Quantum measurement theory with angular momentum conservation

F. Selleri

Dipartimento di Fisica, Universita di Bari
I.N.F.N. Sezione di Bari
Via Amendola 173
I-70126 BARI

ABSTRACT. The quantum theory of measurement contains well-known difficulties. For instance, if the object and the apparatus are both described with state vectors the existence of additively conserved quantities is not compatible with a correct functioning of the apparatus. This conclusion ("Wigner-Araki-Yanase theorem") is studied in detail for the case of spin measurements of a spin -1/2 particle at rest, where conceptual simplicity allows detailed calculations to be made. Planck's constant is shown to assume a new and unforeseen role: it fixes the size of instrumental error necessarily present in every spin measurement process.

1. Introduction.

Wigner [1] and Araki-Yanase [2] pointed out the existence of a difficulty of the conventional quantum-mechanical framework when applied to the description of measurement processes. In general these processes lead to a very simple correlation between object and apparatus, reflected in the structure of the overall state vector. If, however, there is an additive conserved quantity for the whole system object plus apparatus, then the usual ("text-book") quantum theory cannot be accepted. A modified theory, in which a systematic error of the measuring apparatus is introduced from the beginning, allows instead the existence of additive conservation laws. This modified theory should be accepted as the only legitimate description of quantum measurements as soon as it is recognized that conservation laws of different types are always at work in all physical processes including measurements (energy, momentum, angular momentum, electric charge...).

In the present paper these general considerations are applied to the concrete example of spin measurements on a spin-1/2 particle at rest. State vectors representing an apparatus having registered the "wrong" result are introduced, matrix elements are calculated and the conditions necessary for angular momentum conservation during the measurement process are written down.

Planck's constant is shown to assume a new and surprising role: it determines the size of the error necessarily performed in every spin-measurement process.

2. Review of spin-1/2 theory.

In the present section well known quantum mechanical results for spin-1/2 theory are reviewed, mainly for notation purposes.

The spin operator can be written

$$\vec{S} = (\hbar/2)\,\vec{\sigma} \tag{1}$$

where σ_1 , σ_2 , σ_3 are the usual Pauli matrices, which have the well-known 2×2 representation. As a consequence, the spin component along the direction defined in ordinary space by the unit vector $\hat{a}$:

$$\hat{a} = (\sin\theta \cos\phi\ ,\ \sin\theta \sin\phi\ ,\ \cos\theta) \tag{2}$$

θ and ϕ being polar and azimuthal angle, respectively, can be written

$$\vec{S}\cdot\hat{a} = (\hbar/2)\begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{pmatrix} \tag{3}$$

This operator has two eigenstates $u_{\pm}$ with eigenvalues $\pm\hbar/2$, respectively. That is

$$\begin{cases} \vec{s}\cdot\hat{a}\ \ u_+ = +(\hbar/2)\ u_+ \\ \vec{s}\cdot\hat{a}\ \ u_- = -(\hbar/2)\ u_- \end{cases} \tag{4}$$

The normalized eigenstates are defined up to arbitrary phase factors. It is a simple matter to show that

$$u_+ = e^{i\phi_1}\begin{pmatrix} r_+ \\ r_-\, e^{i\phi} \end{pmatrix} \qquad u_- = e^{i\phi_2}\begin{pmatrix} -r_- \\ r_+\, e^{i\phi} \end{pmatrix} \tag{5}$$

where

$$r_{\pm} = \sqrt{(1\pm\cos\theta)/2} \tag{6}$$

and where ϕ_1 and ϕ_2 are arbitrary phases. The usual request that spin-1/2 wave functions should change sign under a 2π rotation of the azimuthal angle ϕ leads to the conditions

$$\begin{aligned} \phi_1 &= \alpha - \phi/2 \\ \phi_2 &= \beta - \phi/2 \end{aligned} \tag{7}$$

with α and β constant (ϕ independent). The previous conditions will however not be used in the present paper, even though nothing of what will be written contradicts them.

The spinors $u_{\pm}$ can be written as superpositions of the eigenstates of the third component of the spin operator

$$u_+^0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\ ;\quad u_-^0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\ ; \tag{8}$$

The relevant relations are

$$\begin{cases} u_+ = e^{i\phi_1}\,[r_+ u_+^0 + r_- e^{i\phi}\, u_-^0] \\ u_- = e^{i\phi_2}\,[-r_- u_+^0 + r_+ e^{i\phi}\, u_-^0] \end{cases} \tag{9}$$

Note that (3) can be written

$$\vec{S}\cdot\hat{a} = (\hbar/2)\ [\sigma_+ \sin\theta\, e^{-i\phi} + \sigma_- \sin\theta\, e^{i\phi} + \sigma_3 \cos\theta] \tag{10}$$

where

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\ ;\ \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\ ;\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{11}$$

From (10), (11) and (8) it follows

$$\begin{cases}(u_+^0, \vec{S}\cdot\hat{a}u_+^0) = (\hbar/2)\cos\theta \\ (u_-^0, \vec{S}\cdot\hat{a}u_-^0) = -(\hbar/2)\cos\theta \\ (u_+^0, \vec{S}\cdot\hat{a}u_-^0) = (\hbar/2)\sin\theta\, e^{-i\phi}\end{cases} \tag{12}$$

These relations will be useful later on.

3. Wigner-Araki-Yanase theory for spin-1/2.

Wigner's 1952 theory [1] can be applied to the spin-1/2 problem as follows: if A_0 is the initial state of an apparatus which has been prepared for measuring the third component of the spin of a spin-1/2 particle, and if $A_\pm$ are final states of the same apparatus having registered the values $\pm\hbar/2$, respectively, of the same observable, the following evolutions hold

$$\begin{cases}A_0 u_+^0 \;\rightarrow\; A_+ u_+^0 + \epsilon_+ u_-^0 \\ A_0 u_-^0 \;\rightarrow\; A_- u_-^0 + \epsilon_- u_+^0\end{cases} \tag{13}$$

where $\epsilon_\pm$ are final states of the apparatus correlated to built-in spin-flip of the particle. The normalization conditions

$$\begin{aligned}(A_+, A_+) + (\epsilon_+, \epsilon_+) = 1 \\ (A_-, A_-) + (\epsilon_-, \epsilon_-) = 1\end{aligned} \tag{14}$$

are assumed to be satisfied by vectors $\epsilon_\pm$ of very small amplitude so that the evolutions (13) reproduce in first approximation the usual quantum theory of measurement in which no error (here: no spin-flip) is assumed to take place.

The relations

$$\begin{aligned}(A_+, A_+) \approx 1 \;\; ; \;\; (\epsilon_+, \epsilon_+) << 1 \;\; ; \\ (A_-, A_-) \approx 1 \;\; ; \;\; (\epsilon_-, \epsilon_-) << 1\end{aligned} \tag{15}$$

represent therefore the idea that the usual quantum theory of measurements is at least approximately correct.

Note that we are not giving an explicit physical interpretation of the states $\epsilon_\pm$. Two interpratations seem possible a priori.

In the first interpretation one says that ϵ_+ represents an apparatus having registered the "wrong" value of the spin component $(-\hbar/2)$ which is therefore correlated, as in (13), with the final spinor u_-^0 (and vice-versa).

In the second interpretation one says that ϵ_+ represents an apparatus having registered the "right" value of the spin component $(+\hbar/2)$, which is **not** correlated with the final spinor u_-^0 by which it is multiplied in (13) (and vice-versa). Perhaps the first interpretation is more natural, since otyherwise one would not see a clear physical difference between A_+ and ϵ_+. Symmetrical considerations hold of course for ϵ_-.

It should however be stressed that ϵ_+ and ϵ_- must be different from zero, if the measurement process is to conserve angular momentum, as it will explicitely be shown in the following sections.

The linearity of the quantum law regulating time evolutions (Schroedinger equation) can be used for deducing the results to be expected when the same apparatus is used for performing measurements of S_3 on the eigenstates of the general operator $\vec{S}\cdot\hat{a}$. One can obtain

$$\begin{aligned}A_0 u_+ &= e^{i\phi_1}[r_+ A_0 u_+^0 + r_- e^{i\phi} A_0 u_-^0] \\ &\rightarrow r_+ e^{i\phi_1}[A_+ u_+^0 + \epsilon_+ u_-^0] + r_- e^{i(\phi_1+\phi)}[A_- u_-^0 + \epsilon_- u_+^0] \\ &= [r_+ A_+ + r_- \epsilon_- e^{i\phi}] e^{i\phi_1} u_+^0 + [r_- A_- e^{i\phi} + r_+ \epsilon_+] e^{i\phi_1} u_-^0 \\ A_0 u_- &= e^{i\phi_2}[-r_- A_0 u_+^0 + r_+ e^{i\phi} A_0 u_-^0] \\ &\rightarrow -r_- e^{i\phi_2}[A_+ u_+^0 + \epsilon_+ u_-^0] + r_+ e^{i(\phi_2+\phi)}[A_- u_-^0 + \epsilon_- u_+^0] \\ &= [-r_- A_+ + r_+ \epsilon_- e^{i\phi}] e^{i\phi_2} u_+^0 + [-r_- \epsilon_+ + r_+ A_- e^{i\phi}] e^{i\phi_2} u_-^0\end{aligned} \tag{16}$$

These results can also be written

$$\begin{cases}A_0 u_+ \rightarrow \Omega_1 u_+^0 + \Omega_2 u_-^0 \\ A_0 u_- \rightarrow \tilde{\Omega}_2 u_+^0 + \tilde{\Omega}_1 u_-^0\end{cases} \tag{17}$$

where

$$
\begin{cases}
\Omega_1 = [r_+A_+ + r_-\epsilon_-e^{i\phi}]e^{i\phi_1} \\
\Omega_2 = [r_-A_-e^{i\phi} + r_+\epsilon_+]e^{i\phi_1} \\
\tilde{\Omega}_1 = [-r_-\epsilon_+ + r_+A_-e^{i\phi}]e^{i\phi_2} \\
\tilde{\Omega}_2 = [-r_-A_+ + r_+\epsilon_-e^{i\phi}]e^{i\phi_2}
\end{cases}
\tag{18}
$$

From the equations (17) and (18) becomes transparent the very general nature of the theory of measurement which is being developed:

1. If one puts everywhere $\epsilon_+ = 0$ and $\epsilon_- = 0$, one obtains the most general formulation of the **ordinary** quantum theory of measurement, often connected with the so-called "statistical interpretation" of quantum mechanics [3].
2. If one puts equal to zero also all the conceivable matrix elements connecting A_+ and A_-, one obtains the usual theory of measurement with the "reduction of the wave-packet" [4].

4. Matrix elements of the apparatus.

In the present section matrix elements of a general operator Γ between the states Ω_1, Ω_2, $\tilde{\Omega}_1$, $\tilde{\Omega}_2$ given in (18) will be calculated. The operator Γ will later be specified to be either the unity operator or some angular momentum operator of the apparatus.

From (18) it follows

$$
\begin{cases}
(\Omega_1,\Gamma\Omega_1) = r_+^2(A_+,\Gamma A_+) + r_-^2(\epsilon_-,\Gamma\epsilon_-) + r_+r_-e^{i\phi}(A_+,\Gamma\epsilon_-) + \text{c.c.} \\
(\Omega_2,\Gamma\Omega_2) = r_-^2(A_-,\Gamma A_-) + r_+^2(\epsilon_+,\Gamma\epsilon_+) + r_+r_-e^{-i\phi}(A_-,\Gamma\epsilon_+) + \text{c.c.} \\
(\tilde{\Omega}_1,\Gamma\tilde{\Omega}_1) = r_+^2(A_-,\Gamma A_-) + r_-^2(\epsilon_+,\Gamma\epsilon_+) - r_+r_-e^{-i\phi}(A_-,\Gamma\epsilon_+) + \text{c.c.} \\
(\tilde{\Omega}_2,\Gamma\tilde{\Omega}_2) = r_-^2(A_+,\Gamma A_+) + r_+^2(\epsilon_-,\Gamma\epsilon_-) - r_+r_-e^{i\phi}(A_+,\Gamma\epsilon_-) + \text{c.c.}
\end{cases}
$$

where c.c. denotes the complex conjugate term. From the previous relations it is not difficult to deduce

$$
\begin{cases}
(\Omega_1,\Gamma\Omega_1) + (\tilde{\Omega}_2,\Gamma\tilde{\Omega}_2) = (A_+,\Gamma A_+) + (\epsilon_-,\Gamma\epsilon_-) \\
(\Omega_2,\Gamma\Omega_2) + (\tilde{\Omega}_1,\Gamma\tilde{\Omega}_1) = (A_-,\Gamma A_-) + (\epsilon_+,\Gamma\epsilon_+)
\end{cases}
\tag{19}
$$

and

$$
\begin{cases}
(\Omega_1,\Gamma\Omega_1) - (\tilde{\Omega}_2,\Gamma\tilde{\Omega}_2) = \cos\theta\,[(A_+,\Gamma A_+) - (\epsilon_-,\Gamma\epsilon_-)] + \sin\theta e^{i\phi}(A_+,\Gamma\epsilon_-) + \text{c.c.} \\
(\tilde{\Omega}_1,\Gamma\tilde{\Omega}_1) - (\Omega_2,\Gamma\Omega_2) = \cos\theta\,[(A_-,\Gamma A_-) - (\epsilon_+,\Gamma\epsilon_+)] - \sin\theta e^{-i\phi}(A_-,\Gamma\epsilon_+) + \text{c.c.}
\end{cases}
\tag{20}
$$

From (18) one can also deduce

$$
\begin{cases}
(\Omega_1,\Gamma\Omega_2) = r_+r_-[e^{i\phi}(A_+,\Gamma A_-) + e^{-i\phi}(\epsilon_-,\Gamma\epsilon_+)] + r_+^2(A_+,\Gamma\epsilon_-) + r_-^2(\epsilon_-,\Gamma A_-)\,, \\
(\tilde{\Omega}_2,\Gamma\tilde{\Omega}_1) = -r_+r_-[e^{i\phi}(A_+,\Gamma A_-) + e^{-i\phi}(\epsilon_-,\Gamma\epsilon_+)] + r_-^2(A_+,\Gamma\epsilon_+) + r_+^2(\epsilon_-,\Gamma A_-)\,.
\end{cases}
$$

It is now a simple matter to obtain

$$
(\Omega_1,\Gamma\Omega_2) + (\tilde{\Omega}_2,\Gamma\tilde{\Omega}_1) = (A_+,\Gamma\epsilon_+) + (\epsilon_-,\Gamma A_-) \tag{21}
$$

and

$$
\begin{aligned}
(\Omega_1,\Gamma\Omega_2) - (\tilde{\Omega}_2,\Gamma\tilde{\Omega}_1) = {} & \cos\theta[(A_+,\Gamma\epsilon_+) - (\epsilon_-,\Gamma A_-)] \\
& + \sin\theta[e^{i\phi}(A_+,\Gamma A_-) + e^{-i\phi}(\epsilon_-,\Gamma\epsilon_+)]
\end{aligned}
\tag{22}
$$

All these relations will be useful in the next section.

5. Angular momentum conservation.

The total angular momentum $\vec{J}$ of particle + apparatus will now be considered in order to study the conservation of its component $\vec{J}\cdot\hat{a}$ (where $\hat{a}$ is a given unit vector) during the measurement process. Technically this means that one must check that d ring the evolutions (17) the expectation value of

$$
\vec{J}\cdot\hat{a} = \vec{M}\cdot\hat{a} + \vec{S}\cdot\hat{a} \tag{23}
$$

remains the same: the angular momentum *of the apparatus* has been called $\vec{M}$. Note that the orbital angular momentum of the particle has been neglected: strictly speaking the following considerations apply therefore to particles at rest.

From the definitions (23) and from equations (17) it follows

$$
\begin{aligned}
(A_0, M_a A_0) + \hbar/2 = {} & (\hbar/2)\cos\theta[(\Omega_1,\Omega_1) - (\Omega_2,\Omega_2)] + \\
& + (\Omega_1,\Omega_2)\,(\hbar/2)\sin\theta\, e^{-i\phi} + \text{c.c.} +
\end{aligned}
$$

$$+ (\Omega_1, M_a\, \Omega_1) + (\tilde{\Omega}_2, M_a\, \tilde{\Omega}_2)\ ,$$

and

$$(A_0, M_a\, A_0) - \hbar/2 = (\hbar/2)\cos\theta[(\tilde{\Omega}_2,\tilde{\Omega}_2) - (\tilde{\Omega}_1,\tilde{\Omega}_1)] + $$
$$+ (\tilde{\Omega}_2,\tilde{\Omega}_1)\,(\hbar/2)\sin\theta\; e^{-i\phi} + \text{c.c.} +$$
$$+ (\tilde{\Omega}_1, M_a\, \tilde{\Omega}_1) + (\tilde{\Omega}_2, M_a\, \tilde{\Omega}_2)\ ,$$

where $\vec{M}\cdot\hat{a} = M_a$.

By summing and subtracting the previous equations one obtains

$$2(A_0, M_a\, A_0) = (\hbar/2)\cos\theta[(\Omega_1,\Omega_1) + (\tilde{\Omega}_2,\tilde{\Omega}_2) - (\Omega_2,\Omega_2) - (\tilde{\Omega}_1,\tilde{\Omega}_1)] +$$
$$+ (\hbar/2)\sin\theta e^{-i\phi}\,[(\Omega_1,\Omega_2) + (\tilde{\Omega}_2,\tilde{\Omega}_1)] + \text{c.c.} \qquad (24)$$
$$+ (\Omega_1, M_a\, \Omega_1) + (\Omega_2, M_a\, \Omega_2) + (\tilde{\Omega}_1, M_a\, \tilde{\Omega}_1) + (\tilde{\Omega}_2, M_a\, \tilde{\Omega}_2)\ ,$$

and

$$\hbar = (\hbar/2)\cos\theta[(\Omega_1,\Omega_1) - (\tilde{\Omega}_2,\tilde{\Omega}_2) + (\tilde{\Omega}_1,\tilde{\Omega}_1) - (\Omega_2,\Omega_2)] +$$
$$+ (\hbar/2)\sin\theta e^{-i\phi}\,[(\Omega_1,\Omega_2) - (\tilde{\Omega}_2,\tilde{\Omega}_1)] + \text{c.c.} \qquad (25)$$
$$+ (\Omega_1, M_a\, \Omega_1) - (\tilde{\Omega}_2, M_a\, \tilde{\Omega}_2) - (\tilde{\Omega}_1, M_a\, \tilde{\Omega}_1) + (\Omega_2, M_a\, \Omega_2)\ .$$

Let us consider Eq. (24) first. The apparatus matrix elements entering in it can all be deduced immediately from Eq. (19) and Eq. (21) by taking Γ either equal to the unit operator, or $\Gamma = M_a$.

The result is

$$2(A_0, M_a\, A_0) = \hbar\cos\theta[(\epsilon_-,\epsilon_-) - (\epsilon_+,\epsilon_+)] +$$
$$+ (\hbar/2)\sin\theta e^{-i\phi}\,[(A_+,\epsilon_+) + (\epsilon_-,A_-)] + \text{c.c.} \qquad (26)$$
$$(A_+, M_a\, A_+) + (\epsilon_+, M_a\, \epsilon_+) + (A_-, M_a\, A_-) + (\epsilon_-, M_a\, \epsilon_-)\ .$$

This condition can in general be satisfied without difficulty: it can even be satisfied in the ordinary quantum-mechanical limit where the amplitudes ϵ_+ and ϵ_- are set equal to zero. In such a case one has from the previous relation

$$2(A_0, M_a\, A_0) = (A_+, M_a\, A_+) + (A_-, M_a\, A_-) \qquad (27)$$

which can be satisfied for instance by assuming that the expectation value of the angular momentum over the states A_0 , A_+ and A_- is always the same, a rather natural condition.

The situation is different for Eq. (25) whose apparatus matrix elements can again be calculated by means of Eq. (20) and Eq. (22), either with $\Gamma = I$, or with $\Gamma = M_a$. The result to the lowest order in the (small) amplitudes ϵ_+ and ϵ_- , using the inessential approximations

$$(A_+,A_-) = 0 = (\epsilon_+,\epsilon_-) \qquad (28)$$

and the relation

$$\vec{M}\cdot\hat{a} = M_+ \sin\theta\; e^{-i\phi} + M_- \sin\theta\; e^{i\phi} + M_3\cos\theta \qquad (29)$$

where

$$M_\pm = (1/2)\,(M_1 \pm iM_2) \qquad (30)$$

can easily be shown to be

$$\hbar = \hbar\cos^2\theta\;[1 + (A_+, M_3 A_+) - (A_-, M_3 A_-)] +$$
$$\sin^2\theta[(A_+, M_+\epsilon_-) - (\epsilon_+, M_+ A_-) + (\epsilon_-, M_- A_+) - (A_-, M_-\epsilon_+)] +$$
$$\sin\theta\cos\theta e^{i\phi}(\hbar/2)[(A_+,\epsilon_-) - (\epsilon_+,A_-) + (\epsilon_+,A_+) - (A_-,\epsilon_-)] + \text{c.c.}$$
$$\sin\theta\cos\theta e^{i\phi}[(A_+, M_- A_+) - (A_-, M_- A_-) + (A_+, M_3\epsilon_-) - (\epsilon_+, M_3 A_-)] + \text{c.c.}$$
$$\sin^2\theta e^{2i\phi}[(A_+, M_-\epsilon_-) - (\epsilon_+, M_- A_-)] + \text{c.c.} \qquad (31)$$

This equality can be satisfied if and only if different powers of $e^{i\phi}$ on the two sides are equal. This implies three results

$$\hbar = \hbar\cos^2\theta\;[1 + (A_+, M_3 A_+) - (A_-, M_3 A_-)] +$$
$$\sin^2\theta[(A_+, M_+\epsilon_-) - (\epsilon_+, M_+ A_-) + (\epsilon_-, M_- A_+) - (A_-, M_-\epsilon_+)] \qquad (32)$$

$$0 = (\hbar/2)[(A_+,\epsilon_-) - (\epsilon_+,A_-) + (\epsilon_+,A_+) - (A_-,\epsilon_-)] +$$
$$(A_+, M_- A_+) - (A_-, M_- A_-) + (A_+, M_3\epsilon_-) - (\epsilon_+, M_3 A_-) \qquad (33)$$

$$0 = (A_+, M_-\epsilon_-) - (\epsilon_+, M_- A_-) \qquad (34)$$

Note that of these three equalities only the second and third ones can be satisfied in the limit $\epsilon_+ = \epsilon_- = 0$, that is in the limit of the usual ("text-book") quantum mechanics. Relation (32) cannot instead be satisfied in the same limit, as expected, since the r.h.s. has a $\cos^2\theta$ dependence which is not present in the l.h.s. This implies that angular momentum is not conserved in the usual quantum theory of measurement. The new amplitudes ϵ_+ and ϵ_- are however such as to restore angular momentum conservation, provided that one puts in (32)

$$(A_+, M_3 A_+) - (A_-, M_3 A_-) = 0 \tag{35}$$

$$(A_+, M_+\epsilon_-) + (\epsilon_-, M_- A_+) - (A_-, M_-\epsilon_+) - (\epsilon_+, M_+ A_-) = \hbar \tag{36}$$

Equation (35) can easily be satisfied: it is enough to assume that the expectation value of M_3 in the apparatus states A_+ and A_- be the same, a rather natural condition. Equation (36) can obviously be written

$$\mathrm{Re}(A_+, M_+\epsilon_-) - \mathrm{Re}(\epsilon_+, M_+ A_-) = \hbar/2 \tag{37}$$

Since Eq. (34) and its complex conjugate summed together give

$$\mathrm{Re}(A_+, M_-\epsilon_-) - \mathrm{Re}(\epsilon_+, M_- A_-) = 0 \tag{38}$$

One can deduce from (37) and (38), remembering (30)

$$\mathrm{Re}(A_+, M_1\epsilon_-) - \mathrm{Re}(\epsilon_+, M_1 A_-) = \hbar/4 \tag{39}$$

Conditions (37), (38) and (39) must necessarily be simultaneously true if angular momentum is to be conserved in the measurement processes as described in the enlarged quantum mechanical framework discussed in the present note and embodied in the evolution relations (17).

6. Discussion.

Angular momentum can be conserved in the description of measurement processes provided by the present theory only if the conditions (37) and (38) are satisfied. The first one gives Planck's constant a new and completely unforeseen role in quantum mechanics: it becomes, so to say, the size of the error which is necessarily performed in every spin measurement. In fact it fixes the difference of the real parts of two matrix elements whose presence is

essential for guaranteeing the conservation law.

Eq. (37) can be interpreted in a rather satisfactory way. It is clear from (13) that the third component of the apparatus angular momentum must be the same for the states A_0, A_+, A_- since they multiply the same state of the spin-1/2 particle. The same does not hold for the spin-flip amplitudes ϵ_+ and ϵ_-. In fact, ϵ_+ can be expected to have one unit of M_3 more than A_0, A_+ and A_-, since it is multiplied by a spin-flipped state of the particle. For symmetrical reasons ϵ_- should have one unit of M_3 less.

Remembering the physical interpretation of M_+ and M_- as third component of angular momentum raising and lowering operators, respectively, one concludes that

$M_+\epsilon_-$ is a state with the same value of M_3 as A_+;

$M_-\epsilon_+$ is a state with the same value of M_3 as A_-.

The **Ansatz**

$$\begin{cases} M_+\epsilon_- = (\hbar/4)\, A_+ \\ M_-\epsilon_+ = -(\hbar/4)\, A_- \end{cases} \tag{40}$$

would for instance immediately satisfy the fundamental condition (37).

Bibliography

[1] E.P. Wigner, Z. Phys. **133**, 101, 1952

[2] M.M. Yanase, Phys. Rev. **123**, 666, 1961
H. Araki and M.M. Yanase, Phys Rev. **120**, 662, 1960

[3] E.g. see: L.E. Ballentine, Rev. Mod. Phys. **42**, 358, 1970

[4] P.A.M. Dirac *The principles of quantum mechanics*, Oxford Press, 1958

(Manuscrit reçu le 6 novembre 1986)