The quantum potential and "causal" trajectories for stationary states and for coherent states

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ABSTRACT. We show for stationary states in a central potential that the quantum action S is a part of the classical action W and derive a new expression for the "quantum potential" U_Q in terms of the other part. The association of momenta of some "trajectories" in the causal interpretation of quantum mechanics by $\vec{p} = \nabla S$ and by $d\vec{p}'/dt = -\nabla(V + U_Q)$ are studied in detail for stationary states. They give different type of orbits which express some flow properties of the quantum mechanical current. For coherent states on the other hand $\vec{p}$ and $\vec{p}'$, as well as the quantum mechanical averages $< \vec{p} >$ and the classical momenta, all four, lead to essentially the same trajectories except for different integration constants. The spinning particle is also considered.

RESUME. Nous avons montré que dans le cas des états stationnaires l'action quantique S est seulement une (première) partie de l'action classique W et nous avons exprimé le "potentiel quantique" U_Q en fonction de la deuxième partie. Les moments associés aux "particules" dans l'interprétation causale par la relation $\vec{p}' = \nabla S$ et par la relation $d\vec{p}/dt = -\nabla(V + U_Q)$ donnent dans les états stationnaires des orbites qui sont très différentes les unes des autres et qui sont très différentes des orbites classiques. Ces orbites provenant de la première relation expriment les propriétés du courant quantique. D'un autre côté, pour les états cohérents, les moments $\vec{p}$ et $\vec{p}'$, ainsi que le moment moyen quantique $\overline{\vec{p}}$ et le moment classique, conduisent essentiellement aux mêmes trajectoires, qui diffèrent par les constantes d'intégration. Nous étudions aussi la particule avec spin dans le champ magnétique.

I. Introduction

During the last decade Bohm's causal trajectories based on the concept of the so called "quantum potential" [1,2] have been evaluated in a number of cases, the double slit [3,4], tunneling through the barrier [5,6], the Stern-Gerlach magnet [7], neutron interferometer [9,10], EPRB experiment [11]. In all these works the trajectories have been evaluated numerically for the time dependent wave packet of the Schrödinger equation.

In the present paper we study explicitly the quantum potential and the associated trajectories for stationary states and for coherent states. We show how the quantum potential arises from the classical action. The classical action is a sum of two parts $S_{c1} = S_1 + S_2$, one part S_1 becomes the quantum action, the other part S_2 is shifted to the quantum potential. We also show that the two definitions of causal trajectories do not coincide for stationary states.

II. Review of Bohm's interpretation versus Schrödinger discovery

In suggesting the interpretation of Quantum theory in terms of "hidden" variables, so-called Causal Interpretation, [2], Bohm treated Schrödinger equation as a postulate. The sequence of steps leading to causal trajectories, is as we are going to show, approximately the inverse of the Schrödinger's original sequence of steps [12,13], one from classical trajectories to wave equation, the other from wave equation to some new associated trajectories which have nothing to do with the original classical trajectories.

Schrödinger's discovery of wave mechanics represents one *specific* synthesis of mechanics, wave theory and Planck hypothesis. Basic mechanical concepts are the classical Hamilton and Lagrange functions for a particle

$$H = (\vec{p}^2/2m) + V(\vec{r}) = T + V \tag{1}$$

$$L = (\vec{p}^2/2m) - V(\vec{r}) = T - V \tag{2}$$

And the basic wave concept is the wave function $\psi(\vec{r}, t)$ required to satisfy the wave (d'Alembert) equation

$$\Delta\psi - \ddot{\psi}/u^2 = 0 \tag{3}$$

where u is the wave (phase) velocity. The synthesis of mechanical and wave notions are realized now by deriving a relationship between the phase velocity and the kinetic and total energies of the particle

$$u = E/\big(2m(E-V)\big)^{1/2} = E/(2mT)^{1/2} \tag{4}$$

in two different ways.

The first method uses the Hamilton-Jacobi equation for the action W

$$W = \int_{to}^{t} (T - V)dt \tag{5}$$

which taken as a function of the upper limit t and of the final value of the coordinates x, y, z satisfies the Hamilton-Jacobi partial differential equation

$$\partial W/\partial t + (\nabla W)^2/2m + V(\vec{r}, t) = 0 \tag{6}$$

If the potential does not depend on time, the equation (6) can be solved by setting

$$W(\vec{r}, t) = -Et + w(\vec{r}) \tag{7}$$

This substitution leads to

$$(\nabla W)^2/2m + V(\vec{r}) = E \tag{8}$$

or

$$|\overrightarrow{\mathrm{grad}}\, w| = [2m(E-V)]^{1/2} \tag{9}$$

The relation

$$\vec{p} = m\vec{v} = \overrightarrow{\mathrm{grad}}\, W = \overrightarrow{\mathrm{grad}}\, w \tag{10}$$

is satisfied in the above procedure.

In this method the relation between the mechanical and the wave motion is expressed by assuming the action W to be proportional to the phase of the wave

$$\psi(\vec{r}, t) = A\exp(iW/K) = A\exp(-iEt/K) + iw/K) \tag{11}$$

Further to the motion of the particle one associates the motion of surfaces of constant W and therefore the propagation of the wave. If W_0 is the value of W on a particular surface for a given t, and $dW_0 = Edt$, then the infinitesimal normal

$$dn = dW_0/[2m(E-V)]^{1/2} \tag{12}$$

determines the velocity of motion of any surface at any one of its points by

$$u = dn/dt = E/[2m(E-V)]^{1/2} \tag{13}$$

which is the desired eq.(4).

The second method uses [14] the analogy between the principle of least action

$$\delta \int_A^B [2m(E-V)]^{1/2} ds = 0 \tag{14}$$

and Fermat's principle in optics

$$\delta \int_A^B ds/u = 0 \tag{15}$$

This analogy leads to

$$u = C/[2m(E-V)]^{1/2} \tag{16}$$

Then one identifies the group velocity of the wave of frequency ν given by

$$1/v_g = d(\nu/u)/d\nu \tag{17}$$

with the particle velocity $v = p/m$. The result is the value of the constant C,

$$C = E$$

i.e. the relation (4) again.

Schrödinger then looked for those solutions of the wave equation (3) whose time dependence is of the form

$$\psi(\vec{r}, t) = \phi(\vec{r}) \exp(-iEt/K) \tag{18}$$

where the constant K must have the physical dimension of action (energy.time). Now since the frequency of the wave is obviously

$$\nu = E/2\pi K \tag{19}$$

Schrödinger “could not resist the temptation” [13] to use Planck’s relation

$$E = h \cdot \nu \tag{20}$$

i.e. to put K equal to $h/2\pi = \hbar$.

By combining the latter relation with (4) one finds for the wavelength

$$\lambda = h/[2m(E - V)]^{1/2} = h/p \tag{21}$$

which is the de Broglie relation originally derived *relativistically* [15], Schrödinger’s derivation is nonrelativistic.

By introducing equations (4), (18) and (20) into d’Alembert equation (3) Schrödinger found the equation for amplitude $\phi(\vec{r})$

$$\nabla^2\phi(\vec{r}) + (\hbar^2/2m)(E - V) \cdot \phi(\vec{r}) = 0 \tag{22}$$

In order to get the equation for a function $\psi(r, t)$ whose time dependence is not restricted to the dependence of the form $\exp(-iEt/\hbar)$ Schrödinger later in the fourth paper of Ref. [12] used the property of the stationary solution

$$E\psi(\vec{r}, t) = i\hbar\partial\psi(\vec{r}, t)/\partial t \tag{23}$$

to eliminate the term $E\phi(\vec{r})$ from (22) and arrived at

$$-(\hbar^2/2m)\nabla^2\psi + V\psi = i\hbar\partial\psi/\partial t \tag{24}$$

“Causal interpretation”

As stated at the beginning Bohm took the latter equation as a postulate and applied the following reasoning. Since every complex function can be written as,

$$\psi(\vec{r}, t) = R(\vec{r}, t) \exp(iS(\vec{r}, t)/\hbar) \tag{25}$$

the equation (24) is equivalent to two partial differential equations for the real functions $R(\vec{r}, t)$ and $S(\vec{r}, t)$

$$\partial R(\vec{r}, t)/\partial t = -(1/2m)[R\nabla^2 S(\vec{r}, t) + 2\nabla R\nabla S(\vec{r}, t)] \tag{26}$$

$$\partial S(\vec{r}, t)/\partial t = -[(\nabla S(\vec{r}, t))^2/2m + V(\vec{r}) - (\hbar^2/2m)\nabla^2 R(\vec{r}, t)/R(\vec{r}, t)] \tag{27}$$

These equations were used earlier by de Broglie [1,16]. Madelung gave them a hydrodynamical interpretation [17].*They were also used by Fenyes in a Stochastic interpretation [18]. The corresponding set of equations for the functions $R(\vec{r})$ and $s(\vec{r})$ associated with stationary solution

$$\psi(\vec{r},t) = R(\vec{r})\exp(-iEt/\hbar)\exp(is(\vec{r})/\hbar) \tag{28}$$

read

$$R(\vec{r})\nabla^2 s(\vec{r}) + 2\nabla R(\vec{r})\nabla s(\vec{r}) = 0 \tag{29a}$$

$$E = (\nabla s(\vec{r}))^2/2m + V(\vec{r}) - (\hbar^2/2m)\nabla^2 R(\vec{r})/R(\vec{r}) \tag{29b}$$

Equation (29b) and the corresponding equation (27) look like the Hamilton-Jacobi equation for the functions $s(\vec{r})$ and $S(\vec{r},t)$, respectively, with an additional term $-(\hbar^2/2m)\nabla^2 R(\vec{r})/R(\vec{r})$. De Broglie gave [1] to this term the name "quantum potential"

$$U_Q = -(\hbar^2/2m)\nabla^2 R(\vec{r})/R(\vec{r}) \tag{30}$$

With the action $S(\vec{r},t)$ which satisfies (26) and (27) Bohm associates now a "momentum" by

$$\vec{p} = \vec{p}(\vec{r},t) = \overrightarrow{\mathrm{grad}}\, S(\vec{r},t) \tag{31}$$

and in this way obtains the *causal* interpretation of quantum mechanics in which particles move along one of the trajectories determined by integrating the equation

$$m\frac{d\vec{r}}{dt} = \vec{p}(\vec{r},t) \tag{32}$$

To each initial coordinate corresponds one trajectory.

We see that the equation (24) is the last in Schrödinger and the first one in Bohm's approach. The relation (31) is the last one in Bohm's approach whereas the corresponding relation (10) is one of the first in Schrödinger one. Relations (11) and (28) are the corresponding ones in the two approaches. Bohm's approach does not contain the relation (4) whereas in Schrödinger one it is the relation which couples elements of mechanical and wave theory. Hamilton-Jacobi equation for the function W is at the beginning of Schrödinger procedure whereas "quantum

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* NDLR. On trouvera ci-après une traduction de cet article, souvent cité, mais peu accessible.

Hamilton-Jacobi equation” for the function S is near the end of Bohm procedure.

The fact that S does not satisfy the classical $H - J$ equation but quantum one indicates that Schrödinger equation is not consistent with all relations used in its “derivation”. That might be the hidden reason for the existence of different interpretations of quantum mechanics whose common feature is that they all contain the Schrödinger equation itself.

The statistical interpretation [19] neglects the whole classical background physical picture which led Schrödinger to his discovery. This interpretation denies the existence of waves as well as the existence of trajectories. Only the equation (24) remains in which absolute value square of $\psi(\vec{r}, t)$ is the density of probability of $\vec{r}$ whereas an overall phase of ψ has no physical interpretation.

The old quantum theory [20] is closer to Schrödinger’s wave picture than the statistical interpretation. It assumes that classical trajectories exist. The Sommerfeld conditions of quantizations reduce the set of all trajectories to a subset in which these conditions are satisfied. Since the Schrödinger wave equation also reduces the set of all allowed energies of the system to a subset, one may say that in this theory the classical Hamiltonian (1), the relation between momentum and classical action (10), the expression for phase velocity (4) and Schrödinger equation are the basic equations. Waves do not destroy classical trajectories, they only reduce the whole set of trajectories to a subset. In such an approach one does not require that the phase of $\psi(\vec{r}, t)$ is equal to the action. That means that one neglects only the relation (11).

In the interpretation of Bohm, finally, the trajectories exist, but the set of trajectories is not a subset of the classical set because the action is not a classical action i.e. neither the classical Hamilton-Jacobi equation is satisfied nor the energy relation (1). In fact, we shall show that these new “trajectories” are entirely different from the classical trajectories.

Bohm takes the phase of the wave function as the analogue of the classical action (we will call it “quantum action”) and keeps the relation (10). But now there is the new relation (31) between “momentum of the particle” and “quantum action”. In Bohm’s theory particles are real, but since the relation (1) is not valid the expression (4) for phase velocity is not valid either.

III. Remarks on Bohm’s interpretation

Already in 1952 Halpern [21] criticized the contention of Bohm that his interpretation is a mechanical interpretation of the wave equation :

"To obtain the solution of a mechanical problem it is necessary to have an action S which depends on the coordinates of the system, the physical parameters entering into the expression for kinetic and potential energy and f nonadditive integration constants. Bohm has failed to show that such a function can be found... Until a way to find these integration constants has been devised, the similarity with the Hamilton-Jacobi theory is purely extraneous, and one cannot talk about a mechanical interpretation of the wave equation".

Bohm replied [22] that Hamilton-Jacobi theory was being used only for the purpose of indicating in a simple way how one might arrive at a causal interpretation of the quantum theory, while the theory itself was to be based directly on the equations of motion

$$md^2\vec{r}/dt^2 = -\nabla\{U_Q(\vec{r}) + V(\vec{r})\} \tag{33}$$

where $V(\vec{r})$ is the classical potential, and $U_Q(\vec{r})$ is the "quantum potential". Guided by Hamilton-Jacobi theory, one guesses tentatively, writes Bohm, that if the momentum were equal to $\vec{p} = \nabla S(\vec{r})$, this function would satisfy the equations of motion. Now, Bohm takes $\vec{p}$ to be the function of $\vec{r}$ and t (whereas in Newton's formulation $\vec{p}$ is equal to $\dot{\vec{r}}$ and depends on t only). Hence

$$d\vec{p}/dt = \partial\vec{p}/\partial t + (\vec{v}\cdot\nabla)\vec{p} = \partial(\nabla S)/\partial t + \nabla(\nabla S)^2/2m \tag{34}$$

But since S satisfies (27), Bohm finds

$$d\vec{p}/dt = -\nabla\{V + U_Q\} \tag{35}$$

and concludes that the equation (33) follows. Bohm further concludes that this verifies that a particle traveling with velocity $\vec{v} = \nabla S/m$ will satisfy the equations of motion (26) and (27). Therefore according Bohm the condition $\vec{p} = \nabla S$ is a consistent subsidiary condition.

Eqs. (34) and (35) should however be written as a covariant derivative

$$\frac{D\vec{p}}{Dt} = -\nabla(V + U_Q) \quad , \quad \frac{D}{Dt} = \partial_t + \vec{v}\cdot\vec{\nabla} \tag{36}$$

At first sight Bohm's answer seems acceptable. But in his original work [2] as well as in his answer [22] Bohm does not always keep track of the other equation (26) and in this way gives the impression that $U_Q(\vec{r})$ may be treated on the same footing as the external potential. This impression is misleading because S and R are coupled also through the equation (26) and therefore $U_Q(\vec{r})$ in eq.(27) is not a function given in advance and independent of S or $\vec{p}$.

The most direct way to see the importance of the equation (26) is to consider the one dimensional stationary case. Equations (29) become

$$Rd^2s(x)/dx^2 + 2(dR/dx)\cdot(ds/dx) = 0 \tag{37}$$

$$(ds/dx)^2/2m + V(x) - (\hbar^2/2m)(d^2R/dx^2)/R = E \tag{38}$$

The first relation results

$$R(x) = C/(ds/dx)^{1/2} \tag{39}$$

$$U_Q(x) = -(\hbar^2/4m)[(d^3s/dx^3)/(ds/dx) - 3(d^2s/dx^2)/2(ds/dx)^2] \tag{40}$$

The substitution of the latter expression into "quantum Hamilton-Jacobi equation" gives

$$E = s'^2/2m + V(x) + (\hbar^2/4m)[s'''/s' - (3/2)(s''/s')^2] \tag{41}$$

It is very hard to see any ressemblance of equation (41) to the Hamilton-Jacobi equation and consequently it is hard to justify the relation $p = dS/dx$ on this basis.

Furthemore, by taking covariant derivative (36) instead of dp/dt one has replaced the ordinary differential equation for $p(t)$, of Newton, by a partial differential equation for $p(x,t)$. Thus the assertion in Bohm's reply [22] that Newton's form of mechanics is the methodological basis of his interpretation is not valid.

The main consequence of the above observations is that, while classical trajectories depend on initial velocity $\vec{v}_0$ and initial position $\vec{r}_0$, the trajectory calculated in the causal interpretation from $m\dot{\vec{r}} = \vec{p} = \overrightarrow{\mathrm{grad}}\, S(\vec{r},t)$ depends only on the initial position $\vec{r}_0$ [23,24,25,26].

The best way to see the main features of this "causal interpretation" is to determine explicitly the "associated trajectories" for solvable quantum systems. This is the subject of the next section. We shall also show that the two ways of introducing "causal" orbits by eqs.(32) and (33) respectively are in fact not equivalent at least for stationary states.

IV. Causal trajectories in the stationary states of the central potential (hydrogen atom, spherical oscillator,...)

It is well known that stationary solutions of the Schrödinger equation in the case of a central potential $V(r)$

$$[-(\hbar^2/2m)\nabla^2 + V(r)]\psi = E\psi \tag{42}$$

have the form

$$\psi(r,\theta,\phi) = R(r)\Theta(\theta)\Omega(\phi) \tag{43}$$

where functions

$$R(r) = AF_n^l(r)$$
$$\Theta(\theta) = P_l^M(\cos\theta)$$
$$\Omega_M(\phi) = \exp(iM\phi) \tag{45}$$

are the solutions of three separate differentiel equations

$$(1/R)[d(r^2dR/dr)/dr + (2m/\hbar^2)r^2(E-V)]R = -C \tag{45a}$$

$$(1/\Theta)[(1/\sin\theta)d(\sin\theta d\Theta/d\theta)/d\theta - C\sin^2\theta] = M^2 \tag{45b}$$

$$(1/\Omega)d^2\Omega/d\phi^2 = -M^2 \tag{45c}$$

By Bohm's definition the quantum action is proportional to the phase of the wave function. Since the functions $R(r)$ and $\Theta(\theta)$ are real we conclude that

$$s(\vec{r}) = \hbar M\phi \tag{46}$$

We see in agreement with Halpern remark [21] that $s(\vec{r})$ does not depend on three independent non-additive constants.

Using (43) and (45) one easily determines the quantum potential

$$U_Q(\vec{r}) = -(\hbar^2/2m)\nabla^2(R(r)\Theta(\theta))/R(r)\Theta(\theta) = E - V - \hbar^2M^2/2mr^2\sin^2\theta \tag{47}$$

Taking into account that

$$\overrightarrow{\mathrm{grad}}\, s = \overrightarrow{\mathrm{grad}}\, \hbar M\phi = (\hbar M/r\sin\theta)\vec{e}_\phi \tag{48}$$

we see that s satisfies "quantum Hamilton-Jacobi equation"

$$E = (\overrightarrow{\mathrm{grad}}\, s)^2/2m - (e^2/r) - (\hbar^2/2m)\nabla^2\big(R(r)\Theta(\theta)\big)/R(r)\Theta(\theta) \tag{49}$$

In spherical coordinates $m\dot{\vec{r}}$ reads

$$m\dot{\vec{r}} = mr\dot{\phi}\sin\theta \cdot \vec{e}_\phi \tag{50}$$

Hence equating (48) to (50) on the basis of the definition (31) of "momentum" we obtain

$$\dot{\phi} = \hbar M/mr^2\sin^2\theta \quad , \quad r = r_0 \quad , \quad \theta = \theta_0 \tag{51}$$

and finally

$$\phi = (\hbar M/mr_0^2\sin^2\theta)\cdot t + \phi_0 \quad , \quad \theta = \theta_0 \quad , \quad r = r_0 \tag{52}$$

Therefore, according to the causal interpretation in the stationary states the electron moves along the circles lying in the planes parallel to the $x - y$ plane for any central potential.[1] For the same magnetic quantum number M these trajectories do not distinguish between different potentials. They have to be distinguished by their probability distribution.

For kinetic energy we find

$$T = \vec{p}^2/2m = p_\phi^2/2m = m^2r^2\dot{\phi}^2\cdot\sin^2\theta/2m \tag{53}$$

By substituting this equality into (47) we obtain

$$E = V(r) + T + U_Q(\vec{r}) \tag{54}$$

i.e. quantum energy is a sum of kinetic energy, Coulomb potential energy and quantum potential energy.

The trajectories derived from $\vec{p} = \nabla s$ for the hydrogen atom in particular are quite different from Bohr's (semiclassical) trajectories. Bohr's trajectories form a subset in the set of Kepler's orbits (this particular set satisfies the conditions of quantization). The nucleus lies in the plane of the orbit. This is not the case with these "causal" orbits whose planes in general do not contain the nucleus. This is due to the difference between quantum action $s(\vec{r})$ given in (46) and the classical action given by

$$\begin{aligned} w(\vec{r}) = w_1(r) + w_2(\theta) + w_3(\phi) = &\int [2m(E - V) - \alpha_2^2/r^2]^{1/2}dr \\ &+ \int (\alpha_2^2 - \alpha_3^2/\sin^2\theta)^{1/2}d\theta + \alpha_3\phi \end{aligned} \tag{55}$$

[1] The circles for hydrogen atom have also been mentioned by Belinfante [26] and L. de Broglie [31].

Bohr trajectories satisfy the relation $\vec{p} = \overrightarrow{\mathrm{grad}}\, w$ and the conditions of quantization, whereas Bohm trajectories (Fig. 1) satisfy $\vec{p} = \overrightarrow{\mathrm{grad}}\, s = \overrightarrow{\mathrm{grad}}\, \hbar M\phi = \overrightarrow{\mathrm{grad}}\, w_3$, with $\alpha_3 = \hbar M$.

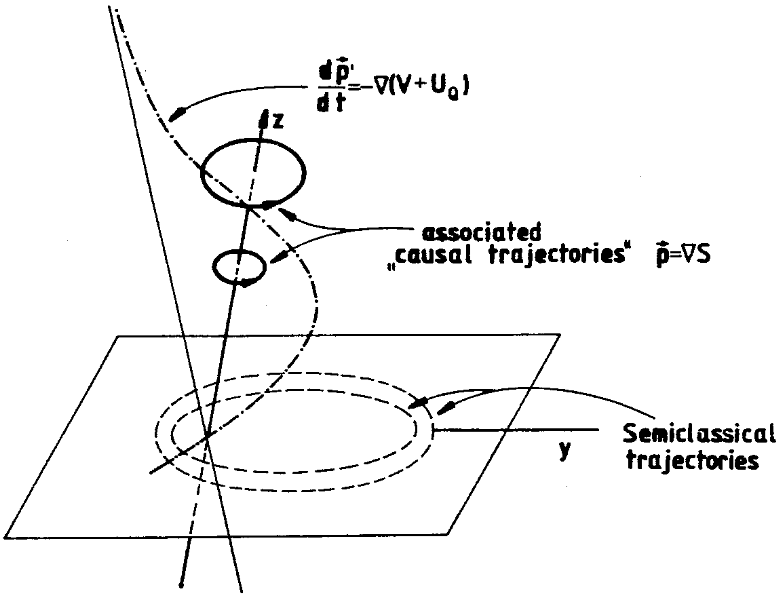

Figure 1. The draft of semiclassical (- - -) and of two types of "causal" trajectories ; $\vec{p} = \nabla S$ (—-), $d\vec{p'}/dt = -\nabla(V + U_Q)$ $(- \cdot - \cdot - \cdot -)$ for the central potential.

In order to see further the difference between the quantum and classical actions we write the Hamilton-Jacobi equation for the function $w(\vec{r})$ taking into account its structure given in (55)

$$(\nabla w_1)^2/2m + (\nabla w_2)^2/2m + (\nabla w_3)^2/2m + V = E \tag{56}$$

From Eq.(55) we find

$$(\nabla w_1)^2 = (\partial\omega_1/\partial r)^2 = -(\alpha_2^2/r^2) - 2m(V - E) \tag{57a}$$

$$(\nabla w_2)^2 = (\partial w_2/\partial\theta)^2/r^2 = \alpha_2^2/r^2 - \alpha_3^2/r^2 \sin^2\theta \tag{57b}$$

$$(\nabla w_3)^2 = (\partial w_3/\partial\phi)^2/r^2 \sin^2\theta = \alpha_3^2/r^2 \sin^2\theta \tag{57c}$$

$$(\nabla w_1)^2 + (\nabla w_2)^2 = 2m(E - V) - \alpha_3^2/r^2 \sin^2\theta \tag{58}$$

By comparing (55) with (46) and (58) with (47) we conclude that in the central potential the "quantum action" is just a part of classical action, whereas quantum potential is associated with the rest of the classical action as follows

$$s(\vec{r}) = w_3 \quad , \quad \alpha_3 = \hbar M \tag{59}$$

$$U_Q(\vec{r}) = [(\nabla w_1)^2 + (\nabla w_2)^2]/2m \tag{60}$$

The fact that the "associated" trajectories are all the same for any central potential is evidently the consequence of the property that these orbits depend only on the initial coordinates and not on the initial velocities.

In the *ground state* of the central potential, in particular, we have $l = 0$, $M = 0$ and consequently $s = 0$, $p = 0$ which means that the "particle" is immobile. Similarly, in the case of standing waves

$$u_n^+(x) = a^{-1/2} \cos \pi n x/2a \quad , \quad n = 1, 3...$$

$$u_n^-(x) = a^{-1/2} \sin \pi n x/2a \quad , \quad n = 2, 4...$$

$$E_n = \hbar^2 \pi^2 n^2/8ma^2$$

between two impenetrable potential walls at $x = -a$ and $x = a$, and for the bound states in the rectangular potential hole

$$V(x) = \begin{cases} -U & |x| < a \\ 0 & \text{elsewhere} \end{cases}$$

the "associated particle" is at rest and cannot distinguish between different physical situations.

Now we come to the second definition of the associated causal momenta by $d\vec{p'}/dt = -\nabla(V + U_Q)$. In the central potential for the states given in (43) we obtain

$$\frac{d\vec{p'}}{dt} = \nabla(\frac{\hbar^2 M^2}{2m} \cdot \frac{1}{r^2 \sin^2\theta} - E_n) \tag{61}$$

where M and E_n are the quantum numbers of the particular stationary state under consideration. This is an interesting dynamical system in its own right. The potential is singular along the z-axis with an inverse cube force law (a two-dimensional version of the Dirac's charge-monopole system). The equations of motion are

$$m\ddot{\vec{r}} = \gamma\nabla(1/\rho^2) \quad , \quad \rho^2 = x^2 + y^2 \quad , \quad \gamma = \hbar^2 M^2/2m$$

We have two integrals of motion for this dynamical system, the total energy and the z-component of the angular momentum. In cylindrical coordinates

$$\epsilon = \frac{m}{2}(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2) + (E_n - \frac{\gamma}{\rho^2})$$

$$L_z = m\rho^2\dot{\phi} \tag{62}$$

Furthermore, $\ddot{z} = 0$, or $\dot{z} = v_z(0)$, $z = v_z(0)t + z(0)$. Consequently, we have one first order equation for ρ

$$\dot{\rho}^2 + (\frac{L_z^2}{m^2} - \frac{2\gamma}{m})\frac{1}{\rho^2} = K \quad , \quad K = 2\frac{\epsilon}{m} - v_z^2(0) - \frac{2E_n}{m}$$

This equation can be integrated and gives

$$\rho = [K(t+C)^2 + (L_z^2 - \hbar^2 M^2)/m^2]^{1/2} \tag{63a}$$

Whence

$$\phi = \frac{L_z}{K}(\frac{K}{L_z^2 - \hbar^2 M^2})^{1/2} \operatorname{arctg} \frac{t+C}{(\frac{Km^2}{L_z^2 - \hbar^2 M^2})^{1/2}} + C' \tag{63b}$$

The projection of the orbits in the z =const. plane are

$$\rho = \frac{\lambda}{\sqrt{K}}[\mathrm{tg}^2(\frac{\lambda m}{L_z}(\phi + \phi_0)) + 1]$$

and schematically shown in Fig. 1 (dotted lines). For $L_z = \hbar M$ (angular momentum of the associated particle = quantum angular momentum) the equations simplify considerably: $\rho = (m/L_z)[1/(\phi_0 - \phi)]$. There are also special circular orbits ρ =const., $\dot{\phi}$ =const. .

V. Passage through a step-like barrier in the causal interpretation

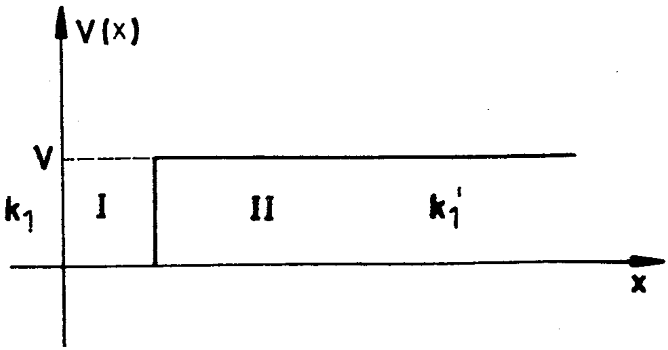

Figure 2. Step-like magnetic barrier.

In the case of the potential shown on Fig. 2. the wave functions in regions I and II have the form

$$\Psi_1^I(x) = A_{01}e^{ik_1x} + \tilde{A}_1 e^{-ik_1x} \equiv R_1^I e^{is_1^I/\hbar}$$

$$\Psi_1^{II}(x) = A_1' e^{ik_1'x} = R_1^{II} e^{is_1^{II}/\hbar} \qquad (64)$$

where

$$k_1'^{\,2} = k_1^2 - (2m/\hbar^2)V \qquad (65)$$

$$\tilde{A}_1 = r_1 A_{01} \quad , \quad A_1' = t_1 A_{01}$$

$$r_1 = (k_1 - k_1')/(k_1 + k_1') \quad , \quad t_1 = 2k_1/(k_1 + k_1') \qquad (66)$$

When k_1' is real ($\hbar^2 k^2/2m > V$) the quantum action and its amplitude in regions I and II are

$$R_1^I = [2(A_{01}^2 + \tilde{A}_1^2)]^{1/2}$$

$$R_1^{II} = t_1 A_{01}$$

$$S_1^I = \hbar\,\mathrm{arctg}((1 - r_1)\,\mathrm{tg}\,k_1 x/(1 + r_1)) \quad , \quad x < 0$$

$$S_1^{II} = \hbar k_1' \cdot x \quad , \quad x \geq 0 \qquad (67)$$

The expressions for "momentum" follow directly

$$p_1^I = dx/dt = \hbar k_1(1 - r_1^2)/(1 + r_1^2 + 2r_1 \cos 2k_1 x) \quad , \quad x < 0$$

$$p_1^{II} = dx/dt = \hbar k_1' \quad , \quad x \geq 0 \tag{68}$$

It is easy to see that p_1^I and p_1^{II} are always positive. That means that for $k_1 > 0$ the particle moves only in the positive x-direction.

The integration of equations (68) gives the relation between x and t

$$(1 + r_1^2)x + (r_1/k_1)\sin 2k_1 x + C^I = (\hbar k_1/m)(1 - r_1^2)t \quad , \quad x < 0$$

$$x = (\hbar k_1'/m)t + C^{II} \quad , \quad x \geq 0 \tag{69}$$

We determine the integration constants from the conditions $t = 0$, $x = x_0$; $t = t_0$, $x = 0$.

$$t_0 = -m[(r_1/k_1)\sin 2k_1 x_0 + (1 + r_1^2)x_0]/\hbar k_1(1 - r_1^2)$$

$$\begin{aligned}(1 + r_1^2)x + (r_1/k_1)\sin 2k_1 x - (r_1/k_1)\sin 2k_1 x_0 \\ - (1 + r_1^2)x_0 = (\hbar k_1/m)(1 - r_1^2)t \quad , \quad t < t_0\end{aligned}$$

$$x = (\hbar k_1'/m)t + k_1'[(r_1/k_1)\sin 2k_1 x_0 + (1 + r_1^2)x_0]/k_1(1 - r_1^2) \quad , \quad t > t_0 \tag{70}$$

If k_1 is imaginary ($\hbar^2 k_1^2/2m < V$)

$$k_1' = i\sqrt{(2m/\hbar^2)V - k_1^2} = i\rho_1' \tag{71}$$

the wave functions in regions I and II are

$$\psi_1^I(x) = e^{ik_1x}A_{01} + A_{01}e^{-ik_1x}e^{-2i\,\mathrm{arctg}(\rho'{}_1/k_1)}$$

$$\psi_1^{II}(x) = A_{01}e^{-\rho_1'x}(2k_1/(k_1^2 + {\rho_1'}^2))^{1/2}e^{-i\,\mathrm{arctg}(\rho_1'/k_1)} \tag{72}$$

whereas quantum action and associated momentum take the form

$$S_1^I = -2\hbar\,\mathrm{arctg}(\rho_1'/k_1) \quad , \quad S_1^{II} = -\hbar\,\mathrm{arctg}(\rho_1'/k_1)$$

$$p_1^I = 0 \quad , \quad p_1^{II} = 0 \tag{73}$$

Therefore, a curious result is obtained : If the initial energy is less than the potential energy in region II the particle remains at rest at the initial position in region I.

For the second definition of causal trajectories, eq.(35), we find that, since quantum potential in regions I and II

$$U_{QI}(x) = \hbar^2 k_1^2/2m \quad , \quad U_{QII}(x) = -(\hbar^2 {\rho_1'}^2/2m)$$

is independent of x, the equation $d\vec{p}/dt = -\nabla(V + W_Q)$ gives arbitrary constant values for momentum in the two regions

$$p_I = C_1 \quad , \quad p_{II} = C_2 \tag{76}$$

We note that in this physical situation the causal interpretation does not agree with the statistical interpretation since according to the latter one there is a finite probability for the particle reflection at the boundary of the two regions. On the other hand Dewdney found that among the numerically computed trajectories in the time dependent states there exist both reflected and transmitted trajectories [8]. Dewdney's result may be understood by analysing the trajectory associated with the simplest wave packet which is a superposition of two energy eigenstates :

$$\psi_I(x,t) = C_1 R_1^I e^{i(s_1^I - E_1 t)/\hbar} + C_2 R_2^I e^{i(s_2^I - E_2 t)/\hbar} = R_I e^{i s_I/\hbar}$$

$$\psi_{II}(x,t) = C_1 R_1^{II} e^{ik_1' x} e^{-iE_1 t/\hbar} + C_2 R_2^{II} e^{ik_2' x} e^{-iE_2 t/\hbar} = R_{II} e^{i s_{II}/\hbar} \tag{77}$$

Here R_I, s_1^I and s_1^{II} are given in (67) whereas R_{II}, s_2^I and s_2^{II} are obtained from (67) replacing k_1 by k_2 :

One directly obtains s_I, s_{II}, p_I and p_{II}

$$s_I(x,t) = \hbar \cdot \mathrm{arctg}\, \frac{C_1 R_1^I \sin(s_1^I - E_1 t)/\hbar + C_2 R_2^I \sin(s_2^I - E_2 t)/\hbar}{C_1 R_1^I \cos(s_1^I - E_1 t)/\hbar + C_2 R_2^I \cos(s_2^I - E_2 t)/\hbar}$$

$$s_{II}(x,t) = \hbar \cdot \mathrm{arctg}\, \frac{C_1 R_1^{II} \sin(p_1' x - E_1 t)/\hbar + C_2 R_2^{II} \sin(p_2' x - E_2 t)/\hbar}{C_1 R_1^{II} \cos(p_1' x - E_1 t)/\hbar + C_2 R_2^{II} \cos(p_2' x - E_2 t)/\hbar} \tag{78}$$

$$p_I = \frac{(C_1 R_1^I)^2 p_1 + (C_2 R_2^I)^2 p_2 + C_1 R_1^I C_2 R_2^I \cos\{[s_1 - s_2 - (E_1 - E_2)t]/\hbar\}[p_1 + p_2]}{(C_1 R_1^I)^2 + (C_2 R_2^I) + 2 C_1 R_1^I C_2 R_2^I \cos[s_1 - s_2 - (E_1 - E_2)t]/\hbar}$$

$$p_{II} = \frac{(C_1 R_1^{II})^2 p_1' + (C_2 R_2^{II})^2 p_2' + C_1 R_1^{II} C_2 R_2^{II} \cos[(p_1' - p_2')x - (E_1 - E_2)t]/\hbar[(p_2' + p_1')]}{(C_1 R_1^{II})^2 + (C_2 R_2^{II}) + 2 C_1 R_1^{II}) + 2 C_1 R_1^{II} C_2 R_2^{II} \cos[(p_1' - p_2')x - (E_1 - E_2)t]/\hbar}$$

The “momenta” now oscillate in time which may perhaps be interpreted as due to reflected waves. But the explicit interpretation of trajectories remains.

VI. The causal motion of a spinning neutral particle inside the magnetic field

The spinning neutral particle in the magnetic field is described by the two-component spinor $\hat{\psi} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ which satisfies the Pauli equation

$$(-\frac{\hbar^2}{2m}\nabla^2 + \mu\hat{\sigma} \cdot \vec{B})\hat{\psi} = i\hbar\frac{\partial\hat{\psi}}{\partial t} \tag{80}$$

In the causal interpretation the spinor $\hat{\psi}$ is written [7,27] in the form

$$\hat{\psi}(\vec{r},t) = Re^{iS/\hbar}\begin{pmatrix}\phi^1\\ \phi^2\end{pmatrix} = Re^{iS/\hbar}\hat{\phi} \quad , \quad \hat{\psi} = Re^{is/\hbar}e^{-iEt/\hbar}\hat{\phi} \tag{81}$$

$$S = s - Et$$

One assumes

$$\phi^a = r^a e^{i\theta^a/\hbar} \quad , \quad a = 1,2 \quad , \quad \theta^1 = -\theta^2$$

$$\hat{\phi}^+\hat{\phi} = 1$$

$$|r^1|^2 + |r^2|^2 = 1 \tag{82}$$

Lochak [25] and Dewdney et al. [7] showed that the Pauli equation is equivalent to four partial differential equations for variables S, R and $\vec{s}$

$$\partial S/\partial t - i\hbar\phi^+\partial\phi/\partial t + (m\vec{v}^2/2) + U_Q + U_s + (2/\hbar)\mu\vec{B}\cdot\vec{s} = 0 \tag{83a}$$

$$\partial\rho/\partial t + \nabla(\rho\vec{v}) = 0 \tag{83b}$$

$$D\vec{s}/Dt = \vec{T} + (2/\hbar)\mu\vec{B}\times\vec{s} \tag{83c}$$

Here, $\rho = R^2$ is the probability density

$$\vec{v} = (1/m)(\nabla S - i\hbar\phi^+\nabla\phi) = (\hbar/im)\psi^+\nabla\psi/\psi^+\psi \tag{84}$$

is a velocity field, $\vec{s}$ a "spin density vector"

$$\vec{s} = (\hbar/2)\hat{\phi}^+\vec{\sigma}\hat{\phi} = (\hbar/2)\psi^+\hat{\sigma}\psi/\psi^+\psi \tag{85}$$

U_s is a spin-dependent addition

$$U_s = (1/2m)\partial_i\theta^j\partial_i\theta^j \tag{86}$$

to the quantum potential

$$U_Q = -(\hbar^2/2m)\nabla^2 R/R$$

where $\vec{T}$ is a quantum torque

$$\vec{T} = (1/m\rho)\vec{s}\times\partial_i(\rho\partial_i\vec{s}) \tag{87}$$

Because of the flow derivative $D\vec{s}/Dt$ (as we have noted already in (36) in connection with $D\vec{p}/Dt$) a quantum torque arises in the spin equation (83c) so that this equation has nothing to do with either the classical spin precession, nor with Heisenberg quantum spin equation. Finally we find that the equation corresponding to the equation (36) in the spin case is

$$\frac{D\vec{p}}{dt} = -\nabla(U_Q + U_s + \mu\frac{2}{\hbar}\vec{B}\vec{s}) + i\hbar[(\nabla\hat{\phi}^+)\frac{\partial\hat{\phi}}{\partial t} - \frac{\partial\hat{\phi}^+}{\partial t}(\nabla\hat{\phi})] \tag{84a}$$

We shall now determine S, s^a and $\vec{v}$ for stationary states in a magnetic field along z-axis.

The general eigenstate of (80) which propagates in the positive x-direction is

$$\hat{\psi}(\vec{r}) = \alpha' e^{i\vec{k'}\vec{r}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta'' e^{i\vec{k''}\vec{r}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{88}$$

where

$$\hbar^2 k^2/2m = \hbar^2 {k'}^2/2m - \mu B \quad , \quad \hbar^2 k^2/2m = \hbar^2 {k''}^2/2m + \mu B \tag{89}$$

Barut et al. showed [28] that by writting the state (88) in the form

$$\hat{\psi}(\vec{r}) = e^{i(\vec{k'}+\vec{k''})\cdot\vec{r}/2} \begin{pmatrix} \alpha' e^{-i(\vec{k''}-\vec{k'})\cdot\vec{r}/2} \\ \beta'' e^{i(\vec{k''}-\vec{k'})\cdot\vec{r}/2} \end{pmatrix} \tag{90}$$

one makes explicit the representations of the translation and rotation groups to which this state for given $\vec{k'}$ and $\vec{k''}$ belongs. The form (90) is consistent with the following transformation relation

$$\hat{\psi}(\vec{r}_2) = D_{\vec{k}_t}(\vec{r}_2 - \vec{r}_1)R(\phi)\hat{\psi}(\vec{r}_1) \quad , \quad \vec{k}_t = (\vec{k'} + \vec{k''})/2 \tag{91}$$

By comparing the form (91) and the requirement (82) of the causal interpretation, we see that symmetry requirements and the requirements of the causal interpretation lead to the same result in the identification of the two parts in the wave function, one of which ($Re^{is/\hbar}$) describes the external degrees of freedom and the other ($\hat{\phi}$) describes the internal spin degrees of freedom.

We see that the action $s(\vec{r})$ is given by

$$s(\vec{r}) = (\hbar/2)(\vec{k'} + \vec{k''}) \cdot \vec{r} \tag{92}$$

By substituting the latter expression into (84) we obtain for momentum

$$\vec{p} = m\vec{v} = \hbar[\vec{k'}|\alpha'|^2 + \vec{k''}|\beta''|^2] \tag{93}$$

which is in the direction of the current density [29]

$$\vec{j} = Re(\frac{\hbar}{i}\hat{\psi}^+\nabla\hat{\psi}) \tag{94}$$

The trajectories associated with "Newton equation"

$$\frac{d\vec{p'}}{dt} = -\nabla(U_Q + U_s + \frac{2}{\hbar}\mu B_z s_z) + i\hbar[(\nabla\hat{\phi}^+)\frac{\partial\hat{\phi}}{\partial t} - \frac{\partial\hat{\phi}^+}{\partial t}(\nabla\hat{\phi})]$$

are straight lines characterized by

$$\vec{p'} = \vec{p_0}$$

because in the stationary state (90) the second parenthesis vanishes whereas U_Q, U_s and z-component of the spin vector $\vec{s}$ are independent of $\vec{r}$

$$\begin{aligned} U_Q &= 0 \\ U_s &= \frac{\hbar^2(\vec{k''} - \vec{k'})^2}{4m} \\ \vec{s} &= 2\,Re\,\alpha'^*\beta'' e^{i(\vec{k''}-\vec{k'})\cdot\vec{r}}\cdot\vec{i} + 2Im\alpha'\beta'' e^{i(\vec{k''}-\vec{k'})\cdot\vec{r}}\cdot\vec{j} \\ &\quad + (|\alpha'|^2 - |\beta''|^2)\cdot\vec{k} \end{aligned}$$

The general relation of causal interpretation to quantum current is discussed in the next Section.

VI. Currents and flows

We think that the best way to interprete the so called causal "associated motions" is to say that they express certain flow properties of the current density (see also Ref.30). We have the following general relations in spinless case

$$Im\,(\frac{\hbar}{i}\psi^+\nabla\psi)/\psi^+\psi = -\hbar\nabla R/R \tag{95a}$$

$$Re\,(-\frac{\hbar}{i}\psi^+\partial_t\psi)/\psi^+\psi = (1/2m)(\nabla S)^2 + (V + U_Q) \tag{96}$$

$$Re(\frac{\hbar}{i}\psi^+(\partial_t + \vec{v}\cdot\nabla)\psi)/\psi^+\psi = (1/2m)(\nabla S)^2 - (V + U_Q) \qquad (96b)$$

(with $\vec{v} = \nabla S/m$).

Thus $\vec{p} = \nabla S$ indicates the flow lines of the wave mechanical current density, whereas the real part of the expectation value of the time displacement operator $i\hbar\partial_t$ and flow displacement operator $(\partial_t + \vec{v}\cdot\nabla)$ (i.e. covariant derivative) give the "Hamiltonian" and "Lagrangian" of a "particle" whose momentum is $\vec{p} = \nabla S$, moving in the total potential $V + U_Q$.

In the case with spin we obtain

$$\frac{\hbar}{i}\psi^+\nabla\psi/\psi^+\psi = (\nabla S - i\hbar\hat{\phi}^+\nabla\hat{\phi}) + \frac{\hbar}{i}\nabla R/R \qquad (97)$$

using the parametrization (81). Or, if we treat each component separately by writing

$$\hat{\psi} = \begin{pmatrix} R_1 e^{iS_1/\hbar} \\ R_2 e^{iS_2/\hbar} \end{pmatrix}$$

we have

$$\begin{aligned}\frac{\hbar}{i}\hat{\psi}^+\nabla\hat{\psi}/\hat{\psi}^+\hat{\psi} =& \frac{R_1^2}{R_1^2+R_2^2}\nabla S_1 + \frac{R_2^2}{R_1^2+R_2^2}\nabla S_2 \\ & - i\hbar(\frac{R_1}{R_1^2+R_2^2}\nabla R_1 + \frac{R_2}{R_1^2+R_2^2}\nabla R_2) \qquad (98)\end{aligned}$$

These equations show that one associates the "causal momentum" to the translational motion of the spin 1/2 particle and not to the individual spin components.

VII. Coherent states

These are time dependent states which have some properties very close to the classical trajectories. Hence it would be interesting to find the two associated orbits for such states.

We consider Schrödinger's original coherent states [32] for the harmonic oscillator characterized by a complex number A ("amplitude")

$$\psi_A(x,0) = < x \mid A > = < x \mid \sum_{n=0}^{\infty}(\frac{A}{2})^n\frac{1}{n!} \mid n > \qquad (99)$$

where $| n >$ are the energy eigenstates of the Hamiltonian

$$\hat{H} = \frac{\hbar\omega_0}{2}(-\frac{\partial^2}{\partial x^2} + x^2) \tag{100}$$

Here x is the dimensionless variable related to the coordinate q of the oscillator by

$$x = q\sqrt{m\omega_0/\hbar}$$

Hence we find

$$\psi_A(x,t) = \psi_{A(t)}(x,0)e^{-i\omega_0 t/2} =< x \mid A(t) > e^{-i\omega_0 t/2} \tag{101}$$

where $A(t) = Ae^{-i\omega_0 t} = |A|e^{i(\phi-\omega_0 t)}$. Explicitly we obtain for the normalized coherent states

$$\psi_A(x,t) = (\frac{m\omega_0}{\hbar\pi})^{1/4} e^{-[x-|A|\cos(\phi-\omega_0 t)]^2/2} \cdot$$
$$e^{-i[(\omega_0 t/2)+(|A|^2/4)\sin 2(\phi-\omega_0 t)-|A|x\sin(\phi-\omega_0 t)]}$$

Hence the quantum action (phase) is

$$S = -\hbar[(\omega_0 t/2) + (|A|^2/4)\sin 2(\phi - \omega_0 t)\cdot |A|x\cdot\sin(\phi-\omega_0 t)] \tag{102}$$

The associated motion is then

$$p = \nabla S = \sqrt{\hbar m\omega_0}|A|\sin(\phi-\omega_0 t) \quad , \quad q = \sqrt{\hbar m\omega_0}|A|\cos(\phi-\omega_0 t) \tag{103}$$

The quantum potential is

$$U_Q = (\hbar\omega_0/2)[1 - (x - |A|\cos(\phi-\omega_0 t))^2] \tag{104}$$

Hence the associated motion of the second kind is

$$dp'/dt = -d(V + U_Q)/dq = -\omega_0\sqrt{m\hbar\omega_0}\cos(\phi-\omega_0 t) \tag{105a}$$

which gives

$$q' = |A|(\hbar/m\omega_0)^{1/2}\cos(\phi-\omega_0 t) + C_1 t + C_2 \tag{105b}$$

We compare these with the quantum mechanical averages in the coherent state

$$\begin{aligned} \overline{p} &=< \psi \mid \hat{p} \mid \psi >= \sqrt{\hbar m\omega_0}|A|\sin(\phi-\omega_0 t) \\ \overline{q} &=< \psi \mid \hat{q} \mid \psi >= \sqrt{(\hbar/m\omega_0)}|A|\cos(\phi-\omega_0 t) \end{aligned} \tag{106}$$

and with the classical solutions

$$\begin{aligned} q_{cl}(t) &= (p_0/m\omega_0)\sin\omega_0 t + q_0\cos\omega_0 t \\ p_{cl}(t) &= p_0\cos\omega_0 t - m\omega_0 q_0 \sin\omega_0 t \end{aligned} \tag{107}$$

in which p_0 and q_0 are the initial position and momentum. (Note that classical action is $w = (\omega_0/2)[q\cdot(\alpha^2 - q^2)^{1/2} + \alpha\sin^{-1}(q/\alpha)]$ where $\alpha^2 = 2E/\omega_0^2$.

Thus in the coherent state all four concepts of momenta and position of "particles" come most close to each other. But even then there are differences, namely in the number of constants of integration.

Conclusion

The "causal interpretation" of quantum mechanics by quantum potential has a priori an appealing feature. Distinct from the statistical interpretation, it emphasizes the importance of the phase of the wave function and in this way aims to solve the difficulties in the characterization of quantum mechanical reality. But as we showed, the methodological basis of this interpretation does not lie in classical mechanics. Also, for explicit quantum systems in stationary states the resulting orbits of the "causal" interpretation have nothing to do with semiclassical picture rather they model certain flow patterns of the quantum current. In coherent states which are time-dependent, however, the orbits of the classical motion, the average quantum motion and those coming from $\vec{p} = \nabla S$ and $d\vec{p'}/dt = -\nabla(V + U_Q)$ all coincide except for different integration constants. But in general the "causal" trajectories cannot give a full picture of the behavior of the quantum system. They must be supplemented by complicated probability densities of such trajectories.

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