Examples of explicit position-velocity coexistence and their physical implications in a 'minimal' stochastic interpretation of quantum mechanics - Part II Discussion of concrete time-dependent solutions of the Schrödinger equation

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ABSTRACT. The time-evolution of solutions to the Schrödinger equation in the presence of potentials $U(x) = -Fx$, $F = \text{const}$ (a constant force field that may be equal to zero in the specific case of free motion) and $U(x) = m\omega^2 x^2/2$ (the harmonic oscillator case) is considered for arbitrary initial conditions in the case of normalizable states of motion and for a class of initial conditions for nonnormalizable states. For a wide variety of cases the behaviour of position distributions in the relevant quantum ensembles coincides (at all times or in the limit of large t) with that of classical ensembles with a one-to-one position-velocity correspondence at any given moment t. This fact leads to a number of inferences about the physical sense of the well known uncertainty relations and the nature of measurement procedures. In particular, it follows that, generally, lawful measurement procedures are in no way (quasi)instantaneous and that there are no grounds to interpret position-momentum uncertainty relations as describing physical properties of individual members of quantum ensembles.

IV. Explicit position-velocity coexistence in certain normalizable nonstationary quantum states

As the uncertainty relation (1.3) applies, by its very way of deriving (cf. any book on QM), to QM states that are normalizable to unity, that is, to $\psi(x,t)$ for which

$$\int_{-\infty}^{\infty} \mid \psi(x,t) \mid^2 dx = 1, \tag{4.1}$$

we shall demonstrate first the necessity of its SI reinterpretation by examining the time-evolution of such states. (One may find a similar example in ref. 10, but in a stationary setting and without an explicit solution). It will be evident from our consideration, besides, in what objective sense the relevant velocities represent macroscopic magnitudes in the case under consideration.

[*Notational convention* : From now on x and v will denote time-independent arguments, whereas the corresponding Newtonian functions will be explicitly written as $x(v,t)$ or $v(x,t)$. Besides, in analogy with "c-numbers" and "q-numbers", we shall write "c-ensembles" and "q-ensembles" for classical and quantum ensembles, correspondingly].

Examine first the case of free motion [that is, all basic physical fields $U(x,t)=0$ at $t \geq 0$ plus the no-analyser requirement $U_A(x,t)=0$, $t \geq 0$]. Let the initial wave function (at $t=0$) be given by

$$\psi_{in} = \psi(x,0), \tag{4.2}$$

where $\psi(x,0)$ is of an *arbitrary* form, the only requirement imposed on it being given by eq. (4.1) at $t=0$. Will the conduct of the q-ensemble of particles (moving along the x-axis), corresponding to any such function (4.2), approach the conduct of an appropriate c-ensemble in the limit $t \to \infty$?

The possible ways of action for finding the answer to this question were described in Sec. III and we shall directly employ them from now on. Formulae (3.10,11) give here

$$\psi(x,t) = \left(\frac{m}{2\pi i\hbar t}\right)^{1/2} e^{\frac{imx^2}{2\hbar t}} \int_{-\infty}^{+\infty} e^{\frac{-imxx'}{2\hbar t}} e^{\frac{imx'^2}{2\hbar t}} \psi(x',0)dx' \tag{4.3}$$

Assume now, just for the sake of convenience, that $\psi(x,0)$ is 'centered' at the origin $x=0$. Replace x by $x(v,t)=vt$ (which expression will be valid for a free classical particle with an initial position $x=0$ and velocity v) and set $t \to \infty$. Eq. (4.1) at $t=0$ guarantees a spatially restricted $\psi(x,0)$, so one may replace, in the limit $t \to \infty$, $\exp(imx'^2/2\hbar t)$ by unity under the integral in (4.3) and arrive at

$$\lim_{t\to\infty} \psi(vt,t) = \sqrt{\frac{m}{it}}\ \exp(imv^2t/2\hbar)a_p(0) \tag{4.4}$$

where we have made use of formula (3.2) for a_p at $t = 0$. [In the case of free motion $\mid a_p(0) \mid^2 = \mid a_p(t) \mid^2$].

Taking the square of the modulus of expression (4.4) and multiplying it by $\mid \partial x(v,t)/\partial v \mid = t$, we obtain

$$\begin{aligned} \lim_{t\to\infty} \mid \psi(vt,t) \mid^2 t &= m \mid a_p(0) \mid^2 = m \mid a_p(t) \mid^2 \\ &= R_{QM}(v,t) = R_{QM}(v,0) \end{aligned} \tag{4.5}$$

[cf. eq. (3.1)]. In such a way, in the case of free motion, the time-evolution of the q-ensemble of particles tends in the limit $t \to \infty$ to that of a c-ensemble of particles, all of which are located initially at point $x = 0$ and whose $R(v,0)$ [$= R(v,t)$] coincides with $R_{QM}(v,0)$.

This result, obtained in our ensemble approach, is an expression of the fact that one may employ the essentially macroscopic and classical time-of-flight technique for measuring the velocity of microparticles [11] (cf. also the somewhat more detailed discussion below). We have obviously, besides, a corroboration for the case just examined of the QM postulate (3.1,2).

It is worth pointing out that the inverse proportionality of $\mid \psi[x(v,t),t] \mid^2$ to t for $x(v,t) = vt$, obtainable from eq. (4.4) in the limit $t \to \infty$, may be regarded as a consequence of the fact that the dimensions of a region in x-space containing a host of particles with velocities in the interval $[v, v+\Delta]$ must increase linearly with time (at least when $t \to \infty$).

The general result obtained above does not offer an immediate answer to questions concerning the possible role of the c-ensemble idealization (all c-particles located at $x = 0$ when $t = 0$), the possibility to violate ineq. (1.3) in (individual) measurement, etc. We may certainly elicit, in principle, the answers from the general information that we already have at our disposal but it is probably better to obtain more vivid answers by examining specific explicit expressions for the time-evolution of position densities in c- and q-ensembles. So we are going to discuss now an explicit solution to the Schrödinger equation for $U(x) = 0$ that is well known from a purely mathematical point of view but whose ensemble interpretation gives nontrivial consequences in a number of cases (cf. also the subsequent considerations and, in particular, Appendix A). Namely, examine an initial wave packet of the Gaussian form

$$\psi(x,0) = \frac{1}{\sqrt{\sigma\sqrt{\pi}}} \exp[-(x-X)^2/2\sigma^2] \exp(iPx/\hbar) \tag{4.6}$$

where $P = \text{const}$ is an arbitrary fixed real number and $\sigma = \text{const} > 0$ characterises the spread of $\psi(x,0)$ along x about a fixed point X, $X \in (-\infty, \infty)$. In our case of $U(x,t) = 0$ [and also in the case of $U(x,t) = U(x) = -Fx$, where F represents a (t,x)-independent force field] we may put $X = 0$, $P = 0$ without any loss of generality. (The first is obvious while the second can always be obtained by a transition to a suitable frame of reference [20]). We can therefore confine ourselves here to

$$\psi(x,0) = \frac{1}{\sqrt{\sigma\sqrt{\pi}}} \; \exp(-x^2/2\;\sigma^2) \tag{4.7}$$

[eq. (4.6) will be employed in Sec. V]. The expression for $\psi(x,t)$ will then be

$$\begin{aligned}\psi(x,t) = \left(\frac{m\sigma}{i\hbar t\sqrt{\pi}}\right)^{1/2} \cdot \frac{\exp(imx^2/2\hbar t)}{(1 - im\sigma^2/\hbar t)^{1/2}} \cdot \\ \exp\left[-\frac{m^2\sigma^2x^2}{2\hbar^2t^2} \cdot \frac{1}{1 + m^2\sigma^4/\hbar^2t^2}\left(\frac{1 + im\sigma^2}{\hbar t}\right)\right]\end{aligned} \tag{4.8}$$

Replacing x by $x(v,t) = vt$, we get

$$\mid \psi(vt,t) \mid^2 = \frac{m\sigma}{\sqrt{\pi}\hbar t} \cdot \frac{1}{\left(\frac{1+m^2\sigma^4}{\hbar^2t^2}\right)^{1/2}} \; \exp\left[-\left(\frac{m\sigma v}{\hbar}\right)^2 \; \frac{1}{\frac{1+m^2\sigma^4}{\hbar^2t^2}}\right] \tag{4.9}$$

Multiplying (4.9) by $\mid \partial x(v,t)/\partial v \mid = t$ and taking the limit of the expression so obtained as $t \to \infty$ we obtain, in agreement with the above considerations, a time-independent velocity distribution

$$R(v) = \frac{m\sigma}{\sqrt{\pi}\hbar} \cdot \exp[-(m\sigma v/\hbar)^2] = R_{QM}(v) \tag{4.10}$$

[As we already know, the second eq. (4.10) may be checked directly with the aid of eqs. (3.1,2) ; the integration itself is standard here]. When assigned to a c-ensemble of particles located at the origin at $t = 0$, this $R(v)$ generates an

$$r(vt,t) = \frac{m\sigma}{\sqrt{\pi}\hbar t} \; \exp[-(m\sigma v/\hbar)^2], \tag{4.11}$$

to which expression (4.9) tends at $t \to \infty$ for an arbitrary given v. We have thus at our disposal now an explicit complete realization for a specific case [eq. (4.7)] of our general consideration.

Let us demonstrate now the inessential character of the δ-like idealization of $r(x,t)$ at $t=0$ in our specific case. (The consideration may be extended to the general case too). Assume that, as in the q-ensemble, the particles in the c-ensemble are smoothly distributed in a region of characteristic length $d_0 = \sigma$ at $t=0$, their v-distribution being given once again by eq. (4.10), in which the characteristic v-spread is equal to $d_v = \hbar/m\sigma$. From the viewpoint of obtaining a well defined $R(v)$, the velocity v will be well defined if the objective uncertainty Δ_0 in its definition is small compared to the characteristic spread d_v, that is, when $\Delta_0/d_v \ll 1$. In our case this (classical) uncertainty will come from the fact that the initial positions of the particles in the realistic c-ensemble are determined with a precision $\sim d_0 = \sigma$, so the objective error in the definition of v in this ensemble will be given by $\Delta_0(t) = d_0/t$, where d_0, by definition, is time-independent. Consequently, in the limit of small $\Delta_0(t)/d_v = d_0/td_v$, the realistic $r(vt,t)$ will be such that its form will exhibit a well defined $R(v)$ given by eq. (4.10), that is, the form of the realistic $r(vt,t)$ will tend to that of the idealized r (with a one-one $x-v$ correspondence) in the limit $t \to \infty$, so the idealization of our basic c-ensemble does not modify the physical essence of the problem. In particular, the tendency of the behaviour of the above realistic c-ensemble and that of the q-ensemble to the same one-one $x-v$ correspondence rule in the limit of large t shows that time-of-flight technique makes possible the violation of ineq. (1.3) to an arbitrary extent in individual measurements. Really, the objective uncertainties in the macrovelocity definition are the same ($\sim d_0/t$) in the limit of large t for the realistic c-ensemble and our q-ensemble and the product $d_0 d_0/t = \sigma^2/t$ tends to zero as $t \to \infty$.

A picture of ensemble evolution that is of the same kind as the one just examined may be found in the case $U(x) = -Fx$ too, $F = \text{const}$. Consider first a normalized initial wave packet of an arbitrary form (4.2). Eqs. (3.10) and (3.12) lead to

$$\psi(x,t) = \sqrt{\frac{m}{it}}\, A(x,t)\, \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \exp\left[-\frac{i}{\hbar}\left(\frac{mx}{t} - \frac{Ft}{2}\right)x'\right]. \qquad (4.12)$$
$$\exp(imx'^2/2\hbar t)\psi(x',0)dx'$$

where $A(x,t) = \exp[i/\hbar(mx^2/2t + Ftx/2 - F^2t^3/24m)]$: obviously, $\mid A \mid = 1$,

$t \geq 0$. The single-particle solution of the respective classical problem [with $x(0) = x_0$ and $v(0) = v_0$] is

$$v = v(x_0, v_0, t) = v_0 + Ft/m \tag{4.13}$$

$$x = x(x_0, v_0, t) = x_0 + v_0 t + Ft^2/2m = x_0 + vt - Ft^2/2m \tag{4.14}$$

[Eq. (4.13) must be interpreted here as an auxiliary law establishing a one-one time-dependent correspondence between the (independent) velocity argument v and the variable v_0 that plays the role of an initial velocity]. Having in mind that the term $\exp(imx'^2/2\hbar t)$ in (4.12) can be put equal to unity at $t \to \infty$ for the same reason as in the case of free motion and replacing, as usually, x in (4.12) with $x(v,t)$ [as determined by the second eq. (4.14)] we see that at $t \to \infty$ the square of the modulus of (4.12) is equal to

$$\lim_{t \to \infty} \mid \psi[x(v,t),t] \mid^2 = m \mid a(p_0) \mid^2 /t \tag{4.15}$$

where $p_0 = mv_0$ and

$$a(p_0) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \exp(-imv_0 x'/\hbar)\psi(x',0)dx' \tag{4.16}$$

Multiplying then, in accordance with our prescriptions in Sec. III, eq. (4.15) by $\mid \partial x(v,t)/\partial v \mid = t$, we arrive at

$$R(v,t) = m \mid a(p_0) \mid^2 \tag{4.17}$$

for v's satisfying the correspondence law (4.13). Evidently, this is the result which permits us to say that in the limit $t \to \infty$ the q-ensemble, evolving from an arbitrary initial normalized state $\psi(x,0)$, will behave as an appropriate c-ensemble. Indeed, in arbitrary c-ensembles of particles (including idealized ones) with an initial velocity distribution

$$R(v_0) = m \mid a(p_0) \mid^2, \tag{4.18}$$

we shall have

$$R(v,t) = R(v_0) \tag{4.19}$$

for v and v_0 connected via (4.13) due to the fact that all classical v_0 obtain the same increase Ft/m with time in our homogeneous field F, so that the lengths of the corresponding velocity intervals Δ does not vary with t i.e. interval $(v_0, v_0 + \Delta)$ is transformed into $(v_0 + Ft/m, v_0 + Ft/m + \Delta)$ with the course of time]. Consequently, (4.19) coincides with (4.17).

The only thing that remains to be done is to check whether the QM postulate (3.1,2) conforms to the just found equivalence of q- and c-ensembles in the limit $t \to \infty$ and in the presence of the homogeneous force field F. This can be done by replacing $\psi(x,t)$ in (3.2) by expression (4.12) and integrating first over argument x. A straightforward application of formula (5.2) of next section leads to the necessary result :

$$R_{QM}(v,t) = R_{QM}(v_0,0) = R(v_0) \tag{4.20}$$

for v and v_0 connected via (4.13) and all $t > 0$. All this means that time-of-flight measuring technique is applicable in the presence of our force field F too since it will evince a classical v-distribution in the limit of large t. Examine as a concrete illustration of this statement the explicit solution for the specific case of a Gaussian $\psi(x,0)$ given by (4.7). Standard manipulations yield here

$$\mid \psi(x,t) \mid^2 = \frac{m\sigma}{\sqrt{\pi}\hbar t} \frac{1}{\left(\frac{1+m^2\sigma^4}{\hbar^2 t^2}\right)^{1/2}} \exp\left[-\frac{\sigma^2 m^2}{\hbar^2}\left(\frac{x}{t} - \frac{Ft}{2m}\right)^2 \frac{1}{1+\frac{m^2\sigma^4}{\hbar^2 t^2}}\right] \tag{4.21}$$

Replacing x in (4.21) by $x(v,t)$ from (4.14) (with $x_0 = 0$, as the idealized c-ensemble may certainly be centered at the origin), we get

$$\lim_{t\to\infty} \mid \psi[x(v,t),t] \mid^2 = \frac{m\sigma}{\sqrt{\pi}\hbar t} \exp[-(mv_0\sigma/\hbar)^2] \tag{4.22}$$

and after multiplying this expression by $\mid \partial x(v,t)/\partial v \mid = t$ we arrive at an expression for $R_{QM}(v,t) = R(v,t)$, $t \to \infty$, coinciding with expression (4.10) for the case of free motion (in the latter expression v plays the role of our v_0). This is certainly the result (4.17) for our concrete case.

Evidently, all the results for a Gaussian $\psi(x,0)$ in the case of free motion can be directly obtained from eq. (4.21) by putting there $F = 0$. Therefore, from the viewpoint of a transition (at large t) to an explicitly classical behaviour of q-ensembles, the cases $F = 0$ and $F \neq 0$ differ only insignificantly for Gaussian ψ_{in}'s, the essential feature in both cases being that a v-distribution of a classical nature emerges when $m^2\sigma^4/\hbar^2t^2 \to 0$. A similar assertion is valid for an arbitrary ψ_{in}.

In the cases examined until now we had an 'irreversible relaxation' of the behaviour of q-ensembles to that of relevant idealized c-ensembles in the limit $t \to \infty$. This is certainly not the general case : as evident from expression (3.13) for the propagator in the case of a harmonic potential field $U(x) = m\omega^2x^2/2$, the corresponding nonstationary wave functions $\psi(x,t)$ have a strictly periodic behaviour with time for arbitrary initial conditions. Nonetheless, limiting situations in which the behaviour of q-ensembles is almost identical to that of an idealized c-ensemble practically all the time are possible here too. In order to see this examine a normalized state ψ satisfying an arbitrary initial condition (4.2). Eqs. (3.10) and (3.13) yield

$$\begin{gathered}\psi(x,t) = (m\omega/2\pi i\hbar\sin\omega t)^{1/2}\exp(im\omega\cot\omega t\cdot x^2/2\hbar)\\ \int_{-\infty}^{\infty}\exp(im\omega x'^2\cot\omega t/2\hbar)\exp(-im\omega xx'/\hbar\sin\omega t)\psi(x',0)dx'\end{gathered}\tag{4.23}$$

The single-particle solution of the corresponding classical problem [with initial conditions $x(0) = x_0$ and $v(0) = v_0$] is

$$x(x_0,v_0,t) = x_0\cos\omega t + (v_0\sin\omega t)/\omega \tag{4.24}$$

$$v(x_0,v_0,t) = \partial x(x_0,v_0,t)/\partial t = -x_0\omega\sin\omega t + v_0\cos\omega t \tag{4.25}$$

Examine now an idealized c-ensemble of particles located at a fixed point $x_0 = X$ in the harmonic field at $t = 0$ and having an initial velocity distribution $R(v_0,0)$. The obvious equality

$$R(v,t) \mid dv \mid = R(v_0,0) \mid dv_0 \mid, \tag{4.26}$$

valid for such a c-ensemble [v and v_0 being connected via (4.25)] yields

$$R(v,t) = R(v_0,0)/\mid\cos\omega t\mid \tag{4.27}$$

[From (4.25) it follows that, for a fixed $x_0 = X$, $\mid \partial v/\partial v_0 \mid = \mid \cos\omega t \mid$; eqs. (4.26) and (4.27) for our specific c-ensemble hold at all t, with the exception of a discrete set of moments $t = t_n$, $n = 0,1,2,...$,

$$t_n = (2n+1)\pi/2\omega, \tag{4.28}$$

at which all particles in the c-ensemble have the same velocity $\pm X\omega$ and eq. (4.26) loses sense due to the δ-like course of the velocity distribution at these moments of time].

Let the initial wave packet $\psi(x, 0)$ be centered at our fixed point $x_0 = X$ and let us begin to contract its spread about X (certainly, preserving normalization). Replace, as usually, x in (4.23) by expression (4.24). For initial states with sufficiently small spreads about X all the exponential terms under the integral in (4.23) will be almost all the time practically constant along the 'effective length' of $\psi(x, 0)$, with the exception of the term $\exp(-imv_0 x'/\hbar) = \exp(-ip_0 x'/\hbar)$, in which v_0 may be large. This will not be true only in a small vicinity of time-points $t = \tau_n$, $n = 0, 1, 2, ...$,

$$\tau_n = n\pi/\omega, \tag{4.29}$$

at which $\sin\omega t = 0$ [so that all particles in the idealized c-ensemble have the same position $\pm X$ at τ_n –cf. (4.24)]. The length of these time-intervals, however, will clearly tend to zero as the spread of $\psi(x, 0)$ tends to zero. Multiplying, in accord with our prescription, the expression for $\mid \psi[x(x_0, v_0, t), t] \mid^2$ by

$$\mid \partial x(x_0, v_0, t)/\partial v \mid = \mid \partial x(x_0, v_0, t)/\partial v_0 \cdot \partial v_0/\partial v \mid = \mid \operatorname{tg}\omega t/\omega \mid,$$

we obtain for almost all t the practically precise equation

$$\mid \psi_t \mid^2 \mid \partial x_t/\partial v \mid \approx m \mid a(p_0, 0) \mid^2 / \mid \cos\omega t \mid = R_{QM}(v_0, 0)/ \mid \cos\omega t \mid \tag{4.30}$$

Consequently, almost all the time the behaviour of the position density of such a q-ensemble will be practically identical with that of an idealized c-ensemble, 'concentrated' at $x_0 = X$ at moment $t = 0$ and having a velocity distribution $R(v_0, 0) = R_{QM}(v_0, 0)$ at that moment. We have to point out, however, that this correspondence between q- and c-ensembles (being absent only in the vanishing intervals around τ_n, $n = 0, 1, ...$) becomes physically meaningless also in small time intervals [of length tending to zero as the spread of $\psi(x, 0)$ tends to zero] around time points t_n,$n = 0, 1, ...$ This may be seen by checking up the validity of the QM postulate (3.1,2), employing expression (4.23) for $\psi(x, t)$. The corresponding expression for $a(p, t)$ is

$$\begin{aligned} a(p, t) = 1/(2\pi\hbar\cos\omega t)^{1/2} \exp(iB) \int_{-\infty}^{\infty} \exp(-ipx'/\hbar\cos\omega t) \cdot \\ \exp(-im\omega x'^2/2\hbar\sin\omega t\cos\omega t)\psi(x', 0)dx', \end{aligned} \tag{4.31}$$

where B is an inessential real term. If one replaces v in $p = mv$ with expression (4.25), one sees that for sufficiently narrow initial states $\psi(x', 0)$ expression

$$R_{QM}(v,t) = R_{QM}(v_0, 0)/ \mid \cos\omega t \mid \tag{4.32}$$

is precise enough, with the exception of small time intervals about points t_n, τ_n that tend to zero as the dispersion of $\psi(x', 0)$ tends to zero. We therefore have practically all the time (with the just said exception) a clear-cut one-one $x - v$ correspondence, hence a clear-cut classical macrovelocity concept in these q-ensembles, so time-of-flight measuring technique is applicable here too inside the large time intervals of interest and it will evince there the QM predictions for $R_{QM}(v,t)$.

An initial condition that makes possible a concrete numerical realization of the above general consideration is given in (4.6) (with $P = 0$ just for the sake of simplicity, in order to evade unnecessarily long formulae that lead to the same final conclusions). The QM state of motion at $t > 0$ will then be

$$\begin{aligned}\psi(x,t) = \sigma \left(\frac{m\omega}{2\pi i\hbar\sigma\sqrt{\pi}\sin\omega t}\right)^{1/2} &\left[\frac{2\pi(1 + im\omega\sigma^2\cot\omega t/\hbar)}{1 + m^2\omega^2\sigma^4\cot^2\omega t/\hbar^2}\right]^{1/2} \\ \exp\left(-\frac{X^2}{2\sigma^2} + \frac{im\omega x^2\cot\omega t}{2\hbar}\right)& \\ \exp\left\{\frac{1}{2}\frac{\sigma^2}{1 + m^2\omega^2\sigma^4\cot^2\omega t/\hbar^2}\right.&\cdot \\ \left\{\frac{X^2}{\sigma^4} - (m\omega x/\hbar\sin\omega t)^2 + 2m^2\omega^2 Xx\cot\omega t/\hbar^2\sin\omega t\right.& \\ \left.\left. + i\left[-\frac{2m\omega Xx}{\hbar\sigma^2\sin\omega t} + \frac{m\omega\sigma^2\cot\omega t}{\hbar}\left(\frac{X^2}{\sigma^4} - \frac{m^2\omega^2x^2}{\hbar^2\sin^2\omega t}\right)\right]\right\}\right\}&\end{aligned} \tag{4.33}$$

In the limit $\sigma \to 0$ practically all particles in the q-ensemble will be localized at moment $t = 0$ in a vanishingly small interval of length $\sim \sigma$ centered at point $X = x_0 = \text{const}$. Correspondingly, all particles in the idealized c-ensemble are localized at X at moment $t = 0$. Our prescription gives now

$$\mid \psi_t \mid^2 \mid \frac{\partial x_t}{\partial v} \mid \approx \frac{\sigma m}{\hbar\sqrt{\pi} \mid \cos\omega t \mid} \exp\left[-\frac{\sigma^2 m^2}{\hbar^2}(v_0^2 - \omega^2 X^2\cot^2\omega t)\right], \tag{4.34}$$

where certain inessential terms are neglected having in mind that we are interested in very small σ's. Eq. (4.34) will be invalid or physically meaningless in small time intervals (of length tending to zero in the limit $\sigma \to 0$) about points τ_n and t_n, $n = 0, 1, 2, ...$, in agreement with the above consideration. The second term in the exponent has a fixed value, given t, whereas v_0 is a free variable, so in the limit $\sigma \to 0$ we have

$$\mid \psi_t \mid^2 \mid \frac{\partial x_t}{\partial v} \mid \approx \frac{\sigma m}{\hbar \sqrt{\pi} \mid \cos \omega t \mid} \exp[-(\sigma m v_0/\hbar)^2], \tag{4.35}$$

meaningful for practically all times, with the exception of the mentioned vanishing time intervals. The right-hand side of (4.35) is certainly the v-distribution at t of an idealized c-ensemble whose v-distribution at $t = 0$ is $R(v_0, 0) = [\sigma m \exp(-\sigma^2 m^2 v_0^2/\hbar^2)]/\hbar\sqrt{\pi}$. On the other hand, one obtains using (4.33) that

$$\begin{aligned} R_{QM}(v,t) = {} & \frac{\sigma m}{\sqrt{\pi}\hbar \mid \cos \omega t \mid (1 + \frac{m^2\omega^2\sigma^4}{\hbar^2} \operatorname{tg}^2 \omega t)^{1/2}} \cdot \\ & \exp\left[-\frac{1}{\sigma^2} (\frac{v}{\omega} \sin \omega t + X)^2 - \frac{\sigma^2 m^2 v^2}{\hbar^2} \cos \omega t \right] \cdot \\ & \exp\left\{ \frac{1}{\sigma^2\omega^2(1 + \frac{m^2\omega^2\sigma^4}{\hbar^2} \operatorname{tg}^2 \omega t)} \left[v(1 - \frac{m^2\omega^2\sigma^4}{\hbar^2}) \sin \omega t + X\omega \right]^2 \right\} \end{aligned} \tag{4.36}$$

A straightforward analysis shows that, after replacing v in (4.36) by expression (4.25) (with $x_0 = X$), $R_{QM}(v,t)$ will be given in the limit of small σ's and in the large time intervals of interest by the right-hand side of eq. (4.35), which fact demonstrates the agreement of the QM postulate (3.1,2) with the concept of macrovelocity in the case examined.

The employing in the case of $U(x) = m\omega^2 x^2/2$ of 'realistic' c-ensembles with the same initial velocity distribution and initial dispersion of the same order of magnitude as in the corresponding q-ensembles leads to the same inference as in the previous cases. Namely, $r(x,t)$ behaves analogously to the relevant $\mid \psi(x,t) \mid^2$, exhibiting well defined macrovelocities (whose distribution practically coincides with the microvelocity distribution of the idealized c-ensemble with which the q-ensemble is compared) for all t, with the exception of the mentioned small time intervals of vanishing length for vanishing initial position dispersions. The concept of macrovelocity thus is of the same nature in q-ensembles and relevant realistic c-ensembles. A feature of a similar kind is easily noticeable in all cases examined : The time necessary for $\mid \psi(x,t) \mid^2$ to exhibit a well defined macrovelocity distribution decreases with the decrease of the initial position dispersion d_0 ($= \sigma$ in the Gaussian case) and in this sense the limit $d_0 \to 0$ may be called "quasiclassical". Somewhat paradoxically, the formal limit $\hbar \to 0$ acts in the opposite sense, contrary to current notions on the applicability of a quasiclassical consideration. The reason for this may be found in the fact that we examine suitable time-variable positions in the q-ensemble (In the usual WKB method, say, one has a static position distribution picture) and, besides, in the fact that $\hbar$ participates as a parameter in both q- and c-initial velocity distributions (and one may certainly introduce formally all kinds of parameters in c-velocity distributions at $t = 0$).

We must pause here to mention a fact which appears to be already known and which should be derivable from the general formulae in the recent review paper by Littlejohn [21] but which is certainly worth a simple discussion on the basis of the formulae obtained up to now. Namely, in the case of initial Gaussian wave packets [eq. (4.6)] one can construct classical phase space densities $\rho(x,v,t)$ that yield position densities $r(x,t) = \int \rho(x,v,t)dv$ and velocity densities $R(v,t) = \int \rho(x,v,t)dx$, which behave in *exactly* the same way as the relevant QM density distributions in all potential fields examined until now. In order to see this examine a well known corollary of a theorem due to Liouville which says that

$$\rho(x,v,t) = \rho(x_0,v_0,0) \tag{4.37}$$

in a c-ensemble of particles, where x and v are the position and velocity of a particle at moment t, corresponding to position x_0 and velocity v_0 of the same particle at $t = 0$. Let the initial QM state be given by (4.6) (once again with $P = 0$ for the sake of simplicity). The initial QM velocity distribution corresponding to this $\psi(x,0)$ is given by (4.10), so examine a classical phase space density ρ of the form

$$\rho(x_0,v_0,0) = \frac{m}{\pi\hbar}\exp[-(x_0-X)^2/\sigma^2]\exp[-(m\sigma v_0/\hbar)^2] \tag{4.38}$$

at $t = 0$ ($\hbar$ playing once again the role of a certain parameter in the c-distribution). [In the case of $U(x) = -Fx$, as we know, we may additionally set $X = 0$ without any loss of generality]. With the aid of eqs. (4.13,14) one arrives at

$$\rho(x, v, t) = \rho(x - vt + Ft^2/2m, v - Ft/m, 0) \tag{4.39}$$

Using this expression, one easily sees that $\int_{-\infty}^{\infty} \rho(x, v, t)dv$ coincides with expression (4.21) for $\mid \psi(x, t) \mid^2$, and $\int_{-\infty}^{\infty} \rho(x, v, t)dx$ gives exactly $m\sigma/\sqrt{\pi}\hbar \exp[-(m\sigma v_0/\hbar)^2] = (m\sigma/\sqrt{\pi}\hbar) \exp\{-[m\sigma(v - Ft/m)/\hbar]^2\}$, that is, the velocity distribution in the q-ensemble at an arbitrary $t > 0$. (When $F = 0$ one arrives at the formulae for free motion adduced by Littlejohn). For the harmonic case (where, generally, $X \neq 0$) eqs. (4.24) and (4.25) lead to

$$\rho(x, v, t) = \rho(x \cos \omega t - \frac{v}{\omega} \sin \omega t, v \cos \omega t + x\omega \sin \omega t, 0) \tag{4.40}$$

Correspondingly, the integration of this ρ over v gives exactly $\mid \psi(x, t) \mid^2$ with $\psi(x, t)$ from eq. (4.33) and integration of ρ over x yields precisely expression (4.36) for $R_{QM}(v, t)$.

The identical behaviour of the position and velocity densities in these q- and c-ensembles is, in itself, a clear indication that the problem of position-velocity coexistence should not be resolved in an essentially different manner in QM and Newtonian mechanics. (Indeed, as argued in the preceding sections, we have no logically consistent way of forming inferences other than consideration and comparison of relevant ensemble behaviours). But as one is inclined nowadays of asserting –under the impact of the CI– that only experiment can determine the applicability of certain classical concepts in QM and as well known simple thought experiments (e.g. Heisenberg's electron microscope experiment, etc) seem to rule out position-velocity coexistence, one may suspect that the identical behaviour of the said q- and c-ensembles is, in a way, deceptive : We have no direct experimental evidence of position-velocity coexistence in these cases, hence no grounds to immediately assert the unacceptability of the CI. Such an opinion may find further support in the fact that in models as Bohm's (cf. Appendix A) the QM ensemble picture for these cases is not purely Newtonian (a nonzero $\hbar$-dependent "quantum potential" exists there for a specific velocity definition). But the employing of convenient q-ensembles with a clear-cut one-one $x - v$ correspondence removes objections against the possibility of position-velocity coexistence based on arguments appealing to experiment since these ensembles give evidence for the existence of experimental methods for velocity measurements by position registering (time-of-flight technique) and the capability of these methods of reproducing the exact q-ensemble predictions for velocity distributions.

In fact, I believe that the consideration in this section offers a concrete illustration of de Broglie's physically lucid idea about wave packet separation with the course of time in a process of measurement. The natural evolution of our QM states, however, needs no special analysers here in order to exhibit a well defined (macro)velocity distribution, each velocity corresponding to a definite part of the overall wave packet.

Let us adduce now more systematically the inferences that follow from our consideration in this section.

(i) Our discussion reveals the essence of the ensemble concept of velocity in QM and demonstrates that it is possible to employ time-of-flight macrovelocity measuring technique (based on position detection and conforming to the no-analyser requirement) in normalizable q-ensembles. More exactly, the last is a consequence of the fact that the process of an unperturbed evolution of q-ensembles with known initial normalizable ensemble states objectively reveals with the course of time a definite macrovelocity distribution that can be registered with the aid of the said technique [the latter being at that theoretically capable of reproducing (thus confirming) the ensemble predictions of QM]. Consequently, one may regard time-of-flight technique as a commonplace measuring device and may compare the CI statements about measurements in general with the properties of this concrete device.

(ii) As a consequence of this one sees that ineq. (1.3) applies, generally, to positions and macrovelocities in the entire q-ensemble and has no relevance to the concept of microvelocity. The product of the objective experimental uncertainties for positions and macrovelocities can be made arbitrarily small for an individual particle (cf. also ref. 10). In other words, (1.3) applies, generally, to the result of *ensemble measurement* and not to that of *measurement* (cf. Sec. II for definitions). Hence general statements as "the electron cannot have simultaneously a definite position and momentum" (cf. any textbook on QM) may not be justified even within a 'minimal' SI.

(iii) Measurement can play an inessential role in q- and c-ensembles : it may just give information about objectively existing facts in these ensembles that may be confirmed, at a will, by simple detection. Besides :

(iv) Measurement may not be regarded, generally, as an instantaneous process. Really, the preparatory intervals of time $(0, T)$ (T being sufficiently large) necessary for velocity measurements via the time-of-flight technique [determining, by *measurement*, individual macrovelocities in the time-evolved ensemble corresponding to the initial wave packet $\psi(x, 0)$ or equivalently –by *ensemble measurement*– macrovelocity distributions assigned to the said ensemble] now play the same role as the one attributed to analysers. Namely, both analysers and/or time-intervals $(0, T)$ are necessary for a good separation of certain physical quantities, so they are an inseparable part of the process of measurement, making it noninstantaneous : This process just begins at a given moment –in our case $t = 0$– by, say, 'switching off' all the external fields at that moment. In other words, everything that could be treated as a measuring device must be removed at the chosen moment of time, in contrast to the CI notion of an instantaneous application of such a device at that moment. Only the last stage of measurement, i.e. simple detection, may be regarded as a quasi-instantaneous process. Clearly, everything in this point applies mutatis mutandis to measurements in a realistic c-ensemble too (the q- and c-pictures of the evolution of position and velocity distributions being even explicitly identical for Gaussian wave packets). Hence

(v) From the viewpoint of a time-of-flight velocity measuring technique positions and macrovelocities cannot coexist at small $t \geq 0$ in both q- and realistic c-ensembles due to the very nature of the concept of macrovelocity. Namely, a large part of the ensemble members are still to be found in a small vicinity d_t of point $x = 0$ at these t's and the initial spread d_0 of position densities make impossible a good mathematical definition of macrovelocity when d_t is of the order of magnitude of d_0.

(vi) Ensemble measurement of both q- and c-ensembles introduces a certain asymmetry in treating the past and future (with respect to this measurement) of a physical system (more exactly, of its statistical ensemble image). Namely, with regard to the past it evinces a (predicted) probability distribution of a physical magnitude while with regard to the future it creates a new *ensemble* state. In our case the last will be due to the fact that position detection of particles can certainly modify the velocity distributions in both q- and c-ensembles because of a possible direct perturbing influences of the position detector whose performance may be based on a variety of physical phenomena. Therefore, there is nothing specifically quantum-mechanical in this kind of a past-future asymmetry contrary to widespread notions based on Heisenberg's ideology. (There exist recent discussions of Heisenberg's viewpoint in the literature [10],[22]. This viewpoint has penetrated textbooks too –see, e.g., § 7 in ref. 20). Consequently :

(vii) The specific QM features of the problem examined in this section (which, according to the 'minimal' SI are not to be found in any of the preceding points) must be sought in an explanation of the concrete fixed numerical value of the constant term in the right-hand side of ineq. (1.3). A possible explanation may probably be that in the process of preparation of the wave packet with an initial form $\psi(x,0)$ there exist phenomena, forces, and so on which are not taken into consideration in the usual classical approach to the problem and in which $\hbar$ plays an important role. [In fact, as we know, the 'nonminimal' SI's mentioned in the Introduction (cf. also appendix A) represent attempts at explaining the QM behaviour of microparticles along such lines. We are not going to make here (within the 'minimal' SI) any concrete assumptions about the nature of the possible additional factors and their time-evolution but shall just point out that this interpretation is completely open for new physics (and mathematics –see Appendix B) that may lead to future developments].

To summarize, the CI rests on an admittedly universal theory of measurement in which measurements are instantaneous and "complementary" magnitudes do not coexist. Besides, the CI asserts that immediately after the instantaneous measurement process the measured magnitude has a definite value for the measured system. The above viewpoint, however, finds no corroboration when the objective evolution of the QM state of motion (combined with its ensemble interpretation) is considered for certain solvable cases : It turns out that uncertainty relations are, generally, ensemble concepts while lawful experimental procedures (theoretically capable of reproducing the q-ensemble predictions for certain magnitudes and arbitrary normalizable initial conditions) are essentially noninstantaneous, may be destructive for the measured magnitude (the same applying to the relevant classical case) and are based exactly on the idea of coexistence of "complementary" magnitudes for individual members of the ensemble.

V. Explicit position-velocity coexistence in certain nonnormalizable nonstationary quantum states

The reinterpretation of ineq. (1.3) in the previous section and the explicit demonstration of position-macrovelocity coexistence (within the frame of the 'minimal' SI) in q-ensembles corresponding to normalizable ψ's poses an obvious question. Namely, are there QM problems in which explicit position-velocity coexistence (from the viewpoint of the 'minimal' SI) may be objectively interpreted as an explicit position-microvelocity coexistence for individual particles ?

The possible answer to this question should obviously be sought in problems of two types : (A) problems in which the characteristic initial spread of QM states is exactly equal to zero, so that there would be no necessity of ascribing macroscopic sense to particle velocities, and (B) problems in which the characteristic initial spread of the QM states is infinite, so that the concept of macrovelocity in the above sense just does not exist. A sufficient objective condition for the admissibility of explicit position-microvelocity coexistence in the 'minimal' SI would be an identical behaviour of q- and c-ensemble position and velocity distributions for all $t \geq 0$ in problems of kinds (A) and (B).

Let us recall here that the discussion in Sec. IV already contains a clear indication that type (A) problems may offer an explicitly Newtonian picture of position-microvelocity coexistence for all $t \geq 0$ in the physical fields of interest. Indeed, we saw there that the time needed of a q-ensemble to evince via its position distribution a clear-cut classical velocity distribution tends to zero as the spread d_0 of $\psi(x, 0)$ tends to zero and one may certainly expect that the limit $d_0 = 0$ itself will give one-one $x - v$ correspondence for all $t \geq 0$. But, whereas in the classical case it is possible to construct normalizable $\rho(x, v, t)$'s even in the case of ensembles 'concentrated' at a given point x_0 (at $t = 0$), the type (A) problems require in the QM case initial states of the form $\psi(x, 0) \sim \delta(x - x_0)$. These states correspond to nonnormalizable c-ensembles in order to carry out the necessary comparisons. A typical initial QM state in a type (B) problem is $\psi(x, 0) = \exp(ipx/\hbar)$, for which $| \psi(x, 0) = 1$, $x \in (-\infty, \infty)$, that is, once again a nonnormalizable state whose evolution must be compared with that of a relevant nonnormalizable c-ensemble.

The consideration of such nonnormalizable QM states is of a definite theoretical interest for other reasons too, say (a) they give an answer to legitimate questions as "what is the state of motion of particles for which either position or momentum are exactly known (recall that there is no theoretical limit in nonrelativistic QM to the precision with which these magnitudes can be defined), and (b) ineq. (1.3) seems to imply total inapplicability of a given classical concept (x or p) in cases when its "complementary" concept (p or x, respectively) is exactly defined. But the corresponding nonnormalizable QM states of motion in these cases, as we know, go outside the range of states to which ineq. (1.3) applies, so special investigation of the properties of the QM systems in question is necessary.

Somewhat unexpectedly from a CI viewpoint, the nonnormalizable QM states that will be examined here will really give a totally classical statistical picture of explicit position-microvelocity coexistence for all times $t \geq 0$.

[One should recall here that in the nonnormalizable case $\mid \psi \mid^2$ has a *relative meaning*, determining for instance via $\mid \psi(x_1, t_1) \mid^2 / \mid \psi(x_2, t_2) \mid^2$ just the ratio of the actual number of particles (and not actual probabilities) in infinitesimal intervals of the type $(x_i - \epsilon,\ x_i + \epsilon)$, x_i fixed, $i = 1, 2$, at moments t_1 and t_2].

Examine first the case of a homogeneous force field F and an initial state

$$\psi(x, 0) = \exp(ip_0 x/\hbar), \tag{5.1}$$

p_0 being an arbitrary initial momentum. Formulae (3.10) and (3.12), with $\psi(x', 0)$ from (5.1), yield via standard integration

$$\psi(x, t) = \exp[ip(t)x/\hbar] \exp\left[\frac{1}{2i\hbar m}(p_0^2 t + p_0 F t^2 + F^2 t^3/3)\right], \tag{5.2}$$

where

$$p(t) = p_0 + Ft \quad , \quad t \geq 0 \tag{5.3}$$

is the momentum at t of a classical particle with an initial momentum p_0 [cf. (4.13)]. Consequently, $\mid \psi(x, t) \mid^2$ remains constant all the time while the variation of the momentum of microparticles conforms to Newton's dynamics. A c-ensemble of (noninteracting) particles with the same initial momentum p_0, mass m and a homogeneous position distribution will behave in exactly the same way at $t \geq 0$ in the above force field. This identity of our q- and c-ensemble pictures at all $t \geq 0$ makes it possible to assert for the case of interest explicit position-microvelocity coexistence in q-ensembles within the frame of the 'minimal' SI.

Examine now an initial QM state

$$\psi(x,0) = \delta(x - x_0) \tag{5.4}$$

in our force field F, x_0 being an arbitrary fixed position along x. It is well known that

$$\lim_{\epsilon \to 0} K(x,t) + \epsilon; x', t) = \delta(x - x'), \tag{5.5}$$

so the time-evolved state $\psi(x,t)$ will essentially coincide here with the propagator itself and the position density will therefore be

$$\mid \psi(x,t) \mid^2 \sim \text{const}/t \quad , \quad t > 0 \tag{5.6}$$

The corresponding $r(x,t)$ in a c-ensemble of particles located at point $x_0 \in (-\infty, \infty)$ at moment $t = 0$ and having a homogeneous velocity distribution $R(v,0)$ at that moment is easily seen to obey the same law (5.6) at all $t > 0$, $R(v,t)$ remaining homogeneous all the time. A straightforward check-up shows that the same applies to $R_{QM}(v,t)$. We can thus assert both explicit position-microvelocity coexistence within the 'minimal' SI and validity of the QM postulate (3.1,2) for q-ensembles obeying condition (5.4) in a homogeneous field F.

We are going to examine now a really instructive example, namely, particle motion in a harmonic potential field $U(x) = m\omega^2 x^2/2$, $x \in (-\infty, \infty)$. As we shall see, there will be one-one $x - v$ correspondence in our problem at all $t \geq 0$, with the exception of the discrete set of moments t_n, τ_k, $(n, k = 0, 1, 2, ...)$, defined in Sec. IV, at which the violation of this correspondence will turn out to be inessential from the viewpoint of the applicability of our approach outlined in Sec. III, so we shall employ the said prescriptions in the usual manner. Consider first a nonnormalizable c-ensemble of noninteracting particles all of which have the same initial velocity v_0, being homogeneously distributed along the x-axis at moment $t = 0$, that is, $r(x,0) =$ const. (The ensemble thus represents a classical situation in which one knows the precise value of the initial momentum $p_0 = mv_0$ of a microparticle in the harmonic field and has no information whatsoever about its position). For the sake of convenience we choose

$$r(x,0) = 1 \tag{5.7}$$

The motion of a given particle in our c-ensemble is determined by eqs. (4.24,25). From (4.24) it follows now (v_0 fixed, x_0 –a free variable) that the particles in the c-ensemble with different values of x_0 will be located at nonoverlapping points $x(x_0, v_0, t)$ at any moment $t \geq 0$, with the exception of the discrete set of moments t_n [eq. (4.28)], $n = 0, 1, ...$, at which all particles will have the same position $x(t_n) = \pm v_0/\omega$. This means that the equation

$$r[x(x_0, v_0, t), t]Dx(x_0, v_0, t) = r(x_0, 0)Dx_0 \tag{5.8}$$

will hold at any $t \neq t_n$; the 'differential' D in (5.8) is defined for the general case via $Df =\mid \partial f/\partial x_0 \mid dx_0$, dx_0 ($= Dx_0$) being an infinitesimal variation of the (free) initial position x_0. With the aid of eqns. (5.7) and (4.24) we obtain

$$Dx(x_0, v_0, t) =\mid \cos \omega t \mid dx_0 \tag{5.9}$$

Hence

$$r[x(x_0, v_0, t), t] = 1/ \mid \cos \omega t \mid \tag{5.10}$$

at $t \neq t_n$. That is, at any moment $t \neq t_n$ we have once again a homogeneous position density distribution, the only difference being that the number of particles per unit length has changed everywhere by a factor of $1/ \mid \cos \omega t \mid$. (It is worth recalling here that there exists no upper bound for the speed of a particle in Newtonian dynamics and that under our assumptions infinity serves as a source of particles).

Eq. (3.3) may be written here in the form

$$R[v(x_0, v_0, t), t]Dv(x_0, v_0, t) = r[x(x_0, v_0, t), t]Dx(x_0, v_0, t) \tag{5.11}$$

The velocity 'differential' in (5.11) is equal to

$$Dv(x_0, v_0, t) = D\partial x(x_0, v_0, t)/\partial t = \omega \mid \sin \omega t \mid dx_0,$$

so that

$$R[v(x_0, v_0, t)t] = 1/\omega \mid \sin \omega t \mid \tag{5.12}$$

at $t \neq \tau_k$ [eq. (4.29)], $k = 0, 1, 2, ...$ Consequently, $R(v, t)$ is homogeneous too and varies with time analogously to $r(x, t)$. At moments τ_k, for which R is indefinite, all particles in the c-ensemble will have equal velocities $\pm v_0$. (For the sake of comparison recall what happened at t_n and τ_k in the consideration in Sec. IV).

Let us see now what will be the result for the appropriate q-ensemble in the same field. The initial wave function $\psi(x,0)$ that corresponds to an initial value $p_0 = mv_0$ of the momentum and to $r(x_0,0) = \mid \psi(x,0) \mid^2 = 1$ will evidently be

$$\psi(x,0) = \exp(ip_0 x/\hbar) \tag{5.13}$$

Eqs. (3.10) and (3.13) lead in this case to

$$\psi(x,t) = \frac{1}{(\cos\omega t)^{1/2}} \ \exp\left[\frac{i}{2\hbar}\left(-m\omega x^2 \operatorname{tg}\omega t + \frac{2p_0 x}{\cos\omega t} - \frac{p_0^2 \operatorname{tg}\omega t}{m\omega}\right)\right] \tag{5.14}$$

We thus see that at moments $t \neq t_n$ the position density as given by $\mid \psi(x,t) \mid^2$ is homogeneous and equal to

$$\mid \psi(x,t) \mid^2 = 1/ \mid \cos\omega t \mid, \tag{5.15}$$

which result exactly coincides with the classical one [cf. eq. (5.10)] at $t \neq t_n$.

In order to determine the nature of the indefiniteness of $\psi(x,t)$ at moments t_n examine the time-evolution of the wave function satisfying the initial condition

$$\psi_1(x,0) = (-2\pi i\hbar/m\omega)^{1/2}\delta(x-x_0) \tag{5.16}$$

where $x_0 = p_0/m\omega = v_0/\omega$. From eqs. (3.10) and (3.13) we get

$$\psi_1(x,t) = \frac{1}{(-\sin\omega t)^{1/2}} \ \exp\left[\frac{im\omega}{2\hbar}(x^2 \cot\omega t - \frac{2xx_0}{\sin\omega t} + x_0^2 \cot\omega t)\right] \tag{5.17}$$

Obviously, the value of $\psi_1(x,t)$ coincides with that of $\psi(x,t')$, where $t' = t + \pi/2\omega$ [cf. eq. (5.14)]. Consequently, the indefinite nature of $\psi(x,t)$ at moments t_n results from the fact that it represents a δ-function of coordinates at these moments, 'centered' alternatingly at points $x = \pm v_0/\omega$. The behaviour of the c- and q-position densities in this problem is thus identical at all $t \geq 0$.

We therefore have sufficient grounds to assert that the q-ensemble in this problem must have the same velocity distribution as that of the c-ensemble at all t, so that we have once again an explicit example of coexistence of the position-microvelocity concepts. The check-up of the validity of the QM postulate (3.1,2) yields a result that agrees with the just said. Really, by standard integration,

$$a(p,t) = \frac{\exp(-i\pi/4)}{(m\omega\sin\omega t)^{1/2}} \exp\left[\frac{i}{2m\omega\hbar}\left(p^2\cot\omega t - \frac{2pp_0}{\sin\omega t} + p_0^2\cot\omega t\right)\right] \tag{5.18}$$

Hence

$$R_{QM}(v,t) = 1/\omega \mid \sin\omega t \mid \tag{5.19}$$

in agreement with eq. (5.12). Evidently, $a(p,t)$ is indefinite at the same moments τ_k at which the classical momentum density is indefinite and this is once again due to a δ-like course of the function in question [compare eqs. (5.17) and (5.18)].

Consequently, the coinciding classical and QM pictures exhibit a periodic interchange of a fixed momentum and a fixed position for all members of the ensemble. In our consideration this fact is not related to any kind of measurement but results from the properties of the physical systems considered whose evolution in the harmonic field is such that precise knowledge of momentum at a given moment of time makes possible the precise prediction of a fixed coordinate at a relevant subsequent moment of time and vice versa.

The implications of the last example are thus quite instructive indeed: One not only sees the admissibility of position-microvelocity coexistence but obtains as well an explicit demonstration of consequtive localizations in coordinate and/or momentum space as a result of particle dynamics, without any need of employing a special projection postulate.

The above consideration readily leads to a prescription for velocity measurement in the QM states in question. Namely, as in the identical classical case, one may measure velocity at $t \neq t_n$ by registering the particle's position at such t. Knowledge of position at t makes possible the reconstruction of the initial position x_0 via formula (4.24). The velocity $v(x_0, v_0, t)$ (v_0 being fixed and known for all particles) can then be calculated with the aid of (4.25). [The identical nature of the indefiniteness at $t = t_n$ in the classical and QM cases makes it possible to determine, in any case, $v(x_0, v_0, t_n)$ by position registering at some $t \neq t_n$ and then computing x_0 and $v(x_0, v_0, t_n)$ in the manner just described]. The ensemble of results so obtained (in other words, the statistics of ensemble measurement) will be precisely the one predicted by both QM and the statistical variant of classical mechanics and, as it was already pointed out, statistical theories admit just ensemble predictions in the general case.

Références

[20] L.D. Landau and E.M. Lifshits, *Quantum Mechanics. The Nonrelativistic Theory*, Nauka, Moscow, 1974.
[21] R.G. Littlejohn, Phys. Rep. **138**, (1986) 193.
[22] Max Jammer, *The Conceptual Development of Quantum Mechanics*, Mc Graw Hill, New York, 1967.

(Manuscrit reçu le 20 juin 1987, révisé le 10 septembre 1987)