The nonlinear extended oscillator: a particle concept beyond quantum mechanics

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ABSTRACT. The proposed particle concept is based on non linear field equations derived from a least action principle and interpreted calssically. Solutions correspond to stable, extended, soliton-like field structures, which as a whole are viewed as "particles". The total field energy of the structure is interpreted as rest energy or mass of the particle. Internal oscillations (or rotations) of the structures lead, for the moving particle, to a wave with a wavelength and a frequency comparable to those of the matter wave of quantum mechanics. Two examples are discussed: the sine-Gordon breather in $1 + 1$ dimensions, and Hopf maps in $3 + 1$ dimensions. The free sine-Gordon breather represents an oscillator in uniform, rectilinear motion. When confined in a square well potential, the breather assumes discrete energy levels identical with those found in quantum mechanics for a particle in a box. Mappings of three dimensional space on a sphere, Hopf maps, also derived from a well defined least action principle, lead to field structures which are topologically stable. The total field energy, the size of the structures, and their internal rotation can be calculated in principle, but, contrary to what holds for the breather, no exact results heve been obtained as yet.

RESUME. Le concept de particule proposé est basé sur des équations de champ non linéaires déduites d'un principe de moindre action et interprétées classiquement. Les solutions correspondent à des structures de champ stables et étendues, du type soliton, qu'on peut considérer globalement comme des "particules". L'énergie totale de champ de la structure est interprétée comme une énergie ou une masse au repos de la particule. Les oscillations (ou rotations) internes des structures conduisent, pour la particule en mouvement, à une onde avec une longueur d'onde et une fréquence comparables à celles de l'onde de matière de la mécanique quantique. Deux exemples sont discutés: le "breather" de sinus-Gordon à $1 + 1$ *dimensions, et les applications de Hopf à* $3 + 1$ *dimensions. Le "breather"* *de sinus-Gordon en mouvement libre représente un oscillateur en mouvement rectiligne et uniforme. Confiné dans un puits de potentiel carré, le "breather" adopte des niveaux d'énergie discrets identiques à ceux trouvés en mécanique quantique pour une particule dans une boîte. Les applications de l'espace à trois dimensions sur la sphère, applications de Hopf, déduites aussi d'un principe de moindre action bien défini, conduisent à des structures de champ qui sont topologiquement stables. L'énergie totale du champ, la taille des structures, et leur rotation interne peuvent être calculées en principe, mais, contrairement à ce qui est vrai pour le "breather", des résultats exacts n'ont pas encore été obtenus.*

Introduction

The purpose of the present paper is to propose and discuss a particle concept which is based on classical, continuous fields. The field introduced here is considered to be the basic fundamental entity from which all particle properties and the known physical fields can be derived as limiting cases or asymptotic approximations. The equations governing the field are essentially nonlinear and of such a type that they allow for soliton-like solutions representing stable extended (bunched) field structures of minimum energy or action. The central idea is to identify such a stable field structure with a stable particle. Examples treated include the sine-Gordon equation and its breather solution, representing a model particle in a one dimensional space, and the Hopf map, which leads to a (more realistic) particle model in three dimensional space. Both descriptions are Lorentz invariant.

In our concept, the rest energy (or rest mass) of a single free particle is defined as the total field energy of such a bunched field. The extension of the field structure is infinite in principle, but the region of essential nonlinearity (and thus of sizeable contribution to the total field energy) may be quite small, so that the larger part of the rest mass can be considered to be well localized. The region mainly containing the mass will turn out to be related to a given elementary length. The field is continuous and regular at all points of space and time. The field structures are not assumed to be static, breather-like internal oscillations or a rotation of the structure as a whole are not excluded. Such structures can be viewed as localized (nonlinear) oscillators or rotators exhibiting a rest mass. The Lorentz invariance of the equations leads directly to solutions discribing such structures in a state of uniform motion along some direction. Momentum and kinetic energy of the particle are then easily derived from the proper Lorentz transformation. The particle concept advocated in the present paper deals essentially with these moving oscillators or rotators : they are viewed as moving particles.

The description given so far is classical. Any particle concept should address, however, the wave nature of matter. Indeed our concept includes, without further assumptions, wave aspects which emerge from two properties of the model, i.e. the internal oscillation and the infinite extension of the field. The particle at rest exhibits an oscillation which is periodic in time with correlated phase all over space. Transformation to motion of this oscillating structure leads to a periodicity of the field in both space and time, i.e. a phase wave. This (nonlinear) phenomenon is governed by a vector k describing the translational invariance of the structure. It is of great importance to note [1] that the momentum of the moving structure turnes out to be proportional to k. Therefore a comparison of k with the wave vector of a harmonic plane wave of equal phase, the ψ-wave of a free particle, is possible, although the two concepts are completely different. In this context we would like to stress the relation of our concept with de Broglie's [2] fundamental ideas on moving oscillators leading to quantum mechanics.

In the present paper, the emphasis is put on the sine-Gordon breather in order to expose and discuss the particle concept in detail. The breather solution is well known and can be expressed in closed form using elementary mathematics, which facilitates the interpretation. Moreover it has been found that a solution in closed form is not restricted to uniform motion, a breather confined in a square well potential can be treated exactly as well [3]. The confined breather assumes discrete energy levels which are identical with the energy levels found in quantum mechanics for a particle in such a potential [3]. These exact results are used to confront our particle concept with the usual quantum mechanical description and to discuss questions like localization, probability interpretation, or momentum in the simplest possible context.

The second example to be treated in this paper concerns bunched field structures in three dimensional space emerging from Hopf maps, i.e. mappings of three dimensional space onto the surface of a sphere. The topology of the resulting stable field structures is well known, it is characterized by an integer, the Hopf index. Explicite solutions yielding the detailed structure are not known, however. Therefore these interesting structures [4] will be discussed only briefly in this paper.

The ideas advocated here are not an attempt to return to a classical description of particles in the sense of classical mechanics, on the contrary they are intended to uncover a deeper level of particle structure and to reach a synthesis of particle aspects and wave aspects. At the outset, the concept differs fundamentally from both classical point mechanics and quantum mechanics. We do not suppose, as a basic assumption, the existence of pointlike particles as given entities with given properties in the way it is done in both of these theories. The notion particle does not enter as a basic element, but the particle emerges as a result of theory : as an extended stable field structure. The wave aspects and the proper energy levels are inherent in the model as well, they follow as a consequence of the transformation properties of the extended structures.

Breather dynamics

In this chapter we treat the sine-Gordon breather, which we consider as a particle in a one dimensional space. Our starting point is the sine-Gordon equation, written in the form

$$u_{xx} - c^{-2} u_{tt} = d^{-2} \sin u. \tag{1}$$

The function $u(x,t)$ is introduced as a classical scalar field depending on the space like coordinate x and the time t. Eq.(1) governs the evolution of this field in space and time. If we identify c with the velocity of light and interpret d as an elementary length, eq.(1) represents a classical field theory of particles. No additional notions like particle, interaction, or matter field are introduced from the outset, all physical properties of the model follow as a consequence of the properties of eq.(1) and its solutions. Eq.(1) is attractive as a model theory in one dimension for several reasons : it is of second order in x and t and reduces to the linear Klein Gordon equation in the limit of $u \ll 1$, it is Lorentz invariant, and its solutions are well known.

Eq.(1) represents the Euler equation of the minimum action principle

$$\delta \int [d^2(u_x^2 - c^{-2}u_t^2) + 4\sin^2(u/2)] dx dt = 0. \tag{2}$$

The structures relevant for our concept are stable structures of minimum action. The energy density is made up of terms in u_x^2, u_c^2, and $\sin^2 u/2$. The corresponding total energy is given by the integral [5]

$$E = \frac{1}{2} G \int [d^2(u_x^2 + c^{-2}u_t^2) + 4\sin^2(u/2)] dx, \tag{3}$$

taken over the entire x-axis. A constant G with the dimension of a force is introduced in eq.(3) for dimensional reasons.

The complete solutions [6] of the sine-Gordon equation can be interpreted as a multitude of (anti)solitons and (anti)breathers, moving with various velocities along the x-axis. Here we focus on a single, isolated breather in motion, to which we ascribe, according to our concept, physical significance. The breather solution [7] of eq.(1) reads

$$u = 4\tan^{-1}\left[\frac{s\sin((r/d)(c(t-t_0)-\beta x)(1-\beta^2)^{-1/2})}{r\cos h((s/d)(x-x_0-vt)(1-\beta^2)^{-1/2})}\right]. \tag{4}$$

The velocity of the breather is v ($\beta = v/c$). A parameter q ($s = \sin q, r = \cos q$) appears in eq.(4), which determines the internal structure, the size and the oscillation frequency of the breather. We assume q to be constant. Our model is therefore based on four constants : the elementary length d, the velocity of light c, the constant G and the number q. The position of the breather and its phase at time $t = 0$ are determined by the integration constants x_0 and t_0. The solution given by eq.(4) can be viewed to describe a bound soliton-antisoliton pair performing an anharmonic yet periodic oscillation, the whole structure being in a state of uniform motion. The structure is centred around the position $x = x_0 + vt$, which we define as its 'site'. The maximum separation b reached by the soliton-antisoliton pair during one periode is a measure of the size of the breather. From eq.(4) we find the relation

$$r \approx 2\exp[-\frac{b}{2d}(1-\beta^2)^{-1/2}], \tag{5}$$

which yields b in terms of r and d.

As seen from eq.(4), the oscillation frequency of the breather is given by

$$\omega_b = \frac{rc}{d}(1-\beta^2)^{-1/2}. \tag{6}$$

The anharmonicity of the oscillation is restricted to a region of the order of b around the center of the breather. Far from the center the oscillation is approximatively harmonic with an amplitude of the form $f(x - vt)$. The solution (4) yields a vector

$$k_b = \frac{rv}{dc}(1-\beta^2)^{-1/2}, \tag{7}$$

being of great importance for our considerations. The dimension of k_b is that of a wave vector, its significance is more general, however : it governs the transformation properties of the breather. During one oscillation periode $\tau = 2\pi\omega_b^{-1}$, the breather advances along the x-axis by an amount

$$x_2 - x_1 = 2\pi v \omega_b^{-1} \approx 2\pi \frac{d^2}{r^2} k_b, \tag{8}$$

or, more precisely, the field structure representing the breather reconstitutes itself in identical shape at a position advanced by the above amount. The vector k_b thus defines breather positions of equal phase and therefore describes what could be called a discrete translational invariance. The vector k_b also determines a length

$$\lambda_b = 2\pi k_b^{-1} = 2\pi \frac{dc}{rv}(1-\beta^2)^{1/2} \sim \frac{1}{v}. \tag{9}$$

The length (9) is not the wavelength of a planar wave, but rather determines the distance of the knots $u = 0$, of the field at any time t. We will see that this property of k enables us to construct bound breather states.

The total energy of the moving breather is given by the integral (3). According to Lamb [5] the result of this integration is

$$E_b = E_0(1-\beta^2)^{-1/2}, \tag{10}$$

where E_0 is the energy of the breather at rest :

$$E_0 = 16 s G d. \tag{11}$$

As $s < 1$, the energy (11) is always smaller than twice the energy of a free soliton, confirming that the breather is a bound state of two solitons. For the larger part, the energy (10) is localized in a space region of the order of the size b centered around the position $x = x_0 + vt$. A rest mass

$$m_b = E_0 c^{-2} \tag{12}$$

can be ascribed to the breather. The mass density is finite for all values of x and falls exponentially to zero far from the site of the breather. The breather can therefore be viewed as a rather well localized amount of energy or mass moving with velocity v. The momentum corresponding to this moving mass reads, according to relativistic mechanics,

$$p_b = E_b v c^{-2}. \tag{13}$$

Summarizing we observe that the moving breather is governed on the one hand by a frequency ω_b and a vector k_b, representing its wave nature, and on the other hand by an energy E_b and a momentum p_b, representing its particle character. The dual nature of the breather has also be emphasized by J.J. Klein [8]. It is very important to realize that there exist proportionalities relating [1] these breather properties. The relations are

$$E_b = \hbar_b \omega_b \tag{14}$$

and

$$p_b = \hbar_b k_b, \tag{15}$$

as becomes clear from eqs. (6), (7), and (10)...(13). The proportionality constant, the action

$$\hbar_b = \frac{16sGd^2}{rc} = E_0 \frac{d}{rc} = m_0 c \frac{d}{r} \tag{16}$$

appearing in both of these relations is a function of the constants introduced earlier. It may be called Planck's constant of breather dynamics. The analogy between the expressions (14) and (15), and the fundamental quantum mechanical laws expressing the wave nature of matter is evident. We will discuss this analogy in detail in the next chapter.

Up to this point we have considered a breather in rectilinear uniform motion. We now treat the breather as a particle confined [3] in a square well potential. This type of potential is characterized by perfectly reflecting walls, in our one-dimensional case by two walls at positions $x = 0$ and $x = L$. We assume that u vanishes at those positions, which implies for the free breather, according to eq.(4), that

$$\pi n k_b^{-1} = L, \tag{17}$$

where n is an integer. Eq.(1) has thus to be solved for the boundary condition $u(0) = 0$ and $u(L) = 0$. It is known that the sine-Gordon equation has a special class of analytic solutions, the multisoliton solution, for which explicite expressions can be obtained by using the inverse scattering formalism [6,9]. We construct a solution by introducing two trains of breathers of proper phase moving in opposite directions. Both trains consist of an infinite number of equidistant breathers (distance $2L$), one train moving with velocity $+v$, the other with velocity $-v$. The breathers of each of the trains oscillate in phase, the phase of the two trains being opposite. The breather sequences are such that the points $x = 0$ and $x = L$, the positions of the walls, are centers of antisymmetry. Periodic breather collisions occur at these positions (and at all equidistant positions outside the domain $0 \leq x \leq L$), where $u = 0$ at all times. We interprete this result as a reflection of breathers at the walls, whereby their phase is reversed. Within the domain, the region of interest, one and only one breather is present at any time, so that our construct indeed describes a confined 'particle', in fact a one particle state.

The above procedure, which is discussed in somewhat more detail elswhere [3], assures the existence of exact solutions with the proper boundary conditions, a very important result for our considerations. In the following we do not further persue the general solution or the details of breather reflection, but restrict ourselves to large potential wells and thus large interbreather distances, i.e. $L \gg b$. An especially simple solution then results : except near the reflecting walls (at distances smaller than several times b), the solution describing the confined breather is close to that describing a free breather, which is in accordance with Bäcklund's transformations [5]. The energy and the momentum are found to be close to the free breather values as well, deviations falling off exponentially with interbreather distance.

We now turn to the energy of a confined breather in a stationary state. Similar to the familiar linear problem, a stationary state is defined as a state of strict periodicity (periode $2L/v$), for which the condition (17) has to be fulfilled. We therefore have discrete values of the wave vector according to

$$k_{b,n} = \pi n L^{-1}. \tag{18}$$

As a consequence of the discreteness of the wave vector discrete energy levels result. In our approximation $L \gg b$ and for velocities $v \ll c$ the energy levels are

$$E_{b,n} - E_0 = E_0 \frac{d^2\pi^2 n^2}{r^2 2L^2} = \frac{\hbar_b^2 \pi^2 n^2}{m_b 2L^2}, \tag{19}$$

as results from eqs.(7), (10), (12), (16), and (18). It should be kept in mind that eq.(19) represent the approximate energy of an *exact* solution of eq.(1). The energy levels (19) are identical with those found in quantum mechanics for a particle of mass m moving in a square well potential. The physical implication of this analogy will be discussed in the next chapter.

Confrontation of particle concepts

We now confront our particle concept with the particle concept underlying quantum mechanics. This is done by comparing the assumptions made and the results obtained in both concepts when treating the simple example of a particle in a square well potential.

We first recall the quantum mechanics of this problem. The Schrödinger equation for a particle of mass m in such a potential yields

$$\psi \sim (\exp ikx - \exp -ikx), \tag{20}$$

with

$$k_n = n\pi L^{-1}. \tag{21}$$

This solution meets the boundary conditions $\psi(0, L) = 0$. Eq.(20) can be viewed as the superposition of two plane waves, representing a real function of the type $\sin(kx)$. The expected mean value of the momentum of the particle in our configuration is zero, a result generally valid for real waves functions. The probability of finding the particle at the coordinate x is

$$2L^{-1}\sin^2 k_n x. \tag{22}$$

The energy levels corresponding to the above solution are

$$E_n - E_0 = \frac{\hbar^2}{m}\frac{\pi^2}{2}\frac{n^2}{L^2}. \tag{23}$$

If we identify m with m_b, and $\hbar$ with $\hbar_b$, the energy levels (23) are identical with the energies (19) of the breather in the same potential.

Comparing the concepts we see that both of them rely on scalar functions, ψ and u, respectively. The difference in interpretation is evident, however. Whereas the wave (20) is a plane wave of constant amplitude, governed by a linear equation and subject to a probabilistic interpretation, the wave (4) represents a bunched field governed by the nonlinear equation (1) and interpreted entirely classical. Far from the (moving) center of the breather, where $u \ll 1$, the solution (4) takes the form

$$u \sim f(x - vt)\sin(k_b x - \omega_b t), \tag{24}$$

which represents a monochromatic wave with strongly varying amplitude. Its wavelength is equal to the length (9), and, in the non relativistic limit, also equal to the de Broglie wave length

$$\lambda_b = \hbar_b m_b^{-1} v^{-1} \tag{25}$$

of the quantum mechanical description.

The identity of the eigenvalues follows directly from the equivalence of 'wave lengths'. Concerning energy levels, we thus have two descriptions, quantum mechanics and the breather model, which lead to the same answer. This result is remarkable in view of the widely hold opinion that quantum mechanics is unique in discribing discrete energies. It is interesting to note that measurements of energy levels would not discern between the two models.

Concerning the position as well as the momentum of the particle or breather, however, the two concepts differ in an essential way. The quantum mechanical formalism deals with pointlike particles and ascribes to them, after measurement, a position at a point on the x-axis. The breather concept, on the other hand, deals with an extended object. The extension is infinite in principle for the free breather, and equal to L for the confined breather. Nevertheless, as we have seen, the mass is rather well localized, because far from the center of the breather the intensity of the field and the corresponding energy density fall off exponentially. We have defined the length b ($\ll L$) as the size of the breather, so that we conclude that our particle, while being extended, possesses a position determined with a precision of the order of $\approx b$ at any time. Other definitions of size will lead to similar conclusions.

Except at the boundaries $x = 0$ and $L = 0$, there are, also for higher orders $n > 1$, no fixed nodes inside the domain $0 < x < L$. (The footnote in ref.[3] on this point is in error). The confined 'particle', represented by the breather, reaches all positions in the domain. A hypothetical measurement of particle position would reveal the following picture. A single measurement would deliver an arbitrary value of x in the domain in both concepts. Repeated measurement would produce a density according to eq.(22) in quantum mechanics, yet a uniform density in the breather concept. There are thus no sites inaccessable to the breather, a result relevant in connection with the problem of particle passage through nodes. D. Bohm and B.J. Hiley [10] recently adressed this problem and concluded that in the quantum mechanics of a particle in a box equiprobability of opposing velocities is not consistent with nodes. In the breather picture this complication is not present.

The momentum of the breather is also well determined. In the square well potential the motion is periodic, the breather momentum is either $+mv$ or $-mv$. Because the mass is localized, the momentum is localized as well. A space region of size b carries the major part of the momentum. We conclude that our 'particle' has a position *and* a momentum, both of them well determined within the limits indicated. The moving breather is monochromatic, i.e. it has an unique length (9), and simultaneously a well defined position. Heisenbergs relation, which is based on linear superposition, does not apply in our nonlinear model. However, a measurement of position by confining the breather to a region $< L$ would destroy the stationary state in a way similar to the quantum mechanical result.

Breather reflexion shows interesting details. The extension of the breather implies that reflexion is not an instantaneous phenomenon but has, if we focus on the main part of the mass, a duration of the order of b/v. In view of the asymptotic parts of the structure u, reflection is, in strict sense, even continuous. The breather interacts permanently with both of the walls !

Expanding on breather extension we note that as an alternative definition of particle size we may separate an inner region of size b', the particle, from the outside region, the field, by defining a fraction of the total breather mass embraced by the length b'. This fraction of the mass will be close to unity for e.g. $b' \approx 2b$. However, either the length b' or the mass fraction is arbitrarily chosen, indicating that a meaningful separation is not possible.

A further, more relevant definition of size concerns the length d/r. This length governs the wave properties of the breather and represents the direct analogue of the Compton wave length of the electron,

$$\lambda_c = hm^{-1}c^{-1} \tag{26}$$

We may equate d/r to λ_c and write the 'Klein Gordon equation of the breather'

$$\psi_{xx} - c^{-1}\psi_{tt} = \frac{m_b^2 c^2}{\hbar_b^2}\psi = \frac{r^2}{d^2}\psi. \tag{27}$$

Eq.(27) describes the quantum mechanics of a particle of mass m_b and thus represents the linear, relativistic invariant counterpart of eq.(1). The results (20)...(23) follow, in the non relativistic approximation, directly from eq.(27) as well. Eq.(27) is not identical with the small field limit of eq.(1), because the relevant length is d/r and not d.

We arrive at the conclusion that the concepts under discussion represent two valid descriptions, both internally consistent, of the motion of a particle or breather. They are not equivalent, however. The Schrödinger equation (or eq.(27)) describes a partial aspect only, i.e. the wave nature, whereas the breather concept describes, apart from the wave nature, also the internal structure, the stability and the discreteness of the mass. Therefore the linear equation is a substitute for the more general nonlinear concept, reproducing the wave aspects only. It has the character of an artefact.

Hence we further conclude that there is a hierarchy (fig.1) of descriptions of the phenomenon 'particle'. The concept proposed is the most fundamental one, special cases of limited validity follow by simplifying assumptions, i.e. the classical relativistic particle picture by neglecting the internal structure of the u-field and by concentrating the total mass in one point ; and the quantum mechanical description by substituting the plane ψ-wave while maintaining the proper phase of the u-wave.

Hopf maps

What has been presented so far is a model useful to exemplify and discuss the particle concept we have in mind. The simplicity of the mathematics used facilitates the discussion of the underlying physical ideas. The model yields a unified description of a rather large number of particle properties and therefore contains valid elements of a realistic theory. Yet the model falls short in an important respect : it is one dimensional. A direct identification of the structure described with a known particle is therefore not possible. As an obvious way to generalize the model to three spacelike dimensions one might think of replacing the left side of eq.(1) by the d'Alambertian operator $\Box u$. However, Derrick's [11] (or Hobart's [12]) theorema excludes stable static solutions of such a three dimensional model. Hence such a direct generalization of eq.(1) does not lead to the desired result.

A viable three dimensional model should meet the following postulates : 1. The model is to be based on a non-linear Lorentz-invariant equation following from a least action principle. 2. The equation should not be subject to Derrick's theorema, thus allow for extended, stable solutions free of singularities. 3. The stability may either be of the non-topological type (like the breather), or, preferably, of the topological type (like the sine-Gordon soliton). 4. The resulting field structure should perform internal oscillations or rotations. 5. The interpretation is classical.

A route to a three dimensional model meeting these postulates is found if we recall that the sine-Gordon system represents a mapping of the x-axis on a unit circle. The Hopf map is a direct generalization of this map, i.e. the mapping of three dimensional space on the (two dimensional) surface of a sphere. The topological properties of Hopf maps are characterized by the Hopf index, an invariant integer indicating the number of times space is mapped on the sphere. We represent the Hopf map by two angular variables, the spherical coordinates (or Eulerian angles) $\theta(x, y, z, t)$ and $\phi(x, y, z, t)$, which we interpret as classical scalar fields. The model is based on the minimum action principle [4]

$$\delta W = 0 \quad , \quad W = B \int w(D) d^3\xi d\tau. \tag{28}$$

In eq.(28) the ξ's are dimensionless Cartesian coordinates expressed in units of a fundamental length $l(\tau = ct/l, \xi_1 = x/l$, etc.). There are thus three fundamental constants : the length l, the velocity of light c, and the constant B having the dimension of an action. The dimensionless scalar density D is defined as

$$D = (\nabla\theta)^2 - \theta_\tau^2 + [(\nabla\phi)^2 - \phi_\tau^2] \sin^2\theta. \tag{29}$$

In the first instance the function w is assumed to read

$$w(D) = D + D^2, \tag{30}$$

but higher order terms of D are not excluded. If the quadratic term in eq.(30) is ommitted, Derrick's [11] reasoning applies, and stable field structures are not obtained. Including a quadratic or higher order term leads to the consequence that the energy density increases stronger than the inverse of the volume element when the structures are scaled down, and stability results. We therefore arrive at the expression (30) as the simplest assumption. It may be remarked that Williams [13] showed that assuming $w = D^{3/2}$ also leads to stable field structures, be it without scale. From the minimum action principle defined by eq.(28, 29, 30), the corresponding Euler equations (not reproduced here) are readily found [14].

As yet, no detailed analytical or numerical solutions of these equations are known, the existence of solutions and their general structure and topological properties is assured, however. The simplest of these structures [14] can be viewed as a closed, twisted 'string' having a core defined as the line $\theta = \pi$. On a closed path both along and around this line, the function ϕ varies by 2π. There are two of these structures, characterized by the Hopf indices ± 1, forming mirror images of each other. One or more structures sharing the same sign of the Hopf index are topologically stable, a pair of structures with indices $+1$ and -1 can annihilate each other, because such a pair is topologically equivalent to $\theta = 0$. The boundary condition is such that far from the structure (at a distance large compared with its radius) $\theta \to 0$.

The energy integral corresponding to eq.(11) has been evaluated by Williams [13] for $w = D^{3/2}$, and estimated [4] for $w = D + D^2$. Both results refer to static structures only. We expect that the minimum action principle (28) will lead to a rotating structure. Further work including numerical solutions of eq.(28) is therefore needed.

Conclusions

In conclusion, we advocate a particle concept based on stable, extended, oscillating field structures derived from a nonlinear action principle and subject to classical interpretation as an alternative to the presently accepted concept, which is based on pointlike particles with fundamental properties not resulting from theory, but introduced as labels. The most important difference between the concepts concerns particle structure and particle extension. In a strict sense, notions like structure or extension are without meaning in quantum mechanics, the formalism presupposes pointlike particles. A structure is exclusivly ascribed to those particles, e.g. the proton, which are assumed to be built up of more elementary, pointlike particles. In quantum mechanics there is literally no room for particle structure. It is important to realize that the theoretical framework determines to a large extent the interpretation of experimental data. Concerning e.g. the electron, the experimental evidence, interpreted within the quantum mechanical framework, confirms its pointlike character. In a different framework a quite different conclusion may result. Our particle concept is based on such an alternative theoretical framework constituting a structural theory from the outset. The resulting stable field configurations not only have a structure, they are structures.

The moving breather is an example of these ideas. Indeed, our result that the breather assumes discrete energy states is a direct consequence of the spatial extension of the field. The interference is a consequence of the simultaneous interaction of the extended field with both walls. In this respect the particle is a non local phenomenon, an insight which may be of relevance for the interpretation of certain recent experiments.

The model based on Hopf maps also fits into this classical theoretical framework, and is expected to lead to analoguous results. (Our model differs from that of Skyrme [15], which is interpreted in the framework of quantum mechanics, although there is a close relation in topological properties. His model is based on the mapping of space on the three dimensional surface of a hypersphere, which leads to extended field structures having point symmetry). Summarizing we conclude that the concept presented unifies particle properties like discrete mass (and built in interaction [14]) with the wave nature of the particle. It represents a new synthesis of discreteness and continuity.

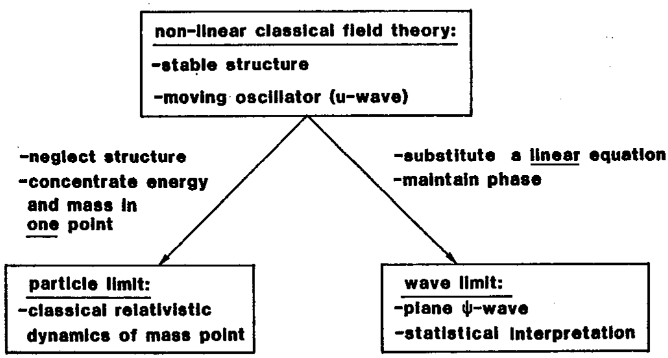

Figure 1. Hierarchy of particle descriptions : The familiar picture of the relativistic mass point on the one hand and the plane wave description of quantum mechanics on the other hand follow from the more general nonlinear description by making the simplifying assumptions indicated in the figure.

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