On the quantum mechanical description of the Stern-Gerlach experiment with spin-orbit coupling

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ABSTRACT. In a previous paper we verified that, ignoring in the hamiltonian the spin-orbit interaction, the Stern-Gerlach experiment could be considered as a measuring operation of the magnetic moment component along the field direction.

In this paper we retake the same subject but also considering now the spin-orbit interaction. Using certain approximations, which seem justified, we came to the conclusion that the experiment leads to the determination of the "effective" magnetic moment along the field direction.

I - Introduction

In a recent paper [1] was presented the theoretical analysis of the interaction of a beam of hydrogen or alkali atoms with the Stern-Gerlach device in the approximation in which are ignored in the Breit hamiltonian, either the relativistic terms or the spin-orbit and spin-spin interactions. Then we concluded that such experiment could be considered as a measuring operation of the component of the atomic magnetic moment along the field direction.

In this paper we retake the theoretical analysis of that same experiment, but now we will consider the additional difficulties which result from considering in the hamiltonian the spin-orbit coupling term, that is the most significative term that had been ignored in the approach considered on the previous paper. Such as in Paper I, the space between the colimator and the detector will be divided into three regions [2], being region II characterized by the presence of a magnetic-field $\vec{B}$, which is null in regions I and III.

[1] See Ref [1]. This paper will be named "Paper I" for simplicity reasons.
[2] See Fig. 1, Paper I.

2 - The state function in region I

Being formally limited to the case in which a furnace emits hydrogen atoms and considering the spin-orbit coupling, the hamiltonian in region I, expressed in terms of the center of mass coordinate $\vec{R}$ and the relative coordinate $\vec{r}$, is written

$$H^I = \frac{\vec{P}^2}{2M} + \frac{\vec{p}^2}{2m} - \frac{e^2}{r} + \alpha(r)\vec{l}\cdot\vec{\sigma} = H_0^I(\vec{P}) + H_1^I(\vec{r},\vec{p}) \tag{1}$$

with $\alpha(r) = (e\hbar/2mc)^2 r^{-3}$, $M = m_1 + m_2$, $m = m_1 m_2/(m_1 + m_2)$ and where, as usually, $\vec{P}$, $\vec{p}$, $\vec{l}$ and $\vec{s} = \vec{\sigma}/2$ represent respectively the linear momentum, the orbital angular momentum and the angular spin momentum operators. As done in Paper I the global atom movement is associated with the movement of the "particle" of mass M and not having any reasons to admit that this "particle" has been correlated in the beginning with the "particle" of mass m the form of the hamiltonian H^I allows to write the most general solution of the evolution equation in region I

$$H^I\psi^I(\vec{R},\vec{r},t) = i\hbar\frac{\partial}{\partial t}\psi^I(\vec{R},\vec{r},t) \tag{2}$$

through the expression

$$\psi^I(\vec{R},\vec{r},t) = \phi^I(\vec{R},t)\chi^I(\vec{r},t) \tag{3}$$

which implies

$$H_0^I(\vec{P})\phi^I(\vec{R},t) = i\hbar\frac{\partial}{\partial t}\phi^I(\vec{R},t) \tag{4}$$

$$H_1^I(\vec{r},\vec{p})\chi^I(\vec{r},t) = i\hbar\frac{\partial}{\partial t}\chi^I(\vec{r},t) \tag{5}$$

Just as in Paper I it is convenient to expand the spinor $\chi^I(\vec{r},t)$ in a complete set of eigenfunctions of H_1^I, and the presence in this operator of the spin-orbit term, $\alpha(r)\vec{l}\cdot\vec{\sigma}$, justifies that we here use the representation $\{H_1^I, l^2, s^2, j^2, j_Z\}$ (where $\vec{j} = \vec{l} + \vec{s}$ is the total angular momentum operator), obtaining, in this way, for ψ^I

$$\psi^I(\vec{R},\vec{r},t) = \phi^I(\vec{R},t)\sum_{nljm_j}\theta^I_{nljm_j}(t)Q_{nljm_j}(\vec{r}) \tag{6}$$

We won't express here, for the reason of so well known [2], the eigenfunctions $Q_{nljm_j}(\vec{r})$ and the correspondent eigenvalues E_{nlj} expressions ; however, it seems that, it is worthy to remind that m_j represents the eigenvalues of the j_Z operator, in $\hbar$ units and that, in the case here considered, j can only have the values $l+1/2$ or $l-1/2$.

Since that the function ψ^I satisfies the evolution equation (2), we shall have then

$$E_{nlj}\theta^I_{nljm_j}(t) = i\hbar\frac{d}{dt}\theta^I_{nljm_j}(t) \tag{7}$$

and, consequently,

$$\theta^I_{nljm_j}(t) = a_{nljm_j}e^{-iE_{nlj}t/\hbar} \tag{8}$$

where a_{nljm_j} are constants. The expression (6) is now written

$$\psi^I(\vec{R},\vec{r},t) = \phi^I(\vec{R},t)\sum_{nll\pm1/2\,m_j} a_{nljm_j}e^{-iE_{nlj}t/\hbar}Q_{nljm_j}(\vec{r})$$

3 - The eigenfunctions and eigenvalues of the hamiltonian in region *II*

The adduced reasons in Paper I make us now to impute to the hamiltonian in region II the expression

$$\begin{aligned} H^{II} &= \frac{\vec{P}^2}{2M} + \frac{\vec{p}^2}{2m} - \frac{e^2}{r} + \alpha(r)\vec{l}\cdot\vec{\sigma} + \mu_0 B(\vec{R})(l_Z+\sigma_Z) \\ &= H^I_0 + H^{II}_1 = H^I_0 + H^I_1 + \mu_0 B(\vec{R})(l_Z+\sigma_Z) \end{aligned} \tag{10}$$

where μ_0 is the Bohr magneton, $B(\vec{R})$ the intensity of the magnetic field and l_Z and $s_Z = 2\sigma_Z$, respectively, the operators corresponding to the orbital angular momentum and to the spin angular momentum components in the OZ direction imputed to the field.

As the operators H^I_1 and $H^{II}_1 = H^I_1 + \mu_0 B(\vec{R})(l_Z+\sigma_Z)$ don't commute, to determine the eigenvalues and the eigenfunctions of H^{II}_1, it is convenient to consider the matrix which represents this operator in the basis defined by the eigenfunctions Q_{nljm_j} of H^I_1. One can verify easily that the determination of the eigenvectors and eigenvalues of this matrix, demands only the previous computation of the following matrix elements

$$\begin{aligned} A_1 &= \langle n\,l\,l+1/2\,m'_j \mid l_Z+\sigma_Z \mid n\,l\,l+1/2\,m_j\rangle \\ A_2 &= \langle n\,l\,l+1/2\,m'_j \mid l_Z+\sigma_Z \mid n\,l\,l-1/2\,m_j\rangle \\ A_3 &= \langle n\,l\,l-1/2\,m'_j \mid l_Z+\sigma_Z \mid n\,l\,l-1/2\,m_j\rangle \\ A_4 &= \langle n\,l\,l-1/2\,m'_j \mid l_Z+\sigma_Z \mid n\,l\,l+1/2\,m_j\rangle = A_2^* \end{aligned} \tag{11}$$

because all the matrix elements of $l_Z + \sigma_Z$ not diagonal in n and l are zero.

The computation of the elements (11), though extensive, do not present any difficulty and gives us

$$\begin{aligned} A_1 &= 2(2l+1)^{-1}(l+1)m_j\delta_{m_j m_j'} \\ A_2 = A_4 &= -(2l+1)^{-1}[(l+1/2)^2 - m_j^2]^{1/2}\delta_{m_j m_j'} \\ A_3 &= 2(2l+1)^{-1} l m_j \delta_{m_j m_j'} \end{aligned} \tag{12}$$

The matrix representation of the operator H_1^{II} has consequently a form schematically represented on Table I and the determination of the eigenvalues and eigenfunctions of this matrix is reduced to the determination of the eigenvalues and eigenfunctions of a second order generic matrix

$$\begin{bmatrix} \langle n,l,l+1/2,m_j \mid H_1^{II} \mid n,l,l+1/2,m_j\rangle & \langle n,l,l+1/2,m_j \mid H_1^{II} \mid n,l,l-1/2,m_j\rangle \\ \langle n,l,l-1/2,m_j \mid H_1^{II} \mid n,l,l+1/2,m_j\rangle & \langle n,l,l-1/2,m_j \mid H_1^{II} \mid n,l,l-1/2,m_j\rangle \end{bmatrix} \tag{13}$$

So, owing to (12) and since the eigenvalues of H_1^I are E_{nlj}, it is

$$\begin{bmatrix} E_{nll+1/2} + \mu_0 B(\vec{R})A_1 & \mu_0 B(\vec{R})A_2 \\ \mu_0 B(\vec{R})A_2 & E_{nll-1/2} + \mu_0 B(\vec{R})A_3 \end{bmatrix} \tag{13'}$$

and the usual technique of diagonalization gives us the eigenvalues,

$$\begin{aligned} E^{\pm}_{nlm_j}(\vec{R}) =& E_{nll\pm 1/2} \mp \frac{\Delta E_{nl}}{2} + \Delta E_{nl}\Bigg[m_j \xi_{nl}(\vec{R}) \pm \\ & \pm \frac{1}{2}\sqrt{1 + \frac{4m_j}{2l+1}\xi_{nl}(\vec{R}) + \xi^2_{nl}(\vec{R})}\Bigg] \end{aligned} \tag{14}$$

where

$$\Delta E_{nl} = E_{nll+1/2} - E_{nll-1/2} \tag{15}$$

$$\xi_{nl}(\vec{R}) = \mu_0 B(\vec{R})/\Delta E_{nl} \tag{16}$$

and the orthonormalized eigenfunctions

$$Q^{\pm}_{nlm_j}(\vec{r},\vec{R}) = a^{\pm}_{nlm_j}(\vec{R}) Q_{nll+1/2\,m_j}(\vec{r}) + b^{\pm}_{nlm_j}(\vec{R}) Q_{nll-1/2\,m_j}(\vec{r}) \tag{17}$$

with

$$a^{\pm}_{nlm_j}(\vec{R}) = \left\{\frac{1}{2}[1 \pm \gamma_{nlm_j}(\vec{R})]\right\}^{1/2} \tag{18}$$

$$b^{\pm}_{nlm_j}(\vec{R}) = \mp\left\{\frac{1}{2}[1 \mp \gamma_{nlm_j}(\vec{R})]\right\}^{1/2} \tag{19}$$

with the definition

$$\gamma_{nlm_j}(\vec{R}) = \frac{1 + 2m_j(2l+1)^{-1}\xi_{nl}(\vec{R})}{[1 + 4m_j(2l+1)^{-1}\xi_{nl}(\vec{R}) + \xi^2_{nl}(\vec{R})]^{1/2}} \tag{20}$$

Table 1

j m_j	$l+1/2$ $l+1/2$	$l+1/2$ $l-1/2$	$l-1/2$ $l-1/2$	... ...	$l+1/2$ $1/2$	$l-1/2$ $1/2$	$l+1/2$ $-1/2$	$l-1/2$ $-1/2$	... ...	$l+1/2$ $-(l-1/2)$	$l-1/2$ $-(l-1/2)$	$l+1/2$ $-(l+1/2)$
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$l+1/2$ $-(l-1/2)$												
$l-1/2$ $-(l-1/2)$												
$l+1/2$ $-(l+1/2)$												

(In this table were only mentioned the matrix elements different from zero)

4 - The state function in region II

The presence of the additional term $\mu_0 B(\vec{R})(l_Z+\sigma_Z)$ in the expression of the hamiltonian H^{II} prevents the state function ψ^{II} from being written (such as it had already appeared on Paper I) in a similar form as (3). But this function can be expanded in the basis defined by the functions $Q^{\pm}_{nlm_j}(\vec{r},\vec{R})$ taking then the form

$$\psi^{II}(\vec{R},\vec{r},t)=\sum_{nlm_j}[\theta^{II+}_{nlm_j}(\vec{R},t)Q^{+}_{nlm_j}(\vec{r},\vec{R})+\theta^{II-}_{nlm_j}(\vec{R},t)Q^{-}_{nlm_j}(\vec{r},\vec{R})] \tag{21}$$

As ψ^{II} has to satisfy the evolution equation

$$H^{II}\psi^{II}(\vec{R},\vec{r},t)=i\hbar\frac{\partial}{\partial t}\psi^{II}(\vec{R},\vec{r},t) \tag{22}$$

we have

$$\begin{aligned} -\frac{\hbar^2}{2M}\sum_{nlm_j}\Big[&\theta^{II+}_{nlm_j}\nabla^2_{\vec{R}}Q^{+}_{nlm_j}+2\vec{\nabla}_{\vec{R}}\theta^{II+}_{nlm_j}\cdot\vec{\nabla}_{\vec{R}}Q^{+}_{nlm_j} \\ &+Q^{+}_{nlm_j}\nabla^2_{\vec{R}}\theta^{II+}_{nlm_j}+\theta^{II-}_{nlm_j}\nabla^2_{\vec{R}}Q^{-}_{nlm_j}+2\vec{\nabla}_{\vec{R}}\theta^{II-}_{nlm_j}\cdot\vec{\nabla}_{\vec{R}}Q^{-}_{nlm_j} \\ &+Q^{-}_{nlm_j}\nabla^2_{\vec{R}}\theta^{II-}_{nlm_j}\Big]+\sum_{nlm_j}\Big[\theta^{II+}_{nlm_j}E^{+}_{nlm_j}Q^{+}_{nlm_j}+\theta^{II-}_{nlm_j}E^{-}_{nlm_j}Q^{-}_{nlm_j}\Big] \\ &=i\hbar\sum_{nlm_j}\Big[Q^{+}_{nlm_j}\frac{\partial}{\partial t}\theta^{II+}_{nlm_j}+Q^{-}_{nlm_j}\frac{\partial}{\partial t}\theta^{II-}_{nlm_j}\Big] \end{aligned} \tag{23}$$

but as a dependency in $\vec{R}$ of the functions $Q^{\pm}$ results from the inhomogeneity of the magnetic field, which varies negligibly over distances of order the atom's dimension, this situation must be very similar to that in which the field is constant ; therefore, we shall pratically have $\vec{\nabla}_{\vec{R}}Q^{\pm}_{nlm_j}(\vec{r},\vec{R})=0$. In these conditions and since the functions $Q^{\pm}_{nlm_j}$ are all mutually orthogonal, is deduced from (23)

$$[H^{I}_0(\vec{P})+E^{\pm}_{nlm_j}(\vec{R})]\theta^{II\pm}_{nlm_j}(\vec{R},t)=i\hbar\frac{\partial}{\partial t}\theta^{II\pm}_{nlm_j}(\vec{R},t) \tag{24}$$

So, owing to the expression (14) of $E^{\pm}_{nlm_j}$ and introducing the functions $\phi^{II\pm}_{nlm_j}(\vec{R},t)$, defined by

$$\theta^{II\pm}_{nlm_j}(\vec{R},t)=b^{II\pm}_{nlm_j}e^{-iE_{nll\pm1/2}t/\hbar}\phi^{II\pm}_{nlm_j}(\vec{R},t) \tag{25}$$

where the $b^{II\pm}_{nlm_j}$ are constant, (24) will be written more explicitly

$$\begin{aligned}&\left\{H_0^I \mp \frac{\Delta E_{nl}}{2} + \Delta E_{nl}\left[m_j\xi_{nl}(\vec{R}) \pm \frac{1}{2}\sqrt{1+\frac{4m_j}{2l+1}\xi_{nl}(\vec{R})+\xi^2_{nl}(\vec{R})}\right]\right\}\times \\ &\qquad \times\, \phi^{II\pm}_{nlm_j}(\vec{R},t) = i\hbar\frac{\partial}{\partial t}\phi^{II\pm}_{nlm_j}(\vec{R},t)\end{aligned} \tag{26}$$

and due to (25), (21) will be substituted by

$$\begin{aligned}\psi^{II}(\vec{R},\vec{r},t) = \sum_{nlm_j}&\Big[b^{II+}_{nlm_j}e^{-iE_{nll+1/2}t/\hbar}\phi^{II+}_{nlm_j}(\vec{R},t)Q^+_{nlm_j}(\vec{r},\vec{R}) \\ &+ b^{II-}_{nlm_j}e^{-iE_{nll-1/2}t/\hbar}\phi^{II-}_{nlm_j}(\vec{R},t)Q^-_{nlm_j}(\vec{r},\vec{R})\Big]\end{aligned} \tag{27}$$

The continuity of the state function demands that the functions ψ^I and ψ^{II} verify on the $X = A$ plane (see Paper I) the equality

$$[\psi^{II}(\vec{R},\vec{r},t)]_{X=A} = [\psi^I(\vec{R},\vec{r},t)]_{X=A} \tag{28}$$

If we attend, either to the definitions (9), (27) and (17) of ψ^I, ψ^{II} and $Q^\pm$, or to the values assumed by (18), (19) and (20) on the $X = A$ plane

$$\begin{aligned}\left[a^+_{nlm_j}(\vec{R})\right]_{X=A} &= \left[b^-_{nlm_j}(\vec{R})\right]_{X=A} = 1 \\ \left[a^-_{nlm_j}(\vec{R})\right]_{X=A} &= \left[b^+_{nlm_j}(\vec{R})\right]_{X=A} = 0\end{aligned} \tag{29}$$

the condition (28) implies

$$\begin{aligned}&\Big\{\sum_{nlm_j}\Big[b^{II+}_{nlm_j}e^{-iE_{nll+1/2}t/\hbar}\phi^{II+}_{nlm_j}(\vec{R},t)Q_{nll+1/2\,m_j}(\vec{r}) \\ &+ b^{II-}_{nlm_j}e^{-iE_{nll-1/2}t/\hbar}\phi^{II-}_{nlm_j}(\vec{R},t)Q_{nll-1/2\,m_j}(\vec{r})\Big]\Big\}_{X=A} \\ &= \Big\{\phi^I(\vec{R},t)\sum_{nlm_j}\Big[a_{nll+1/2\,m_j}e^{-iE_{nll+1/2}t/\hbar}Q_{nll+1/2\,m_j}(\vec{r}) \\ &+ a_{nll-1/2\,m_j}e^{-iE_{nll-1/2}t/\hbar}Q_{nll-1/2\,m_j}(\vec{r})\Big]\Big\}_{X=A}\end{aligned} \tag{30}$$

and considering the orthogonality of the functions Q_{nljm_j}, the expression (30) is simplified and is written

$$b^{II\pm}_{nlm_j}\left[\phi^{II\pm}_{nlm_j}(\vec{R},t)\right]_{X=A} = a_{nll\pm1/2\,m_j}\left[\phi^{I}(\vec{R},t)\right]_{X=A} \tag{30'}$$

Furthermore, the requirement of assuring the uniqueness of the function $\phi(\vec{R},t)$ on the $X = A$ plane demands that the following condition must be imposed

$$\left[\phi^{II\pm}_{nlm_j}(\vec{R},t)\right]_{X=A} = \left[\phi^{I}(\vec{R},t)\right]_{X=A} \tag{31}$$

and introducing (31) in (30') we obtain

$$b^{II\pm}_{nlm_j} = a_{nll\pm1/2\,m_j} \tag{32}$$

So the state function in region II must finally be written

$$\begin{aligned}\psi^{II}(\vec{R},\vec{r},t) = \sum_{nlm_j}\Big[&a_{nll+1/2\,m_j}e^{-iE_{nll+1/2}t/\hbar}\phi^{II+}_{nlm_j}(\vec{R},t)Q^{+}_{nlm_j}(\vec{r},\vec{R})\\ &+a_{nll-1/2\,m_j}e^{-iE_{nll-1/2}t/\hbar}\phi^{II-}_{nlm_j}(\vec{R},t)Q^{-}_{nlm_j}(\vec{r},\vec{R})\Big]\end{aligned} \tag{33}$$

5 - The state function in region III

In region III the hamiltonian operator regains the form (1) that it had in region I, although the correlation between the functions ϕ and χ roused by the crossing of region II demands that ψ^{III} will be written under a more general form

$$\psi^{III}(\vec{R},\vec{r},t) = \sum_{nljm_j}\theta^{III}_{nljm_j}(\vec{R},t)Q_{nljm_j}(\vec{r}) \tag{34}$$

and since ψ^{III} is defined as the solution of an equation identical to (2), we have

$$(H^{I}_0 + E_{nlj})\theta^{III}_{nljm_j}(\vec{R},t) = i\hbar\frac{\partial}{\partial t}\theta^{III}_{nljm_j}(\vec{R},t) \tag{35}$$

Defining the functions $\phi^{III}_{nljm_j}(\vec{R},t)$ through the expression

$$\theta^{III}_{nljm_j}(\vec{R},t) = b^{III}_{nljm_j}e^{-iE_{nlj}t/\hbar}\phi^{III}_{nljm_j}(\vec{R},t) \tag{36}$$

where the $b^{III}_{nljm_j}$ are constant, $\phi^{III}_{nljm_j}(\vec{R},t)$ must satisfy the equation

$$H_0^I \phi^{III}_{nljm_j}(\vec{R},t) = i\hbar \frac{\partial}{\partial t} \phi^{III}_{nljm_j}(\vec{R},t) \tag{37}$$

and due to the expression of H_0^I we can verify that $\phi^{III}_{nljm_j}(\vec{R},t) = \phi^{III}(\vec{R},t)$. So the expression (34) will be written

$$\psi^{III}(\vec{R},\vec{r},t) = \sum_{nljm_j} b^{III}_{nljm_j} e^{-iE_{nlj}t/\hbar} \phi^{III}(\vec{R},t) Q_{nljm_j}(\vec{r}) \tag{38}$$

However, it is necessary to assure the continuity of the functions $\psi(\vec{R},\vec{r},t)$ and $\phi(\vec{R},t)$ on the $X = B$ plane (see Paper I). For reasons similar to those mentioned on the previous paragraph we will have here

$$b^{III}_{nll\pm 1/2\, m_j} \left[\phi^{III}(\vec{R},t)\right]_{X=B} = a_{nll\pm 1/2\, m_j} \left[\phi^{II\pm}_{nlm_j}(\vec{R},t)\right]_{X=B} \tag{39}$$

the continuity of $\phi(\vec{R},t)$ demanding that it will satisfy the directional equality

$$\left[\phi^{III}(\vec{R},t)\right]_{X=B} = \left[\phi^{II\pm}_{nlm_j}(\vec{R},t)\right]_{X=B} \tag{40}$$

which leads us to introduce the two sets of functions $\phi^{III+}_{nlm_j}$ and $\phi^{III-}_{nlm_j}$. Consequently, from (39) and (40) results

$$b^{III}_{nll\pm 1/2\, m_j} = a_{nll\pm 1/2\, m_j} \tag{41}$$

and the state function in region III will assume finally the form

$$\begin{aligned} \psi^{III}(\vec{R},\vec{r},t) = \sum_{nlm_j} \Big[& a_{nll+1/2\, m_j} e^{-iE_{nll+1/2}t/\hbar} \phi^{III+}_{nlm_j}(\vec{R},t) Q_{nll+1/2\, m_j}(\vec{r}) \\ & + a_{nll-1/2\, m_j} e^{-iE_{nll-1/2}t/\hbar} \phi^{III-}_{nlm_j}(\vec{R},t) Q_{nll-1/2\, m_j}(\vec{r}) \Big] \end{aligned} \tag{42}$$

6 - Reformulation of the state function expressions

The expression (42) of ψ^{III} is too much complicated so that we might draw physically usable conclusions. We must however take in consideration that the quantity ξ_{nl} defined by (16) remains much smaller that the unit, even when extremely high values are imputed to the quantum number n. So, it is permissible to hold back the two first terms of the expansion in power series of ξ_{nl} of the expression $\sqrt{1+4m_j/2l+1\xi_{nl}+\xi_{nl}^2}$ which figures in (26). Then the equation (26), is written in a much more simple form

$$\left[H_0^I+\mu_0 B(\vec{R})m_j\left(1\pm\frac{1}{2l+1}\right)\right]\phi_{lm_j}^{II\pm}(\vec{R},t)=i\hbar\frac{\partial}{\partial t}\phi_{lm_j}^{II\pm}(\vec{R},t) \qquad (43)$$

Let us consider now the definition of Landé-g factor

$$g=1+\frac{j(j+1)+s(s+1)-l(l+1)}{2j(j+1)} \qquad (44)$$

that in the case of the hydrogen atom is simply written

$$g=\frac{2j+1}{2l+1} \qquad (45)$$

Since we can only have here $g=2l+2/2l+1$, when $j=l+1/2$ or $g=2l/2l+1$ when $j=l-1/2$, we verify that (43) can also be written in an equivalent form

$$\left[H_0^I+\mu_0 B(\vec{R})gm_j\right]\phi_{ljm_j}^{II}(\vec{R},t)=i\hbar\frac{\partial}{\partial t}\phi_{ljm_j}^{II}(\vec{R},t) \qquad (46)$$

which puts in evidence the importance of introducing the quantity [3] $\mu_0 g m_j$ in order to describe the wave packets evolution.

Equally, considering again the approach that the small value of ξ_{nl} has allowed to introduce above, functions $Q_{nljm_j}^{\pm}$ defined by (17) are reduced to [4]

$$Q_{nlm_j}^{\pm}(\vec{r},\vec{R})=Q_{nll\pm1/2\,m_j}(\vec{r}) \qquad (47)$$

and consequently, instead of (33) comes finally

$$\psi^{II}(\vec{R},\vec{r},t)=\sum_{nljm_j}a_{nljm_j}e^{-iE_{nlj}t/\hbar}\phi_{ljm_j}^{II}(\vec{R},t)Q_{nljm_j}(\vec{r}) \qquad (48)$$

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[3] One must remind that in the interpretation of the Zeeman effect by the atomic vectorial model, several authors (see, for instance, Ref [3] and [4]) had already introduced the quantity "effective" magnetic moment $-\mu_0(1+\vec{s}\cdot\vec{j}/j^2)\vec{j}=-\mu_0 g\vec{j}$ whose component along the field direction is $-\mu_0 g m_j$.

[4] In this approach (19), (20) and (21) impute the equalities $a_{nlm_j}^{+}(\vec{R})=b_{nlm_j}^{-}(\vec{R})=1$, $a_{nlm_j}^{-}(\vec{R})=b_{nlm_j}^{+}(\vec{R})=0$ and $\gamma_{nlm_j}=1$.

Using this new form of the state function in region II, it is enough to turn again to expression (38) and continue demanding the continuity of the functions $\psi(\vec{R},\vec{r},t)$ and $\phi(\vec{R},t)$ in order to write instead of (42),

$$\psi^{III}(\vec{R},\vec{r},t) = \sum_{nljm_j} a_{nljm_j} e^{-iE_{nlj}t/\hbar} \phi^{III}_{ljm_j}(\vec{R},t) Q_{nljm_j}(\vec{r}) \tag{49}$$

or, defining

$$S_{ljm_j} = \sum_n a_{nljm_j} e^{-iE_{nlj}t/\hbar} Q_{nljm_j}(\vec{r}), \tag{50}$$

we shall still have

$$\psi^{III}(\vec{R},\vec{r},t) = \sum_{ljm_j} S_{ljm_j}(\vec{r},t)\phi^{III}_{ljm_j}(\vec{R},t) \tag{51}$$

If we define the operator corresponding to the component of the "effective" magnetic moment along the field direction by the expression [5]

$$\mu_{jZ} = -\mu_0 \left(1 + \frac{\vec{s}\cdot\vec{j}}{j^2}\right) j_Z$$

it is easy to verify that the S_{ljm_j} are eigenfunctions of this operator to which correspond the eigenvalues $-\mu_0 g m_j$.

So the expression (51) shows that ψ^{III} can be expressed as an addition of products of two different nature functions. In more explicit terms each wave packet $\phi^{III}_{ljm_j}(\vec{R},t)$ finds itself correlated with a certain eigenfunction $S_{ljm_j}(\vec{r},t)$ of operator (52) to which must be associated the quantity component of the "effective" magnetic moment along the field direction.

7 - Conclusions

Such as in Paper I we shall also admit here that the wave packets $\phi^{III}_{ljm_j}$ will give rise to the appearing of spots spatially separated in the detector. So, we conclude from the previous analysis that the detection of an atom in one of

[5] The operator (52) was imputed in Ref. [5] to the component of the magnetic moment along the field direction but such a definition would only be permissible if (52) had the same eigenvalues and eigenfunctions as the operator $-\mu_0(j_Z + s_Z)$ which isn't surely the case.

those spots allows to confer it a certain function $S_{ljm_j}(\vec{r}, t)$, that is, a certain value $-\mu_0 g m_j$ of the component of the "effective" magnetic moment along the field direction.

In these conditions, when we do not ignore in the hamiltonian the spin-orbit interaction term, the Stern-Gerlach device goes on deserving to be considered a measuring apparatus ; but now it measures the component of the effective magnetic moment and not the component of the magnetic moment.

It is true that the Stern-Gerlach experiments accomplished till to-day with beams of hydrogen or alkali atoms only concern states in which $l = 0$ and, in these conditions, the values of those two quantities coincide. However it could be rather disturbing that, generally, the quantity that the Stern-Gerlach apparatus measures is, after all, determined by the approximations introduced in the theoretical analysis.

Although with obvious differences, this situation has some analogies with the well known phenomenon that an atom in a constant magnetic field gives rise either to Zeeman effect or to Paschen-Back effect, depending on the intensity of the field.

In the description of the Stern-Gerlach experiment when the spin-orbit interaction is taken into account, one is led to introduce the quantity ξ_{nl} defined by (16). But it was the numerical estimation of ξ_{nl}, based on the intensity of the magnetic fields used in the Stern-Gerlach devices, that induced us to conclude that $\xi_{nl} \ll 1$ and consequently that the Stern-Gerlach apparatus has to be considered as a measuring device of the "effective" magnetic moment.

So the conclusion reached in our previous paper, where the spin-orbit interaction has not been taken into account, should only be valid for intensity values of the magnetic field far higher than those used in the experiments that have been carried out up to the present day.

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RÉSUMÉ. Dans un article précédent on a vérifié que, en négligeant dans l'hamiltonien l'interaction spin-orbite, l'expérience de Stern-Gerlach peut être considérée comme une opération de mesure de la composante du moment magnétique atomique dans la direction du champ.
On reprend ici ce même problème mais en considérant maintenant l'interaction spin-orbite. En introduisant certaines approximations qui semblent justifiées on arrive à la conclusion que l'expérience permet de mesurer le moment magnétique "effectif" selon la direction du champ.