About the ultraquantum limit

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I. Introduction.

The analysis of Heisenberg-type relations for some nonlinear classical field models leads to the consideration of the "ultraquantum region" [1-5]. Such region is characterized by actions smaller than $\hbar$. In some way that is related to the asymptotic limit of the Quantum Mechanics when $\hbar \to \infty$.

In this paper our purpose is to consider some aspects of the ultraquantum behavior in the framework of the Linear Quantum Mechanics. The analysis is carried out in the Schrödinger and Nelson's stochastic formulations (Section II). In Section III the features of the ultraquantum limit are shown for the nonrelativistic harmonic oscillator and hydrogen atom.

II. The ultraquantum limit.

A. Let us consider a quantum nonrelativistic particle in a potential $V(x)$. The Schrödinger equation is

$$i\hbar\frac{\partial\phi}{\partial t} = -\frac{\hbar^2}{2m}\,\Delta\phi + V(x)\phi \tag{1}$$

If we write the wave equation as

$$\phi = A(x,t)\, e^{\frac{i}{\hbar}S(x,t)} \tag{2}$$

the Schrödinger equation is equivalent to

$$\frac{\partial S}{\partial t} + \frac{(\vec{\nabla}S)^2}{2m} + V = \frac{\hbar^2}{2m}\,\frac{\Delta A}{A} \tag{3}$$

$$m\,\frac{\partial A}{\partial t}+\vec{\nabla}A.\vec{\nabla}S+\frac{A}{2}\,\Delta S=0 \tag{4}$$

The eq. (4) is just the continuity equation for the Schrödinger equation. The classical approximation consists in setting $\hbar$ equal to zero in (3) and the Hamilton-Jacobi equation is obtained

$$\frac{\partial S}{\partial t}+\frac{(\vec{\nabla}S)^2}{2m}+V=0 \tag{5}$$

On the contrary when $\hbar\to\infty\,(S\ll\hbar)$ the eq. (3) can be approximated as follows

$$\Delta A=0 \tag{6}$$

Thus in the ultraquantum limit there is no evolution of the system, it remains "frozen", and the bound states disappear. We must note that the classical limit $(\hbar\to 0)$ is given by an evolution equation.

B. Now let us see the meaning of the ultraquantum limit in the framework of Nelson's stochastic interpretation [6,7] of Schrödinger equation. According that we have an equivalence between Schrödinger equation and the system

$$\begin{cases}\dfrac{\partial\vec{U}}{\partial t}=-\dfrac{\hbar}{2m}\,\vec{\nabla}(\vec{\nabla}.\vec{V})-\vec{\nabla}(\vec{V}.\vec{U})\\[2ex] \dfrac{\partial\vec{V}}{\partial t}=\dfrac{\hbar}{2m}\,\Delta\vec{U}+(\vec{U}.\vec{\nabla})\vec{U}-(\vec{V}.\vec{\nabla})\vec{V}-\dfrac{\vec{\nabla}V}{m}\end{cases} \tag{7}$$

where

$$\vec{U}=\frac{\hbar}{m}\,\vec{\nabla}(\ln A)\qquad \vec{V}=\frac{\hbar}{m}\,\vec{\nabla}S \tag{8}$$

These equations describe a Brownian motion with diffusion coefficient $\nu=\hbar/2m$. Thus when $\hbar\to\infty\Rightarrow\nu\to\infty$, i.e. the system is extended instantaneously in the whole space, which means that there is no evolution. This is the same feature obtained from the Schrödinger formulation.

C. It is important to remark that if the limit $\hbar\to\infty$ is obtained formally for the massive Klein-Gordon and Dirac particles in a potential, then the system behaves as a free massless particle and the bound states are destroyed :

$$\begin{cases}\dfrac{1}{c^2}\,\dfrac{\partial^2\phi}{\partial t^2}-\Delta\phi+\dfrac{m^2c^2}{\hbar^2}\phi+\dfrac{V}{\hbar^2}\phi=0 \quad\to\quad \dfrac{1}{c^2}\,\dfrac{\partial^2\phi}{\partial t^2}-\Delta\phi=0\\[2ex] \dfrac{i}{c}\gamma^0\,\dfrac{\partial\psi}{\partial t}+i\gamma^k\,\partial_k\,\psi-\dfrac{mc}{\hbar}\psi+\dfrac{V}{\hbar}\,\psi=0 \quad\to\quad \dfrac{i}{c}\gamma^0\,\dfrac{\partial\psi}{\partial t}+i\gamma^k\,\partial_k\,\psi=0\end{cases} \tag{9}$$

Thus in the relativistic and nonrelativistic ultraquantum limits the bound states dissapear. On the other hand while in the nonrelativistic case the system becomes "frozen" with no evolution, in the relativistic motion the system evolves as a free massless particle.

Naively we can say that the ultraquantum limit shows two features of the quark confinement :

(1) The nonrelativistic ultraquantum behavior shows a very stable configuration, since the system does not experiment evolution.

(2) The relativistic ultraquantum limit resembles the quarks moving as free particles in the confinement region.

III. The harmonic oscillator and the hydrogen atom.

The harmonic oscillator and the hydrogen atom allow us to compute the ultraquantum limit directly in the associated physical quantities instead of using the Schrödinger equation. The ultraquantum features are well shown by the ground state:

Harmonic Oscillator

Energy : $E_0 = \hbar\,\omega/2$

Spatial extension : $R_0 =< 0 \mid x^2 \mid 0 >^{1/2}= (\hbar/2m\omega)^{1/2}$

Characteristic action : $S_0 = E_0\,R_0/c = \frac{\hbar}{2\sqrt{2}}\left(\frac{\hbar\omega}{mc^2}\right)^{1/2}$

In the ultraquantum limit $E_0 \to \infty$, $R_0 \to \infty$, $S_0 \to \infty$ thus in such region the stationary states are destroyed.

Hydrogen Atom

Energy : $E_0 = me^4/2\hbar^2$

Radius : $R_0 = \hbar^2/me^2$

Characteristic action : $S_0 = E_0\,R_0/c = e^2/2c < \hbar.$

In the ultraquantum region $E_0 \to 0$, $R_0 \to \infty$ but the characteristic action is a given value smaller than $\hbar$. Also the bound states disappear.

It is important to show the differences and analogies between the ultraquantum limit for the above linear quantum systems and the nonlinear classical field models studied in Ref. [5] :

Nonlinear Systems	Harmonic Oscillator	Hydrogen Atom
$E_0 \to 0$	$E_0 \to \infty$	$E_0 \to 0$
$R_0 \to \infty$	$R_0 \to \infty$	$R_0 \to \infty$
$S_0 \to 0$	$S_0 \to \infty$	$S_0 = e^2/2c < \hbar$

Acknowledgments

I am very grateful to Professor Ph. Blanchard for helpful discussions in the stimulating research atmosphere of the BiBoS Project at Bielefeld University.

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Manuscrit reçu le 28 octobre 1986