Broglie's striking idea, I and some of my colleagues have explored some problems on the nonlinear Schrödinger equation (the NLS equation) with external field (i.e., with variable coefficients) [2]-[3]

$$(1)\qquad i\frac{\partial\psi}{\partial t}+\frac{\partial^2\psi}{\partial x^2}+f(|\psi|^2)\psi=U(x,t)\psi$$

Our work is concentrated on the following aspects :

(1) Find out the conditions for the existence of the soliton solutions of the NLS equation (1).

(2) Study the nonlinear eigenvalue problem arising from seeking the soliton and the stationary solutions.

(3) Study the stability of the soliton solution.

(4) Develop a method for seeking the soliton solution.

(5) Study a concrete example : the NLS equation for harmonic oscillator

$$(2)\qquad i\frac{\partial\psi}{\partial t}+\frac{\partial^2\psi}{\partial x^2}+2|\psi|^2\psi=kx^2\psi$$

(6) Study the collision between solitons of the NLS equation (2).

In this paper, the complex-valued solution $\psi(x,t)$ of (1) is frequently written in the form (just imitating de Broglie's method)

$$(3)\qquad \psi(x,t)=\phi(x,t)e^{i\theta(x,t)}$$

where ϕ and θ are real functions. Much benefit can be gotten from this method.

It is well-known that many properties of the solution of the NLS equation (1) are similar to the properties of the solution of the corresponding linear Schrödinger equation

$$(4)\qquad i\frac{\partial\psi}{\partial t}+\frac{\partial^2\psi}{\partial x^2}=U(x,t)\psi$$

For instance, if a solution $\psi=\phi e^{i\theta}$ of (1) vanishes rapidly as $x\to\pm\infty$, we then have

$$(5.1)\qquad \int_{-\infty}^{\infty}\psi^*\psi dx=\int_{-\infty}^{\infty}\phi^2 dx=\text{const.}$$

$$(5.2)\qquad \frac{d}{dt}\langle x\rangle\equiv\frac{d}{dt}\frac{\int_{-\infty}^{\infty}\psi^* x\psi dx}{\int_{-\infty}^{\infty}\psi^*\psi dx}=\frac{2\int_{-\infty}^{\infty}\phi^2\cdot\frac{\partial\theta}{\partial x}dx}{\int_{-\infty}^{\infty}\phi^2 dx}$$

$$(5.3)\qquad \frac{d^2}{dt^2}\langle x\rangle=\frac{-2\int_{-\infty}^{\infty}\psi^*(\frac{\partial U}{\partial x})\psi dx}{\int_{-\infty}^{\infty}\psi^*\psi dx}=-2\langle\frac{\partial U}{\partial x}\rangle$$

(5.3) is just the Ehrenfest theorem.

A solution $\psi_s=\phi_s e^{i\theta_s}$ of (1) is called a soliton, if the amplitude function $\phi_s(x,t)$ of it satisfies the following conditions :

(i) $0 < \int_{-\infty}^{\infty} \phi_s^2(x,t)dx < +\infty$

(ii) There exists a function $s(t)$, $\dot{s} = \frac{ds(t)}{dt} \not\equiv 0$,
such that

$$\phi_s(x,t) = \phi_s(x-s(t)).$$

Note that this is only a weakened definition of soliton. Later on, we will strengthen this definition by taking the stability into account.

If $\psi_s = \phi_s e^{i\theta_s}$ is a soliton solution of (1), from the Ehrenfest theorem (5.3), we see

$$\frac{d^2}{dt^2}\langle x\rangle = \frac{d^2s(t)}{dt^2} = -2\left\langle\frac{\partial U}{\partial x}\right\rangle$$

i.e., $\phi_s(x-s(t))$ is a travelling wave moving according to the second Newton's law of motion.

II. Conditions for Existence of Soliton

Naturally, we can not hope that the equation (1) always has soliton solution for any nonlinear term $f(|\psi|^2)$ and for any external field $U(x,t)$. In this connection, we have the following theorem :

Theorem 1 The NLS equation (1) has a soliton solution, if and only if the following two conditions are satisfied :

(i) There exists such a function $s(t)$ $(\frac{ds}{dt} \not\equiv 0)$ that $U(x,t)$ can be rewritten as

(6) $U(x,t) = V(x-s(t)) - \frac{1}{2}\ddot{s}x + h(t) \equiv V(\xi) - \frac{1}{2}\ddot{s}x + h(t)$,

where $\xi \equiv x-s(t)$, $\ddot{s} = \frac{d^2s}{dt^2}$ and $h(t)$ is an arbitrary definite function of t ;

(ii) The following ordinary differential equation

(7) $\frac{d^2\phi}{d\xi^2} - V(\xi)\phi + f(\phi^2)\phi + \lambda\phi = 0$

has a solution $\phi_\lambda(\xi)$ for a certain eigenvalue λ, such that

$$0 < \int_{-\infty}^{\infty} \phi_\lambda^2(\xi)d\xi < +\infty$$

Wen both (i) and (ii) are satisfied, the soliton solution of (1) has the following form :

(8) $$\begin{cases} \psi_{s\lambda}(x,t) = \phi_\lambda(x-s(t))e^{i\theta_\lambda(x,t)} \\ \theta_\lambda(x,t) = \frac{1}{2}\dot{s}x - \lambda t - \int_0^t [h(t') + \frac{1}{4}\dot{s}^2(t')]dt' + \theta_0 \end{cases}$$

Proof Substituting (8) into (1), and using (6) and (7), we may verify that the conditions (i) and (ii) are sufficient for the existence of the soliton solution. Here, we prove only the necessity of these conditions.

Assume that equation (1) has a soliton solution

(9) $\psi_s(x,t) = \phi(x-s(t))e^{i\theta(x,t)}$

Substituting (9) into (1) and separating the real and the imaginary terms, we obtain

$$\begin{cases} -\phi\frac{\partial\theta}{\partial t} - \phi(\frac{\partial\theta}{\partial x})^2 + \frac{\partial^2\phi}{\partial x^2} + f(\phi^2)\phi = U(x,t)\phi & (10.1) \\ -\dot{s}\frac{\partial\phi}{\partial x} + 2\frac{\partial\phi}{\partial x}\frac{\partial\theta}{\partial x} + \phi.\frac{\partial^2\theta}{\partial x^2} = 0. & (10.2) \end{cases}$$

We point out that

(11) $\frac{\partial\theta}{\partial x} \equiv \frac{1}{2}\dot{s}$

In fact, if $\frac{\partial\theta}{\partial x} \not\equiv \frac{1}{2}\dot{s}$, from (10.2) we have

$$2\frac{\partial\phi}{\partial x}.\phi^{-1} = -\frac{\partial^2\theta}{\partial x^2}(\frac{\partial\theta}{\partial x} - \frac{1}{2}\dot{s})^{-1}$$

Integrating the both sides of this equality, we find

$$\phi^2 = \alpha(t)\left(\frac{\partial\theta}{\partial x} - \frac{1}{2}\dot{s}\right)^{-1}$$

or

$$(12) \qquad \frac{\partial\theta}{\partial x} = \alpha(t)\phi^{-2} + \frac{1}{2}\dot{s}$$

where $\alpha(t)$ is an arbitrary function of t. On the other hand,

$$(13) \qquad \dot{s} = 2\,\frac{\int_{-\infty}^{\infty}\phi^2\cdot\frac{\partial\theta}{\partial x}dx}{\int_{-\infty}^{\infty}\phi^2 dx}$$

Inserting (12) into (13), we have

$$\alpha(t) \equiv 0$$

Hence, from (11), we still get (11).

From (11), we see

$$(14) \qquad \theta(x,t) = \frac{1}{2}\dot{s}x + g(t)$$

and

$$(15) \qquad \frac{\partial\theta}{\partial t} = \frac{1}{2}\ddot{s}x + \dot{g}(t)$$

where g(t) is an arbitrary function of t.

Substituting (11) and (15) into (10.1), we obtain

$$(16) \qquad \frac{\partial^2\phi}{\partial x^2} - [U(x,t) + \frac{1}{2}\ddot{s}x + \dot{g} + \frac{1}{4}\dot{s}^2]\phi + f(\phi^2)\phi = 0$$

Notice that

$$(17) \qquad \frac{\partial^2\phi}{\partial x^2} = \frac{\partial^2\phi(x-s(t))}{\partial x^2} \equiv \frac{d^2\phi}{d\xi^2}, \quad (\xi \equiv x-s(t))$$

From (16) we find that

$$U(x,t) + \frac{1}{2}\ddot{s}x + \dot{g} + \frac{1}{4}\dot{s}^2$$

must equal a function of ξ. Therefore, we may set

$$(18) \qquad U(x,t) + \frac{1}{2}\ddot{s}x + \dot{g} + \frac{1}{4}\dot{s}^2 = V(\xi) - \lambda$$

where λ is a real number. (18) can be rewritten as

$$(19) \qquad U(x,t) = V(\xi) - \frac{1}{2}\ddot{s}x + h(t)$$

where $h(t) = -(\lambda + \dot{g} + \frac{1}{4}\dot{s}^2)$. Equation (16) becomes

$$(20) \qquad \frac{d^2\phi}{d\xi^2} - V(\xi)\phi + f(\phi^2)\phi + \lambda\phi = 0$$

Thus, we have seen that if equation (1) has a soliton solution (9), then U(x,t) can be rewritten in the form (19), and equation (20) must have a solution $\phi(\xi)$ for a certain real number λ, such that

$$0 < \int_{-\infty}^{\infty}\phi^2 d\xi < +\infty$$

These facts have shown the necessity of the conditions (i) and (ii).

In addition, since $\dot{g} = -\lambda - [h(t) + \frac{1}{4}\dot{s}^2]$, we get

$$(21) \qquad g(t) = -\lambda t + \int_0^t [h(t') + \frac{1}{4}\dot{s}^2(t')]dt' + \theta_0$$

Inserting (21) into (14), we obtain (8) finally.

Hence, theorem 1 has been proved.

If an external field U(x,t) can be written in the form (6), it will be called the harmonious field.

<u>Example 1</u> For any continuous functions A(t) and B(t),

$$U_1(x,t) = A(t)x + B(t)$$

is a harmonious field. In fact, we may let

$$s_1(t) = -2\int_0^t dt' \int_0^{t'} A(t'')dt'' + ct + d$$

$$h_1(t) = B(t), \qquad V(\xi) \equiv 0$$

then

$$U_1(x,t) = V_1(\xi) - \frac{1}{2}\ddot{s}_1 x + h_1(t).$$

In the special case,

$$U_1(x,t) = 2\alpha x$$

Chen and Liu has obtained the exact soliton solution of the corresponding NLS equation[10].

Example 2 For any positive number k and for any continuous real functions A(t) and B(t), the external field

(22) $$U_2(x,t) = kx^2 + A(t)x + B(t)$$

is harmonious. In fact, let $s_2(t)$ be the solution of the differential equation

(23) $$\ddot{s} = -4ks - 2A(t)$$

then

(24) $$s_2(t) = \alpha\cos(2\sqrt{k}\,t + \beta) + s_0(t)$$

where $s_0(t)$ is a special solution of (23). Besides, set

(25) $$h_2(t) = -ks_2^2(t) + B(t), \quad V_2(\xi) = k\xi^2.$$

Then we find

(26) $$U_2(x,t) = V_2(\xi) - \frac{1}{2}\ddot{s}_2 x + h_2(t)$$

Unfortunately, in most of more complicated cases, U(x,t) is not harmonious. For instance,

$$U(x,t) = kx^4$$

is inharmonious.

Theorem 1 shows that in the non-relativistic case, only for some harmonious fields, the NLS equation (1) can have

strict soliton solution, but for the inharmonious field, the amplitude $\phi(x,t)$ of a nonlinear wave $\psi = \phi e^{i\theta}$ can not be a travelling wave, and its shape must change more or less as it moves in the field. Perhaps the latter case means that the structure of an elementary particle has to change more or less while it moves in an inharmonious field.

III. Nonlinear Eigenvalue Problem of the NLS Equation

We point out that one can not determine if equation (1) has a soliton solution (8) only by theorem 1. In fact, if we have known that U(x,t) is harmonious, we must further determine whether equation (7) has a suitable solution.

In addition, there is another interesting fact : If we look for a bound stationary state solution $\psi(x,t) = \phi_\lambda(x)e^{-i\lambda t}$ of another NLS equation

(27) $$i\frac{\partial\psi}{\partial t} + \frac{\partial^2\psi}{\partial x^2} + f(|\psi|^2)\psi = V(x)\psi$$

we can find that ϕ_λ and λ must satisfy the following differential equation

(28) $$\begin{cases} \dfrac{d^2\phi}{dx^2} - V(x)\phi + f(\phi^2)\phi + \lambda\phi = 0 \\ 0 < \displaystyle\int_{-\infty}^{\infty} \phi^2 dx < +\infty \end{cases}$$

which is just the same as (7).

Therefore, the research on the nonlinear eigenvalue problem (28) is important either to the soliton problem or to the stationary state problem.

Before the exploration of problem (28), let me quote some basic facts on the corresponding linear problem. The details can be seen in [11]. It is well-known that if $\psi(x,t) = \phi(x)e^{-i\mu t}$ is the bound stationary state solution of the corresponding linear Schrödinger equation

$$(29) \qquad i\frac{\partial\psi}{\partial t} + \frac{\partial^2\psi}{\partial x^2} = V(x)\phi$$

then $\phi(x)$ and μ must be the solution of the linear eigenvalue problem

$$(30) \qquad \begin{cases} \dfrac{d^2\phi}{dx^2} - V(x)\phi + \mu\phi = 0 \\ 0 < \displaystyle\int_{-\infty}^{\infty} \phi^2 dx < +\infty \end{cases}$$

If the continuous real function $V(x)$ satisfies

$$(31) \qquad \lim_{x\to\pm\infty} V(x) = +\infty$$

problem (30) has a series of eigenvalues $\{\mu_n\}$, such that

$$-\infty < \mu_0 < \mu_1 < \ldots < \mu_n < \ldots < +\infty$$

and

$$\lim_{n\to\infty} \mu_n = +\infty$$

the eigenfunction $\phi_n(x)$ corresponding to the (n+1)-th eigenvalue μ_n has exactly n zeros.

If $V(x)$ satisfies

$$(32) \qquad \liminf_{x\to\pm\infty} V(x) = 0$$

problem (30) has a negative discrete spectrum, such that

$$-\infty < \mu_0 < \mu_1 < \ldots < 0$$

The number of the discrete eigenvalues depends only on the function $V(x)$ and can be either finite or countable infinite. This number will be denoted by $n(V)$. If $n(V) = \infty$, the set $\{\mu_n\}$ has only one condensation point — 0. The eigenfunction $\phi_n(x)$ corresponding to the (n+1)-th eigenvalue has also exactly n zeros.

The nonlinear eigenvalue problem (28) is similar to the corresponding linear eigenvalue problem (30) in many aspects. We have proved the following two theorems :

Theorem 2 If the real function $f(x)$ in (28) is continuous on the interval $[0,+\infty)$, and if the continuous real function $V(x)$ satisfies (31), then for any positive number M and any integer $n \geqslant 0$, the nonlinear eigenvalue problem has (at least) an eigenvalue $\lambda_n(M)$ and its eigenfunction $\phi_n(x,M)$ with exactly n zeros, such that

$$\begin{cases} (33.1) \quad \|\phi_n(x,M)\| \equiv \sup\{|\phi_n(x,M)| \mid x\in(-\infty,\infty)\} = M \\ (33.2) \quad \mu_n - \overline{f}(M) \leqslant \lambda_n(M) \leqslant \mu_n - \underline{f}(M) \end{cases}$$

where μ_n is just the (n+1)-th eigenvalue of the corresponding linear problem (30), and

$$\begin{cases} (34.1) \quad \overline{f}(M) = \sup\{f(\rho^2) \mid \rho \in [0,M]\} \\ (34.2) \quad \underline{f}(M) = \inf\{f(\rho^2) \mid \rho \in [0,M]\} \end{cases}$$

Theorem 3 If the real function $f(x)$ in (28) is continuous on the interval $[0,+\infty)$ and $\lim\limits_{x\to 0} f(x) = f(0) = 0$, and if the continuous real function $V(x)$ satisfies (32), and if $n(V) > 0$ for the corresponding linear problem (30), then for any natural number $m \leqslant n(V)$, there exists a positive number ϵ_m, such that, for any positive number M, which is smaller than ϵ_m, the nonlinear problem (28) has an eigenvalue $\lambda_m(M)$ and a corresponding eigenfunction $\phi_m(x,M)$ with exactly m zeros, which satisfy

$$\begin{cases} (35.1) \quad \|\phi_m\| \equiv \sup\{|\phi_m(x,M)| \mid x \in (-\infty,\infty)\} = M \\ (35.2) \quad \mu_m - \overline{f}(M) \leqslant \lambda_m(M) \leqslant \mu_m - \underline{f}(M) \end{cases}$$

where μ_m is just the (m+1)-th eigenvalue of the corresponding linear problem (30), and $\overline{f}(M)$ and $\underline{f}(M)$ are defined also according to (35).

Since the proof of these two theorems needs much pure mathematical inference and occupies much space, here we only briefly sketch the idea of the proof of theorem 2. The proof of theorem 3 is similar to it. The interested reader can find the details in [3] and [4].

At first, we notice that, if let $\tilde{\phi}(x)$ be an arbitrary function, such that

$$||\tilde{\phi}(x)|| \equiv \sup\{|\tilde{\phi}(x)|\,|x \in (-\infty,\infty)\} \leqslant M$$

then for the given $M > 0$ and $n \geqslant 0$, the linear eigenvalue problem

$$\begin{cases} \dfrac{d^2\phi}{dx^2} - [V(x) - F(\tilde{\phi}(x))]\phi + \lambda\phi = 0 \\ 0 < \displaystyle\int_{-\infty}^{\infty} \phi^2 dx < +\infty \end{cases}$$

has an eigenvalue $\lambda_n(\tilde{\phi})$ and its corresponding eigenfunction $\phi_n(x,\tilde{\phi})$, such that

$$\begin{cases} ||\phi_n(x,\tilde{\phi})|| = M \\ \mu_n - \overline{f}(M) \leqslant \lambda_n(\tilde{\phi}) \leqslant \mu_n - \underline{f}(M) \end{cases}$$

Therefore we may establish a mapping Q_n by

$$Q_n(\tilde{\phi}) = \phi_n(x,\tilde{\phi})$$

which maps the closed ball S(M) into its boundary $\partial S(M)$. The ball S(M) is defined as

$$S(M) \equiv \{\phi(x)|\ ||\phi(x)|| \leqslant M\}$$

which is a closed and convex subset of the Banach space $C_0(R)$.

Then we can have further estimation on $\phi_n(x,\tilde{\phi})$:
There exist three positive numbers H(M,n), X(M,n) and V_0, which rely only on M and n, such that for any $\tilde{\phi}(x) \in S(M)$

$$||\phi_n(x,\tilde{\phi})|| + ||\phi_n'(x,\tilde{\phi})|| + ||\phi_n''(x,\tilde{\phi})|| \leqslant H(M,n)$$

and when $|x| > X(M,n)$

$$|\phi_n(x,\tilde{\phi})| < M\, e^{2V_0(X(M,n)-|x|)}$$

$$|\phi_n'(x,\tilde{\phi})| < \min\{M, \frac{M}{|x|-X(M,n)}\}$$

By these evaluations we know that $Q_n(S(M))$ is a compact subset of $\partial S(M)$.

Furthermore, we can prove that $Q_n(\tilde{\phi})$ is continuous, i.e.,

$$\lim_{m\to\infty} ||Q_n(\tilde{\phi}_m) - Q_n(\tilde{\phi})|| = 0$$

and

$$\lim_{m\to\infty} \lambda_n(\tilde{\phi}_m) = \lambda_n(\tilde{\phi})$$

provided $\lim\limits_{m\to\infty} ||\tilde{\phi}_m - \tilde{\phi}|| = 0$.

Hence, Q_n is a compact mapping, which maps the ball S(M) into $\partial S(M) \subset S(M)$. By the Schauder's fixed point theorem [12], we know that Q_n has a fixed point $\phi_n(x)$, i.e.,

$$Q_n(\phi_n(x)) = \phi_n(x)$$

$\phi_n(x)$ is just the eigenfunction being sought.

By theorem 2, the shaded parts in Fig.1 show the regions in the λ-M plane, where the eigenvalues of the nonlinear problem (28) may exist under the condition (31).

By theorem 3, the shaded parts in Fig.2 show the regions where the eigenvalues of (28) may exist under condition (32).

Theorem 2 and theorem 3 show that if the corresponding linear problem (30) has a discrete spectrum, then the nonlinear problem (28) has also a discrete spectrum, provided the norms (33.1) and (35.1) of the eigenfunctions are kept fixed, and the number of zeros of the nonlinear eigenfunction equals the one of the corresponding linear eigenfunction.

If the nonlinear eigenvalue problem (28) has solutions, then it has at least such a kind of eigenfunctions that the eigenfunction has no zero and is shaped like a bell (Fig.3). According to theorem 1, the corresponding soliton solution (8) is also bell-shaped.

It is also possible that the eigenfunction has one, two or more zeros (Fig.4), and the corresponding soliton solution (8) has also one, two or more zeros. However, this kind of "soliton" solutions has not been seen in the common soliton theory.

These facts lead to a question : Can we call the latter solutions "solitons" in a stricter sense ? This question spures us to study the stability of solution (8). By the stability, we may strengthen the definition of soliton.

IV. Stability of Soliton Solution

Generally speaking, a soliton solution should have two properties : its invariable shape and its stability[13]. But in our weakened definition, only the first property is guaranteed evidently. In fact, the stability of soliton is more concerned in the present theory, since the soliton is to be regarded as an elementary particle.

However, since the soliton is a special solution of the engaged nonlinear evolution equation, it is still a complex problem to give a suitable definition on its stability. For some well-known nonlinear equations such as the KdV equation, the sine-Gordon equation, the stability of soliton has caused a variety of interesting discussions[13][14][15][16][17].For the

nonlinear Schrödinger equation (1), because the soliton solutions are complex-valued, and in most of cases the exact analytic form of the soliton solution can not be obtained, the problem of the stability looks more complex.

In spite of that, by bestowing suitable physics significance to the solution $\psi(x,t)$ with the aid of de Broglie's idea, we may find a definition of the stability of the soliton solution (8) in a sense of dynamics.

The functional

$$(36)\quad E(\psi) = \int_{-\infty}^{\infty}\left(\left|\frac{\partial\psi}{\partial x}\right|^2 + U(x,t)|\psi|^2 - F(\psi^2)\right)dx$$

$$= \int_{-\infty}^{\infty}\left(\left(\frac{\partial\phi}{\partial x}\right)^2 + \phi^2\left(\frac{\partial\theta}{\partial x}\right)^2 + U\phi^2 - F(\phi^2)\right)dx$$

where $F(x) = \int_0^x f(x')dx'$, is regarded as the energy of the solution of the NLS equation (1). One can then conjecture that among the solutions of (1) satisfying certain constraints, the soliton solution should make the energy the minimum. From the consideration on dynamics, we can naturally think of the following constraints :

$$\begin{cases} (37.1) & \int_{-\infty}^{\infty}\psi^*\psi dx = \int_{-\infty}^{\infty}\phi^2 dx = M > 0 \\ (37.2) & \langle x\rangle\Big|_{t=t_0} = \dfrac{\int_{-\infty}^{\infty}\phi^2 x dx}{M}\Big|_{t=t_0} = x(t_0) \\ (37.3) & \dfrac{d\langle x\rangle}{dt}\Big|_{t=t_0} = \dfrac{2\int_{-\infty}^{\infty}\phi^2\frac{\partial\theta}{\partial x}dx}{M}\Big|_{t=t_0} = \dot{S}(t_0) \end{cases}$$

Constraint (37.1) requires that all the solutions to be compared with the soliton must have the same "size". Constraint (37.2) requires that all the solutions to be compared with the soliton at a given time $t = t_0$ must have the same "position" in the external field. And constraint (37.3) requires that all the solutions to be compared with the soliton at a given time $t = t_0$ must have the same "mean velocity".

These constraints seem to be reasonable. In fact, we can prove the following theorem :

Theorem 4 If the conditions in theorem 1 are all satisfied, then at any given time t_0, among all the solutions of (1), which all satisfy the constraints (37.1), (37.2) and (37.3), only the soliton solutions can render the energy functional (36) the extremum (the minimum or the saddle point).

Proof Using the Lagrange multipliers $\lambda_1(t)$ and $\lambda_2(t)$ to the extreme value problem under the constraints (37.2) and (37.3) at the given time t_0, we obtain the Euler equations

$$\begin{cases} \dfrac{\partial^2\phi}{\partial x^2} - \phi\left(\dfrac{\partial\theta}{\partial x}\right)^2 - U\phi + f(\phi^2)\phi + \dfrac{\lambda_1}{M}\phi x + \dfrac{\lambda_2}{M}\phi\dfrac{\partial\theta}{\partial x} = 0, & (38.1) \\[2ex] -2\dfrac{\partial}{\partial x}\left(\phi^2\dfrac{\partial\theta}{\partial x}\right) + \dfrac{\lambda_2}{M}\dfrac{\partial\phi^2}{\partial x} = 0 & (38.2) \end{cases}$$

From (38.2) we get

$$\frac{\lambda_2}{M}\phi^2 = 2\phi^2\frac{\partial\theta}{\partial x} + C(t_0) \tag{39}$$

Integrating the both sides of (39), and using (37.3), we can find that $C = 0$, and

$$\frac{\lambda_2(t_0)}{M} = 2\frac{\partial\theta}{\partial x} = \dot{S}(t_0) \tag{40}$$

On the other hand, inserting the formal solution $\psi = \phi e^{i\theta}$ into equation (1), and separating the real and the imaginary parts, we obtain

$$\begin{cases} -\phi\dfrac{\partial\theta}{\partial t} - \phi\left(\dfrac{\partial\theta}{\partial x}\right)^2 + \dfrac{\partial^2\phi}{\partial x^2} + f(\phi^2)\phi = U(x,t)\phi & (41.1) \\[2ex] \dfrac{\partial\phi}{\partial t} + 2\dfrac{\partial\phi}{\partial x}\dfrac{\partial\theta}{\partial x} + \phi\dfrac{\partial^2\theta}{\partial x^2} = 0 & (41.2) \end{cases}$$

Comparing (38.2) with (41.2), we find

$$\frac{\lambda_2}{M} = -\frac{\partial\phi^2}{\partial t}\Big/\frac{\partial\phi^2}{\partial x} \tag{42}$$

And comparing (38.1) with (41.1), we get

$$\phi\frac{\partial\theta}{\partial t} + \frac{\lambda_1}{M}\phi x + \frac{\lambda_2}{M}\phi\frac{\partial\theta}{\partial x} = 0$$

Inserting (42) to this equality, we obtain

$$\frac{\partial\phi^2}{\partial x}\cdot\frac{\partial\theta}{\partial t} + \frac{\lambda_1}{M}\frac{\partial\phi^2}{\partial x}\cdot x - \frac{\partial\phi^2}{\partial t}\cdot\frac{\partial\theta}{\partial x} = 0$$

Integrating its both sides, we find

$$\lambda_1 = -\frac{d}{dt}\int_{-\infty}^{\infty}\phi^2\frac{\partial\theta}{\partial x}\,dx = -\frac{M}{2}\ddot{S}(t_0) \tag{43}$$

Substituting (40) and (43) into (38.1), we get

$$\frac{\partial^2\phi}{\partial x^2} - \left(U(x,t_0) + \frac{\ddot{S}(t_0)}{2}x\right)\phi + f(\phi^2)\phi + \frac{1}{4}\dot{S}(t_0)\phi = 0 \tag{44}$$

Since $U(x,t)$ can be written in form (6), (44) can be rewritten as

$$\frac{d^2\phi}{d\xi^2} - V(\xi)\phi + f(\phi^2)\phi + \lambda(t_0)\phi = 0 \tag{45}$$

where

$$\lambda(t_0) = \frac{1}{4}\dot{S}(t_0) - h(t_0)$$

From (40) and (45), we find that if a solution $\psi = \phi e^{i\theta}$ satisfies the constraints (37) and the Euler equations (38), it must be a soliton solution (8), i.e., only the soliton solution (8) can render the energy (36) the extremum under the constraints (37).

On the other hand, it is easy to see that solution (8) renders

$$\int_{-\infty}^{\infty}\phi^2\left(\frac{\partial\theta}{\partial x}\right)^2 dx$$

which is one part of the energy (36), the minimum under the constraints (37.1) and (37.3) at the given time t_0. This fact shows that the soliton solution (8) can not render the energy the minimum.

This proves the theorem.

Theorem 4 shows that the soliton solution (8) is stable or metastable in the sense of dynamics.

A theorem on the stability of the stationary state solution can be proved more easily.

Theorem 5 If the NLS equation (27) has bound stationary state solution, then among all its solutions satisfying the constraint (37.1), only the stationary solution can render the following energy

$$E(\psi) = \int_{-\infty}^{\infty}\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \phi^2\left(\frac{\partial\theta}{\partial x}\right)^2 + V(x)\phi^2 - F(\phi^2)\right]dx \tag{46}$$

the extremum (the minimum or the saddle point).

By a comparison between the linear and the nonlinear cases, we conjecture that the soliton or the stationary solution, which has no zero, renders the energy the minimum and is stable, and the soliton or the stationary solution, which has at least one zero, is the saddle point of the energy functional and is metastable.

Unfortunitely, since the exact analytic solution can not be obtained in most of cases, and the calculation of the higher order variations of the energy functional is very difficult, up to now this conjecture has not been proved strictly.

In addition, the above-mentioned definition on stability is not clear enough, because there is not a definite and explicit measure to describe this stability in the definition.

In spite of that, we may have a stricter study on the stability of the soliton solution of the simplest NLS equation

$$i\frac{\partial\psi}{\partial t} + \frac{\partial^2\psi}{\partial x^2} + 2|\psi|^2\psi = 0 \tag{47}$$

The nonlinear stability of the soliton solution of (47) has been studied with the inverse scattering method[18]. However, this study has not given a clear measure to describe the stability.

In the study on the nonlinear stability of soliton solution of the KdV equation, Benjamin[17] has developed a generalized measure of the difference between the soliton solution Φ_T and other solution Φ, which permits Φ to be translated along the x axis until the best match with Φ_T is obtained. His generalized measure comes from the following fact. The shape and the "size" of the soliton solution are stable, while its "position" is unstable.

Benjamin's idea on the generalized measure can be used here for the study on the nonlinear stability of the soliton solution (47). The soliton solution of (1) is

$$\psi_s(x,t) = \phi_s(x,t)e^{i\theta_s(x,t)} \tag{48}$$

where

$$\begin{cases} \phi_s(x,t) = A\,\mathrm{sech}[A(x-Bt)] & (48.1)\\ \theta_s(x,t) = \frac{B}{2}(x-Ct) & (48.2)\end{cases}$$

and A ($\neq 0$) and B are arbitrary real numbers, and

$$2BC = B^2 - 4A^4$$

Using the facts revealed in theorem 4, and having more consideration on the physics significance of the solution of (47), we may introduce the following generalized measure of the difference between any two solutions $\psi_1 = \phi_1 e^{i\theta_1}$ and $\psi_2 = \phi_2 e^{i\theta_2}$

$$(49)\quad \rho(\psi_1,\psi_2) = \inf_{\xi\in\mathbb{R}} \min_{\pm\tau_\xi} \left(\int_{-\infty}^{\infty} \left\{ (\pm\tau_\xi\phi_1-\phi_2)^2 + \left(\pm\tau_\xi \frac{\partial\phi_1}{\partial x} - \frac{\partial\phi_2}{\partial x}\right)^2 + \left(\pm\tau_\xi\phi_1\left(\frac{\partial\tau_\xi\theta_1}{\partial x} - \frac{V_1}{2}\right) - \phi_2\left(\frac{\partial\theta_2}{\partial x} - \frac{V_2}{2}\right)\right)^2 \right\} dx \right)^{1/2}$$

where τ_ξ is the shift transformation operator defined as

$$\tau_\xi\phi(x,t) = \phi(x-\xi,t)$$

and

$$(50)\quad V_i \equiv \frac{d}{dt}\langle x\rangle_i \equiv \frac{d}{dt}\frac{\int_{-\infty}^{\infty}\phi_i^2 x\,dx}{\int_{-\infty}^{\infty}\phi_i^2 dx} = \frac{2\int_{-\infty}^{\infty}\phi_i^2\frac{\partial\theta_i}{\partial x}dx}{\int_{-\infty}^{\infty}\phi_i^2 dx}, \qquad i=1,2.$$

Note that V_1 and V_2 are conservative quantities in the case of (47).

The term

$$\left\{\phi_1\left(\frac{\partial\theta_1}{\partial x} - \frac{V_1}{2}\right) - \phi_2\left(\frac{\partial\theta_2}{\partial x} - \frac{V_2}{2}\right)\right\}^2$$

represents the difference between the internal motions of these two solutions. And the terms

$$(\phi_1-\phi_2)^2 + \left(\frac{\partial\phi_1}{\partial x} - \frac{\partial\phi_2}{\partial x}\right)^2$$

can represent the difference between the shapes of ϕ_1 and ϕ_2.

The shift transformation τ_ξ and the operation $\inf_{\xi\in\mathbb{R}} \min_{\pm\tau_\xi}$ in (49) guarantee that the measure is taken through translating ψ_1 along the x axis to the position where the amplitude ϕ_1 and the internal motion of ψ_1 can best match with

the ones of ψ_2.

Just using the generalized measure (49), we have proved that the soliton solution (48) is stable with respect to the small perturbation on the shape of ϕ_s and on its internal motion. Here we neglect the proof of this result, since it includes many careful evaluations which occupies large space. The interested reader can find it in [5].

V. Method of Releasing Energy

We have mentioned that, if all the conditions in theorem 1 are satisfied, we can not yet hope to obtain the soliton solution in exact analytic form for most of the NLS equations. Therefore, we should find some method for seeking the soliton solution.

In section III, we have seen that to find the soliton solution of (1) can be reduced to find the stationary state solution of (27). And in section IV, we have known that the stationary solution of (27) renders the energy (46) the extremum (the minimum or the saddle point) under the constraint (37.1). Besides, it is well-known that for any given solution $\psi(x,t) = \phi e^{i\theta}$ of (27), the quantity (37.1) and the energy (46) are both conservative.

Using the above-mentioned facts, we can develop a method for seeking the soliton solution of the NLS equation, which satisfies all the conditions in theorem 1.

Let $\psi_0(x,t) = \phi_0(x,t)e^{i\theta_0(x,t)}$ be an arbitrary given nonstationary state solution of the corresponding NLS equation (27) with the initial values :

$$(51)\quad \begin{cases} \phi_0(x,0) = \phi_0(x) \\ \theta_0(x,0) \equiv 0 \end{cases}$$

Then for almost all $\tau > 0$,

(52) $$\int_{-\infty}^{\infty} \phi_0^2(x,t)\left(\frac{\partial\theta_0}{\partial x}\right)^2 dx > 0.$$

Let $\psi_1(x,t) = \phi_1(x,t)e^{i\theta_1(x,t)}$ be another solution of (27) with the initial values :

(53) $$\phi_1(x,0) = \phi_1(x) \equiv \phi_0(x,\tau)$$
$$\theta_1(x,0) \equiv 0.$$

Let E_0 and E_1 be the energies of ψ_0 and ψ_1 respectively.

Because of the conservation of the energy for any solution of (27), from (46) and (52) we obtain

(54) $$E_1 = E_0 - \int_{-\infty}^{\infty} \phi_0^2(x,\tau)\left[\frac{\partial\theta_0(x,\tau)}{\partial x}\right]^2 dx < E_0$$

It is easy to see that ψ_0 and ψ_1 satisfy the same constraint (37.1).

Therefore, we can say that the solution ψ_1 is obtained from ψ_0 through releasing its kinetic energy (52) at $t = \tau$, while the constraint (37.1) is maintained. And we can also say that ψ_1 is closer to the stationary state than ψ_0. Repeating the above procedure of releasing energy successively, we can obtain

$$\begin{cases} \psi_0(x,t),\ \psi_1(x,t),\ldots,\psi_n(x,t),\ldots & (55.1) \\ \phi_0(x)\ ,\ \phi_1(x),\ \ldots,\phi_n(x),\ldots & (55.2) \\ E_0 >\ldots\ E_1 >\ldots\ E_n > \ldots & (55.3) \end{cases}$$

where all ψ_n and ϕ_n satisfy the common constraint (37.1).

In many cases, there is a finite limit for the sequence (55.3), for example, we have

Theorem 6 There exists a finite lower bound for the sequence (55.3), if $f(|\psi|^2) = B|\psi|^2$ $(B > 0)$ and if there is a finite lower bound for $V(x)$.

If (55.3) converges, the sequence (55.1) often converges to a stationary state solution of (27), and (55.2) converges to an eigenfunction $\phi_1(x)$ of the nonlinear eigenvalue problem (28), and the eigenvalue can be obtained from

(56) $$\lim_{n\to\infty} \psi_n(x,t) = \lim_{n\to\infty} \phi_n(x,t)e^{i\theta_n(x,t)} = \phi_\lambda(x)e^{-i\lambda t}$$

By theorem 1, we know that, if $\phi_\lambda(x)$ and λ are found, the corresponding soliton solution of (1) can also be obtained according to (8).

The method of releasing energy can be realized numerically through using the computer. Chang Qian-shun has developed a conservative difference scheme for the initial and boundary value problem of the NLS equation[8]. This scheme has been successfully used for the realization of this method[1] [7].

The method of releasing energy can also be used for seeking the approximate analytic solutions of the stationary state or the soliton[6].

Recently the fundamental idea of the method of releasing energy has been generalized to the problem of finding the stationary solutions for a wider range of equations of motion[6].

VI. Harmonic Oscillator

The linear Schrödinger equation for a harmonic oscillator is

(57) $$i\frac{\partial\psi}{\partial t} + \frac{\partial^2\psi}{\partial x^2} = kx^2\psi, \quad k > 0.$$

This equation is carefully discussed in almost every text book on quantum mechanics, since it is the most important example on one dimensional motion and it reveals many fundamental properties on quantum mechanics.

A nonlinear generalization of (57) is

(58) $$i\frac{\partial\psi}{\partial t}+\frac{\partial^2\psi}{\partial x^2}+f(|\psi|^2)\psi = kx^2\psi$$

and the simplest one is

(2) $$i\frac{\partial\psi}{\partial t}+\frac{\partial^2\psi}{\partial x^2}+2|\psi|^2\psi = kx^2\psi$$

From example 2 in section II, we see that the external field $U(x) = kx^2$ is harmonious and the corresponding functions are

$$\begin{cases} s(t) = \alpha\cos(2\sqrt{k}\,t+\beta) & (59.1)\\ h(t) = -ks^2(t) & (59.2)\\ V(\xi) = k\xi^2 & (59.3)\end{cases}$$

where α and β are arbitrary constant real numbers.

The corresponding nonlinear eigenvalue problem is

(60) $$\begin{cases} \dfrac{d^2\phi}{dx^2} - kx^2\phi + f(\phi^2)\phi + \lambda\phi = 0\\ 0 < \displaystyle\int_{-\infty}^{\infty}\phi^2 dx < \infty\end{cases}$$

Since $V(x) = kx^2$ satisfies the condition (31), the conclusion of theorem 2 is applicable to (60), provided $f(x)$ in (60) is continuous on the interval $[0,+\infty)$.

For (2), the estimation (33.2) can be written in an exact form, i.e.,

(61) $$(2n+1)\sqrt{k} - 2M^2 < \lambda_n(M) < (2n+1)\sqrt{k}$$

where

$$\mu_n = (2n+1)\sqrt{k}, \quad n=0,1,2,\ldots$$

is the (n+1)-th eigenvalue of the linear problem

(62) $$\begin{cases} \dfrac{d^2\phi}{dx^2} - kx^2\phi + \mu\phi = 0\\ 0 < \displaystyle\int_{-\infty}^{\infty}\phi^2 dx < +\infty\end{cases}$$

The shaded parts in Fig.5 show the regions in the M-λ plane where the eigenvalue $\lambda_n(M)$ may exist according to (61).

We have seen that all the conditions of theorem 1 have been satisfied by (58) and (2). Therefore, we can conclude that the NLS equation (58) (provided $f(x)$ is continuous on $[0,+\infty)$) and (2) have soliton solutions in the form

$$\begin{cases} \psi_{sn}(x,t) = \phi_{\lambda n}(x-\alpha\cos(2\sqrt{k}\,t+\beta)e^{i\theta_{\lambda n}(x,t)} & (63.1)\\ \theta_{\lambda n}(x,t) = -\alpha\sqrt{k}\,x\sin(2\sqrt{k}\,t+\beta) - \lambda_n t & (63.2)\\ \qquad + \frac{1}{4}\sqrt{k}\,\alpha^2\sin(4\sqrt{k}\,t+2\beta)+\theta_0 & \end{cases}$$

where the eigenvalue $\lambda_n(M)$ and the corresponding eigenfunction $\phi_{\lambda n}(x)$ are the solution of (60). $\psi_{s_0}(x,t)$ is corresponding to the soliton solution with no zero and is stable. $\psi_{sn}(x,t)$ ($n \geqslant 1$) is corresponding to the soliton solution with exactly n zeros and is only metastable.

Using the method of releasing energy, we have obtained the numerical soliton solutions ψ_{s_0}, ψ_{s_1} and ψ_{s_2} under the parameters $k=4$, $\int_{-\infty}^{\infty}\phi^2(x,t)dx = 2$. Fig.3 and Fig.4 just show the eigenfunctions $\phi_{\lambda_0}(x)$, $\phi_{\lambda_1}(x)$ and $\phi_{\lambda_2}(x)$ obtained through releasing energy. Fig.6 shows that the numerical soliton solution ψ_{s_0} moves just like a harmonic oscillator with a frequency

$$\omega = \frac{\sqrt{k}}{\pi}$$

We point out the following interesting fact. The linear Schrödinger equation (57) for the harmonic oscillator is a special case of $f(|\psi|^2) = 0$. So, it has also "soliton" solutions ψ_{sn} (n=0,1,2,...), which all move just like harmonic oscillator with the frequency $\omega = \sqrt{k}/\pi$. This fact on the linear Schrödinger equation (57) had been noted much early (see [19]).

VII. Collision between Solitons

The property of re-emergence of the soliton after a collision with other soliton is used for the definition of soliton in [13]. This property reflects the stability of soliton in a certain sense.

For the sine-Gordon equation and the KdV equation, the re-emergence of two solitons after a collision with each other was found numerically[20],[21] as well as proved theoretically[22],[23].

However, since one can not hope to obtain the soliton solution in the exact and analytic form for most of the NLS equations with external field, it is very difficult to check the re-emergence of two solitons after a collision with each other in exactly analytic form. Therefore, for the present, the computer experiment is the main means for the test of this property.

Using the numerical soliton solutions obtained with the method of releasing energy, Wang Hung-ying has studied the collision between two solitons of the NLS equation (2)[9].

Fig.7 shows the process of a collision between two solitons which have the same "size" and start respectively from a pair of positions symmetric to the origin.

Fig.8 shows the process of a collision between two solitons. One of them is bigger and stays at the origin at the initial time t=0, the other is smaller and starts from the position x = -4.

The numerical results show that the solitons of

the NLS equation (2) do re-emerge after a collision with each other.

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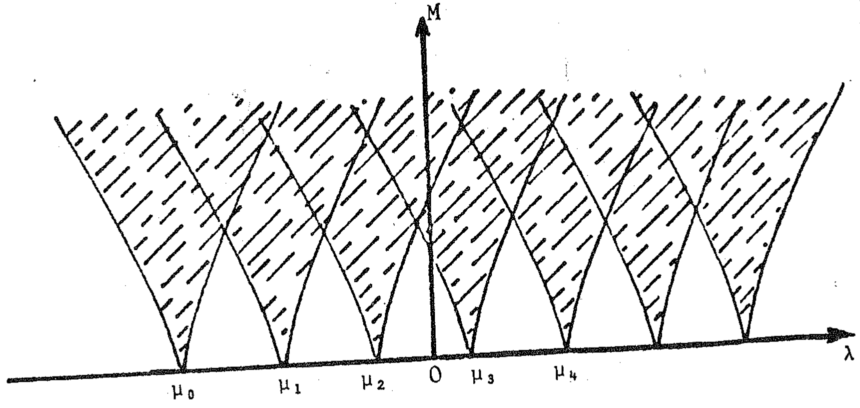


Figure 1

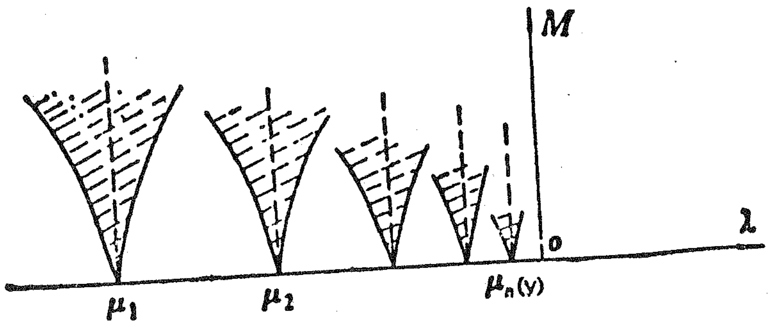


Figure 2

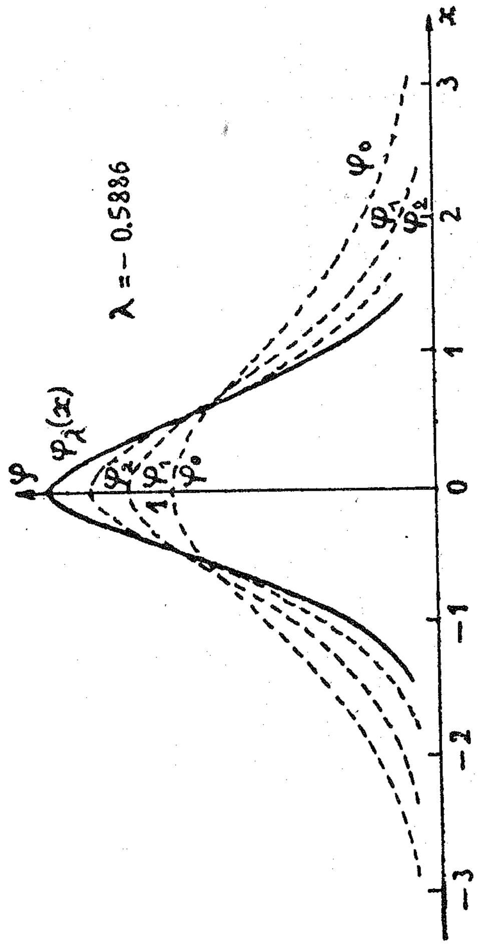
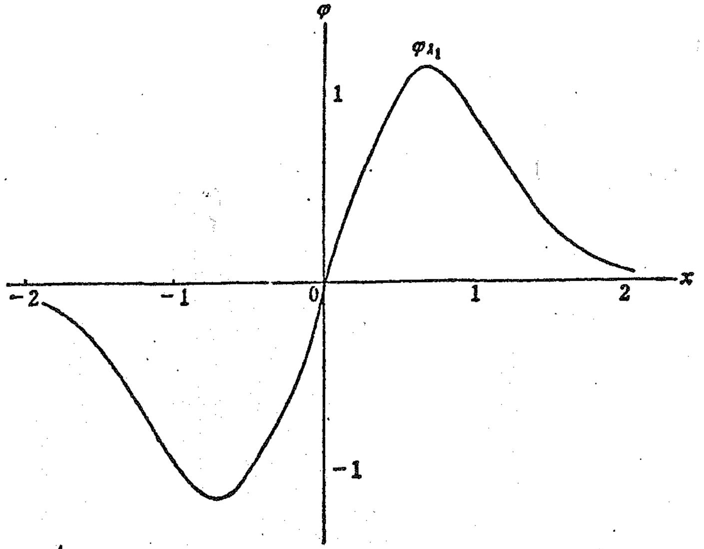
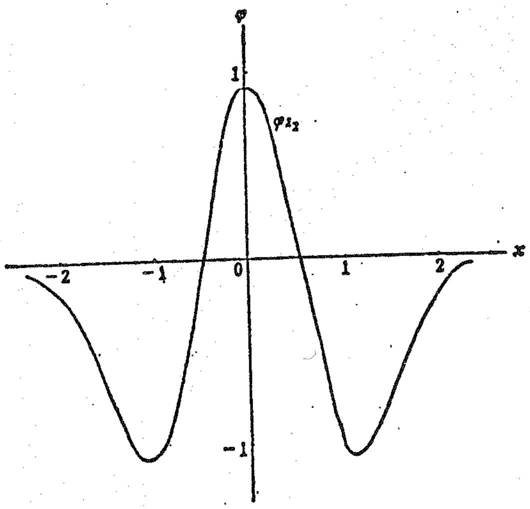


Figure 3

The dotted curves show how $\phi_n(x)$ varies from $\phi_0(x)$ to $\phi_\lambda(x)$ in the processes of releasing energy. $\phi_0(x)$ = sechx.



4-a



4-b

Figure 4

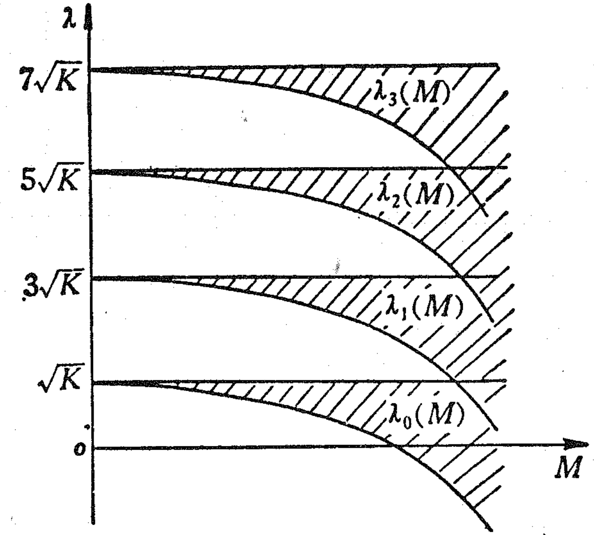


Figure 5

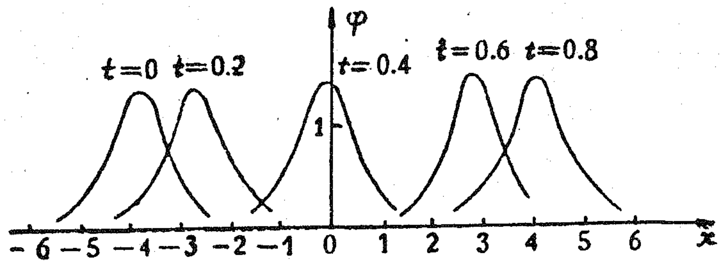


Figure 6. The amplitude $\phi(x,t)$ of the "genuine solution" of (16) are shown at $t = 0$, 0.2, 0.4, 0.6 and 0.8. They are obtained on computer with the initial function $\phi_\lambda(x + 4)$. $\phi_\lambda(x)$ is just the one in Fig. 3. The amplitude moves like a harmonic oscillator, while its shape is maintained.

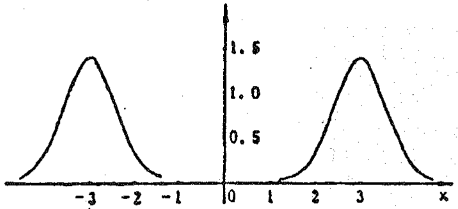


Fig. 7 - I t = 0

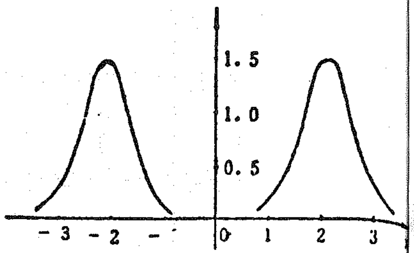


Fig. 7 - II t = 0.2

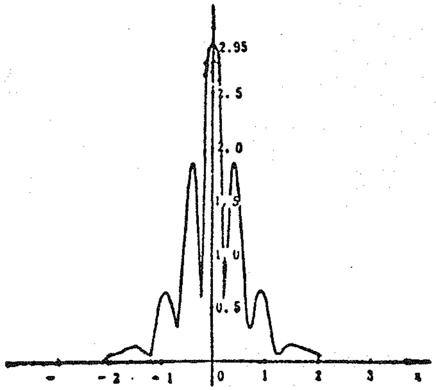


Fig. 7 - III t = 0.4

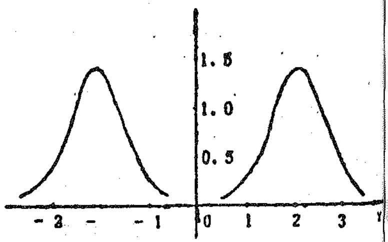


Fig. 7 - IV t = 0.6

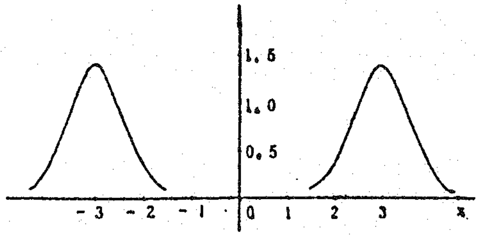


Fig. 7 - V t = 0.8

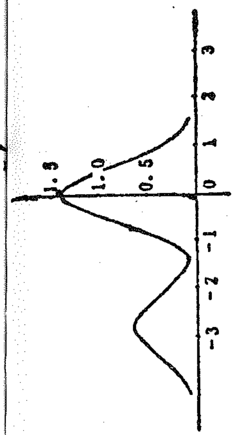


Fig. 8 - II t = 0.2

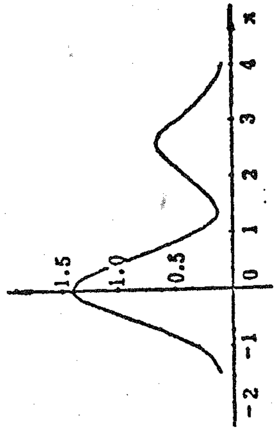


Fig. 8 - IV t = 0.6

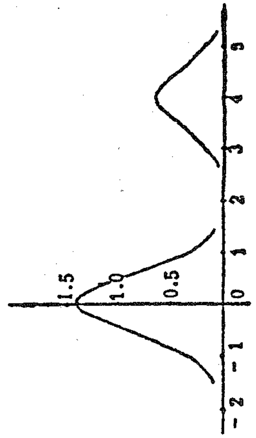


Fig. 8 - V t = 0.8

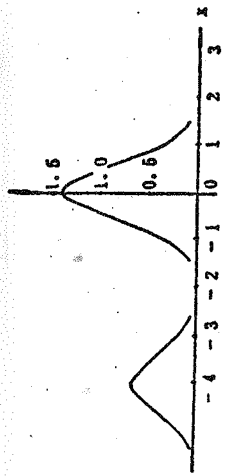


Fig. 8 - I t = 0

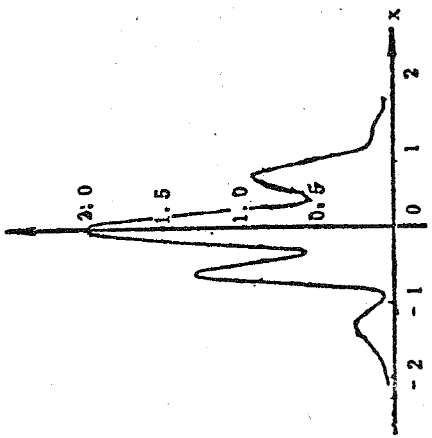


Fig. 8 - III t = 0.4