# Spheroid like solution of non linear Klein Gordon equation

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In the nonlinear field theory of the elementary particles particle-like spherically symmetrical solutions of non linear equations are usually investigated [1,2]. There were also studied stringlike cylindrically symmetrical solutions [3,4] and spatially one dimensional (plane) solutions in order to understand the unusual behavior of solitons which are interpreted as a prototype of elementary particles.

In that way, for a qualitative study of the possibilities of the theory, simple solutions of non linear field equations are ivestigated (e.g. of the non linear Klein-Gordon equation). Moreover the non linearity is chosen according to the suitable mathematical model (sine-Gordon, $\phi^4$-Theory, and so on).

In this paper we consider the possibilities of intermediate type of solutions between spherically and cylindrically symmetrical or plane solutions. We investigated the non linear Klein-Gordon equation with non linearity of the step form [3].

Hence, we begin with the nonlinear complex lagrangian of scalar field (In the following we take $m_0 = \hbar = c = 1$)

$$L(\psi\psi^*) = -\frac{1}{2}\Big\{\frac{\partial\psi}{\partial x_\mu}\frac{\partial\psi^*}{\partial x_\mu} + \psi\psi^* + F(\psi\psi^*)\Big\}$$

where F(x) is a monotonously increasing non linear function.

Most often one considers particle-like and string-like-solutions of non-linear Klein-Gordon equations

$$\Box\psi - (1 + F'(\psi\psi^*))\psi = 0$$

$$\Box\psi^* - (1 + F'(\psi\psi^*))\psi^* = 0$$

$\Box$ is D'Alembertian, and $F'(\psi\psi^*) = d/d(\psi\psi^*)F(\psi\psi^*)$. In the stationary case, when $\psi = v\exp(-i\varepsilon t)$ or $\psi^* = v\exp(i\varepsilon t)$, we have

$$\nabla^2 v - (1 - \varepsilon^2 + F'(v^2))v = 0 \tag{2}$$

We consider the simple case of non linearity admitting analytical solutions for equation (2) using the Terletsky ansatz [3]

$$F'(v^2) = -a^2\theta(v - v_0) \tag{3}$$

where $v_0$, $a$ are constants, but

$$\theta(v - v_0) = \begin{cases} 0, & v < v_0 \\ 1, & v \geq v_0 \end{cases}$$

so in that case, equation (2) yields two equations

$$\nabla^2 v_I - (1 - \varepsilon^2)v_I = 0 \quad , \quad v_I < v_0 \tag{4}$$

$$\nabla^2 v_{II} + (a^2 - 1 + \varepsilon^2)v_{II} = 0 \quad , \quad v_{II} \geq v_0 \tag{5}$$

In the first part of this paper we get an intermediate solution between particle-like and string-like ones.

In that way, we use the prolate spheroidal coordinates $\xi$, $\eta$, $\varphi$ which are connected with the Cartesian coordinates $X$, $Y$, $Z$ by the formulae :

$$X = \frac{d}{2}[(\xi^2 - 1)(1 - \eta^2)]^{1/2}\cos\varphi = \rho\cos\varphi$$

$$Y = \frac{d}{2}[(\xi^2 - 1)(1 - \eta^2)]^{1/2}\sin\varphi = \rho\sin\varphi$$

$$Z = \frac{d}{2}\xi\eta$$

We shall denote the interfocal distance by $d$. In that case the surface $\xi = \text{const} > 1$ is a prolate ellipsoid of revolution $\xi \in [1, \infty)$. The surface $\mid \eta \mid = \text{const} < 1$ is a hyperboloid of revolution $\eta \in [-1, 1]$ consisting of two sheets. The surface $\varphi = \text{const}$, $\varphi \in [0, 2\pi)$ is a plane through the $z$-axis forming angle $\varphi$ with the $X, Z$ plane.

We shall start with equations (4) and (5) in the prolate spheroidal coordinates with the nonlinearity (3) and Laplacian

$$\nabla^2 = \frac{4}{(\xi^2-\eta^2)d^2}\cdot \cdot\left(\frac{d}{d\xi}(\xi^2-1)\frac{d}{d\xi}+\frac{d}{d\eta}(1-\eta^2)\frac{d}{d\eta}+\frac{\xi^2-\eta^2}{(\xi^2-1)(1-\eta^2)}\frac{d^2}{d\varphi^2}\right)$$

In the general case the usual procedure of the separation of variables in the prolate spheroidal coordinates [7] for the equation $(\nabla^2+k^2)W=0$ shows that the eigenfunctions $W_{mn}(\xi,\eta,\varphi)$ are

$$W_{mn}(\xi,\eta,\varphi)=R^{(j)}_{mn}(c,\xi)S_{mn}(c,\eta)e^{\pm im\varphi} \tag{6}$$

where $j=\overline{1,4}$ ; $R^{(j)}_{mn}(c,\xi)$ -the radial prolate spheroidal function of the order $j, S_{mn}(c,\eta)$ -the angular prolate spheroidal function, $c=kd/2$. For the equation $(\nabla^2-k^2)W=0$ eigenfunctions can be obtained also by the transformation $c\to ic$.

We shall find solutions of equations (4) and (5) in the form of series of eigenfunctions (6). It is suggested that these eigenfunctions are independent of angle $\varphi$ so that $m=0$ in (6). Thus when $\varepsilon^2-1<0$ and $a^2-1+\varepsilon^2>0$ we have two solutions

$$\begin{aligned} v_I(\xi,\eta) &= \sum_n A_n R^{(3)}_{on}(ic_1,\xi)S_{on}(ic_1,\eta) \quad , \quad v_I<v_0 \\ c_1 &= \frac{d}{2}\sqrt{1-\varepsilon^2} \end{aligned} \tag{7}$$

$$\begin{aligned} v_{II}(\xi,\eta) &= \sum_{n'} B_{n'} R_{on}(c_2,\xi)S_{on}(c_2,\eta) \quad , \quad v_{II}\geq v_0 \\ c_2 &= \frac{d}{2}\sqrt{a^2-1+\varepsilon^2} \end{aligned} \tag{8}$$

Spheroid-like solutions exist only for the same values of the coordinates $\xi=\xi_0$ (that is on the ellipsoid of revolution $\xi_0$) i.e. when functions $v_1$ and $v_{II}$ satisfy the following boundary condition

$$v_1(\xi_0,\eta)=v_{II}(\xi_0,\eta)=v_0 \tag{9}$$

$$\frac{d}{d\xi}v_1(\xi,\eta)\Big|_{\xi_0}=\frac{d}{d\xi}v_{II}(\xi,\eta)\Big|_{\xi=\xi_0} \tag{10}$$

The coefficients $A_n$ and $B_{n'}$ in (7), (8) can be found using the boundary conditions (9) with the help of the orthonormality in the interval $[-1,1]$ of the angular prolate spheroidal functions

$$\left.\begin{aligned} A_n &= \frac{v_0}{R_{on}(ic,\xi_0)}\int_{-1}^{1} S_{on}(ic,\eta)d\eta \\ B_{n'} &= \frac{v_0}{R^{(3)}_{on'}(c^*,\xi_0)}\int_{-1}^{1} S_{on'}(c^*,\eta)d\eta \end{aligned}\right\} \tag{11}$$

The second boundary condition (10) allows to calculate the possible values of the parameter $\varepsilon$ as a function of $\xi_0$ and of the height of the step $a$.

The values $\varepsilon$ for wich both $v_I$ ad $v_{II}$ have the same values (9) correspond to spheroid-like solution of equation (2). Hence it can be said, that $\varepsilon$ are the eigenvalues of equation (2) in the particular case of the nonlinear function $F'(v^2)$ given in (3).

Let us consider the limit of the spherically symmetrical case when the interfocal distance $d \to 0$ and $\xi \to \infty$. Then $\eta \to \cos\theta$, $ur = d\xi/2$ remains finite. For the radial spheroidal functions we have [6]

$$R^{(3)}_{on}(ic_1,\xi) = h^{(1)}_r(ic\xi)$$

$$R_{on'}(c_2,\xi) = j_r(c_2\xi)$$

with $r$ a positive whole number, $j_r(x)$, $h^{(1)}_r(x)$, Bessel functions of $1^{st}$ and $3^{rd}$ class correspondingly.

By using the properties of modified Bessel functions [8] and the fact that the solutions do not depend on the polar and azimuthal angles we get from (7) and (8).

$$v_I(r) \approx A\exp(k_1 r)/r \quad , \quad v_I < v(r_0)$$

$$v_{II}(r) \approx B\sin(k_2 r)/r \quad , \quad V_{II}(r) \geq v(r_0)$$

where $A$, $B$, $k_1$, $k_2$ are constants. Such solutions were obtained earlier in the paper by Shushurin S.F. [3].

In the other limiting case, when $d \to \infty$, $\xi \to 1$, $\eta \to \pm 1$ while $\rho = (\xi^2 - 1)^{1/2}\frac{d}{2}$, $Z = \frac{d}{2}\eta$ remains finite, we get the ordinary cylindrical coordinates $\rho$, $Z$, $\varphi$. The radial spheroidal functions take the forms [6]

$$\left.\begin{aligned} R^{(3)}_{on}(ic_1, \xi) &= \sqrt{\frac{\pi}{2c_1 i\xi}} H_0\left(ic_1\sqrt{\xi^2-1} - \frac{2n+1}{2}\arctan\sqrt{\xi^2-1}\right) \\ R_{on}(c_2, \xi) &= \sqrt{\frac{\pi}{2c_2\xi}} J_0\left(c_2\sqrt{\xi^2-1} - \frac{2n+1}{2}\arctan\sqrt{\xi^2-1}\right) \end{aligned}\right\} \tag{12}$$

$H_0(x)$ and $J_0(x)$ are spherical Bessel functions of zeroth order. Putting the expressions (12) into (7) and (8) and using the limit $\xi \to 1$, $d \to \infty$ we get

$$\begin{aligned} v_I &\approx A' K_0(K'\rho) \quad , \quad v_I(\rho) < v(\rho_0) \\ v_{II} &\approx B' J_0(K''\rho) \quad , \quad v_{II}(\rho) \geq v(\rho_0) \end{aligned}$$

$J_0(x)$ and $K_0(x)$ are Bessel functions of zeroth order, spherical and modified functions correspondingly. $A'$, $B'$, $K'$, $K''$ some constants.

Similar results of string-like solutions were got in the paper by Edjo O. [4] and Terletsky Ya. [3]. In that way, we have shown that in both limited cases our solutions (7,8) turn into already known particle-like and string-like solutions corresponding to the nonlinearity (3).

Let us again consider the case, when the spheroid $\xi_0$ in the limiting case describes an infinite plane disk. For the solution of the field equation in that limiting case it is convenient to pass from the prolate spheroidal coordinates to the oblate spheroidal coordinates.

In this case, we perform a substitution

$$\begin{aligned} \xi &\to i\xi \\ \eta &\to \eta \\ c_1 &\to -ic_1 \\ c_2 &\to -ic_2 \end{aligned}$$

Then any solution of equation (2) in the prolate coordinates gives rise to a solution of the same equation in the oblate coordinates. Then instead of (7) and (8), we have

$$\left.\begin{aligned} V_I(\xi,\eta) &= \sum_n A_n R^{(3)}_{on}(c_1, i\xi) S_{on}(c_1,\eta) \quad , \quad v_I < v_0 \\ V_{II}(\xi,\eta) &= \sum_{n'} B_{n'} R_{on'}(-ic_2, i\xi) S_{on'}(-ic_2,\eta) \quad , \quad v_{II} \geq v_0 \end{aligned}\right\} \tag{13}$$

In the limit $\xi \to 0$, $d \to \infty$, $\eta \to \pm 1$ (so that both $z = d\xi\eta/2$, $\rho = \sqrt{1-\eta^2}d/2$ remain finite) and if we suppose $\rho$ to be infinite, we get from the starting spheroid $\xi_0$ an infinite plane disk. In that way radial oblate spheroidal functions take the forms [6]

$$\left.\begin{array}{lr} R_{on}^{(3)}(c_1, i\xi) \to \dfrac{1}{c_1\sqrt{\xi^2+1}} \exp(-c_1\xi - 2\chi \arctan\xi) \quad , & z \geq z_0 \\ R_{on'}(-ic_2, i\xi) \to \dfrac{1}{-ic_2\sqrt{\xi^2+1}} \exp(ic_2\xi - 2\chi \arctan\xi) \quad , & z < z_0 \end{array}\right\} \tag{14}$$

Where $z_0$ is the height of the resulting disk. The whole semispace $z > 0$ is divided in two parts one of which $0 < z < z_0$ is interior relative to the disk and the second $z < z_0$ is exterior.

The solutions corresponding to these parts have a form

$$\left.\begin{array}{lr} v_I(z) = A\exp(-z\sqrt{1-\varepsilon^2}) \quad , & z \geq z_0 \\ v_{II}(z) = B\sin(z\sqrt{a^2-1+\varepsilon^2}) \quad , & z < z_0 \end{array}\right\} \tag{15}$$

We can check the solution (15) by taking into account that the stationary equation (2) in ordinary cylindrical coordinates in the absence of the independence on the radial variable $\rho$ takes the form

$$\frac{d^2v}{dz^2} - (1 - \varepsilon^2 + F'(v^2))v = 0 \tag{16}$$

The solutions of equation (16) with the nonlinearity (3) coincide with (15) and satisfy the boundary condition at $z = z_0$

$$A\exp(-z_0\sqrt{1-\varepsilon^2}) = B\sin(z_0\sqrt{a^2-1+\varepsilon^2}) = v_0 \tag{17}$$

$$\tan(\sqrt{a^2+1-\varepsilon^2}z_0) = -\frac{\sqrt{a^2-1+\varepsilon^2}}{\sqrt{1-\varepsilon^2}} \tag{18}$$

Moreover, they satisfy the normalization

$$\int \varepsilon v^2 dV = 1$$

or

$$\int_0^{z_0} \varepsilon B^2 \sin^2(\sqrt{a^2-1+\varepsilon^2}\, z)dz + \int_{z_0}^{\infty} \varepsilon A^2 \exp(-2z\sqrt{1-\varepsilon^2})dz = 1 \tag{19}$$

constants $A$ and $B$ are found from equation (17)

$$A = v_0 \exp(z_0\sqrt{1-\varepsilon^2}) \quad , \quad B = v_0 / \sin(z_0\sqrt{a^2-1+\varepsilon^2})$$

The condition of continuity (18) can be easily brought into

$$\frac{\chi}{az_0} = -\sin\chi \quad , \quad \chi = \sqrt{a^2-1+\varepsilon^2}\, z_0$$

A comparison of these results with the corresponding expressions for a particle-like solution [2] shows that the methods derived in the paper [2] can be successfully applied to calculate in our particular case, the mass, energy, charge and other integrals of motion of the disk-like solution of non linear Klein-Gordon equation.

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