Wave packet anomalies due to a discrete space, discrete time modified Schrödinger equation

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ABSTRACT. By converting both time and space derivatives to finite differences in the Schrodinger equation, we study the effect that these changes have on wave packet propagation. Also, by studying the dependence of the width and the velocity of the wave packet on the initial peaked momentum, we discuss a possible probe to both discrete time and discrete spatial differences in quantum theory.

RESUME. En convertissant les dérivées de temps et d'espace en différences finies dans l'équation de Schrödinger, nous en étudions les effets sur la propagation d'un paquet d'ondes. De même, en étudiant la manière dont la largeur et la vitesse du paquet d'ondes dépendent d'une impulsion intiale concentrée, nous discutons un test possible des différences finies aussi bien d'espace que de temps en mécanique quantique.

Introduction

One of the most profound questions in all of physics, is whether or not, at some level, both space and time become discrete, countable entities that become grainy at some scale. Finkelstein has discussed the idea of the quantum net, wherein the world is actually discrete and countable and the observables that we study are actually averages over a discrete lattice [1]. Also, working along these lines, Bombelli et. al. have discussed the origin of space time as emerging from discrete causual set with the signature $(1,-1,-1,-1)$ being a unique consequence of the algebraic relationship of points [2]. Both t'Hooft [3] and Snyder [4] have advocated the introduction of a space time lattice to eliminate the divergence problems in both Q.E.D. and gravitational theory. In trying to discuss a discrete space time quantum theory there are actually two approaches that emerge. The first is a truly discrete space time lattice theory with a countable number of allowed points, the second approach assumes the existence of a microscopic uncertainty principle which forbids the response of the wave function arbitrarily close to the point of application of the hamiltonian ; and thus, the response is a finite time interval removed from the point of application of the hamiltonian. The second approach is a discrete time difference theory while the actual time variable assumes a continuous set of values. This latter approach has been fully discussed by Santilli [5] and Mignani et. al. [6] within the framework of the hadronic mechanics. The original idea was actually due to Caldirola [7]. In two previous notes, we have discussed the effect that such discrete time differences have on both electron spin resonance [8] and electron spin polarization [9]. We have also discussed the effect that such discrete time differences have on the propagation of a double wave packet with certain distinct signature signaling discrete time effects [10]. In this note, we also introduce discrete spatial differences in addition to discrete time differences, by studying the properties of the wave packet in this picture we find distinct characteristics that the discrete space, discrete time differences introduce into a wave packet. In particular, by studying the characteristic dependence of the width of a wave packet on the initial momentum, we can, at least in principle probe these discrete space, discrete time modifications in quantum theory.

2. Discrete Time, Discrete Space Modification of a Wave Packet

We begin by writing for the discrete time discrete space modification of the spatial and temporal derivatives

$$\frac{\partial\psi}{\partial x} \to \frac{\psi(x+L_0/2)-\psi(x-L_0/2)}{L_0} = \Delta_x\psi \tag{2.1}$$

$$\frac{\partial\psi}{\partial t} \to \frac{\psi(t+T/2)-\psi(t-T/2)}{T} = \Delta_t\psi \tag{2.1}$$

where L_0 and T are discrete space and time intervals. For the modified Schrodinger equation we have

$$-\frac{\hbar^2}{2m}\Delta_x^2\psi = i\hbar\Delta_t\psi \tag{2.2}$$

which gives

$$-\frac{\hbar^2}{2m}\left[\frac{\psi(x+L_0)+\psi(x-L_0)-2\psi(x)}{L_0^2}\right] = i\hbar\left[\frac{\psi(t+T/2)-\psi(t-T/2)}{T}\right] \tag{2.3}$$

This equation can be written according to Jannussis et. al. [11] as

$$-\frac{\hbar^2}{2mL_0^2}\left[e^{L_0\partial/\partial x}+e^{-L_0\partial/\partial x}-2\right]\psi = \frac{i\hbar}{T}\left[e^{T/2\ \partial/\partial t}-e^{-T/2\ \partial/\partial t}\right]\psi$$

$$\left[-\frac{2\hbar^2}{mL_0^2}\sinh^2\frac{L_0}{2}\frac{\partial}{\partial x}\right]\psi = \left[\frac{2i\hbar}{T}\sinh\frac{T}{2}\frac{\partial}{\partial t}\right]\psi \tag{2.4}$$

Inverting the time evolution operator, we have

$$\sinh\frac{T}{2}\frac{\partial}{\partial t} = \frac{Ti\hbar}{mL_0^2}\sinh^2\frac{L_0}{2}\frac{\partial}{\partial x}$$

$$\frac{\partial}{\partial t} = \frac{2}{T}\sinh^{-1}\left[\frac{iT\hbar}{mL_0^2}\sinh^2\frac{L_0}{2}\frac{\partial}{\partial x}\right]$$

giving

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{2i\hbar}{T}\sinh^{-1}\left[\frac{iT\hbar}{mL_0^2}\sinh^2\frac{L_0}{2}\frac{\partial}{\partial x}\right]\psi \tag{2.5}$$

for the discrete time –discrete space modified Schrodinger equation. We now insert the plane wave into Equation (2.5).

$$\psi = Ae^{i/\hbar(Px-Et)} \tag{2.6}$$

to obtain the dispersion relation

$$E = \frac{2i\hbar}{T}\sinh^{-1}\left[\frac{iT\hbar}{mL_0^2}\sinh^2\frac{L_0 iP}{2\hbar}\right]$$

giving

$$E = \frac{2i\hbar}{T}\sinh^{-1}\left[-\frac{iT\hbar}{mL_0^2}\sin^2\frac{L_0 P}{2\hbar}\right] \tag{2.7}$$

We now expand Equation (2.7) as follows

$$\begin{aligned} E \simeq \frac{2i\hbar}{T}\Bigg[&- \frac{iT\hbar}{mL_0^2}\left(\left(\frac{L_0P}{2\hbar}\right) - \frac{1}{3!}\left(\frac{L_0P}{2\hbar}\right)^3 + \frac{1}{5!}\left(\frac{L_0P}{2\hbar}\right)^5\right)^2 \\ &- \frac{1}{3!}\left(-\frac{iT\hbar}{mL_0^2}\left(\left(\frac{L_0P}{2\hbar}\right) - \frac{1}{3!}\left(\frac{L_0P}{2\hbar}\right)^3\right)^2\right)^3\Bigg] \end{aligned} \tag{2.8}$$

We now assume that the lattice parameter L_0 and the initial peak momentum of a gaussian wave packet are such that $L_0P/2\hbar < 1$ giving

$$\begin{aligned} E \simeq &\frac{2\hbar^2}{mL_0^2}\left(\frac{PL_0}{2\hbar}\right)^2 - \frac{\hbar^2}{mL_0^2}\left(\frac{2}{3}\right)\left(\frac{PL_0}{2\hbar}\right)^4 \\ &+ \left[\frac{4\hbar^2}{45mL_0^2} + \frac{2T^2\hbar^4}{m^3L_0^6}\left(\frac{1}{6}\right)\right]\left(\frac{PL_0}{2\hbar}\right)^6 \end{aligned}$$

$$E \simeq \frac{P^2}{2m} - P^4\left(\frac{L_0^2}{24m\hbar^2}\right) + P^6\left[\frac{L_0^4}{720M\hbar^4} + \frac{T^2}{192m^3\hbar^2}\right] \tag{2.9}$$

This is the altered dispersion relation up to the sixth power in P. Now since P will be close to P_0 because of a initial gaussian we may approximate Equation (2.9) or $P^4 \simeq P^2P_0^2$, $P^6 \simeq P^2P_0^4$ giving

$$E \simeq p^2\left[\frac{1}{2m} - P_0^2\left(\frac{L_0^2}{24m\hbar^2}\right) + P_0^4\left(\frac{L_0^4}{720\hbar^4 m} + \frac{T^2}{192\hbar^2 m^3}\right)\right] \tag{2.10}$$

where we generate an effective mass

$$\frac{1}{2m_{\text{Eff}}} = \frac{1}{2m} - P_0^2\left(\frac{L_0^2}{24m\hbar^2}\right) + P_0^4\left(\frac{L_0^4}{720\hbar^4 m} + \frac{T^2}{192\hbar^2 m^3}\right) \tag{2.11}$$

We see from Equation (2.11) that the signature for discrete space-time effects is an effective mass that first increases with the initial peak momentum of P_0 in a quadratic fashion and then decreases in a quartic fashion. If we choose an initial gaussian to have the form

$$\psi(x,0) = \frac{1}{\pi^{1/4}2^{1/4}(\Delta x)^{1/2}} e^{-\frac{(x-x_0)^2}{4(\Delta x)^2}} e^{\frac{iP_0x}{\hbar}} \tag{2.12}$$

and writing the superposition of states at $t = 0$, we have

$$\psi(x,0) = \frac{1}{h^{1/2}} \int [A(P) e^{i/\hbar (Px - Et)} dP]_{t=0} \tag{2.13}$$

giving

$$\psi(x,0) = \frac{1}{h^{1/2}} \int A(P) e^{iPx/\hbar} dP$$

and by Fourier inversion

$$A(P) = \frac{1}{h^{1/2}} \int \psi(x,0) e^{-iPx/\hbar} dx \tag{2.14}$$

giving using Equation (2.12)

$$A(P) = \frac{1}{\pi^{1/4}} \left(\frac{2^{1/4} (\Delta x)^{1/2}}{(\hbar)^{1/2}} \right) e^{-\frac{(P-P_0)^2 \Delta x^2}{\hbar^2}} e^{-\frac{i(P-P_0)X_0}{\hbar}} \tag{2.15}$$

Inserting Equation (2.15) into the wave function at time t, which reads

$$\psi(x,t) = \frac{1}{h^{1/2}} \int A(P) e^{i/\hbar (Px - Et)} dP \tag{2.16}$$

we have with the use of the dispersion relation in Equation (2.10)

$$\begin{aligned} \psi(x,t) =& \pi^{-1/4} \left[\sqrt{2} \Delta x + \frac{i\hbar t}{\sqrt{2} m_{\mathrm{Eff}} \Delta x} \right]^{-1/2} e^{i \left(\frac{P_0 x}{\hbar} - \frac{P_0^2 t}{2 m_{\mathrm{Eff}} \hbar} \right)} \\ & e^{-\frac{(x - x_0 - P_0 t/m_{\mathrm{Eff}})^2 (1 - i\hbar t / m_{\mathrm{Eff}} (2\Delta x^2))}{2(2(\Delta x)^2 + \hbar^2 t^2 / 2\Delta x^2 m_{\mathrm{Eff}}^2)}} \end{aligned} \tag{2.17}$$

We see from Equation (2.17) that the speed of the center of the wave packet varies as P_0/m_{Eff} which will decrease relative to the normal value if the first correction to $1/2m_{\mathrm{Eff}}$ dominates in Equation (2.11) and will increase if the second correction dominates. A transition from a decrease in P_0/m_{Eff} relative to the normal value to an increase as P_0 increases would be a signal for both discrete time and discrete spatial effects. Also, the width in Equation (2.17) varies as

$$2(\Delta x)^2 + \frac{\hbar^2 t^2}{2\Delta x^2 m_{\mathrm{Eff}}^2}$$

which decreases relative to the standard value for small P_0 from Equation (2.11) and increases relative to the standard value for large P_0. We also note from Equation (2.11) that if only discrete time effects are present, then $L_0 \to 0$ and $1/2m_{\rm Eff}$ from Equation (2.11) increases relative to the standard value. Thus, an increase in $P_0/m_{\rm Eff}$ and width as P_0 changes from a small value of P_0 to larger values would signal pure discrete time effects while a decrease in $P_0/m_{\rm Eff}$ and the width relative to the standard value and then an increase as P_0 increases would signal both discrete time and discrete spatial effects. To test these ideas, we might envision an experiment wherein the arrival time of an ensemble of particles might probe the central motion of a wave packet with the anomalous dependence of the arrival time on the initial momentum being a probe to the above discrete space-discrete time effects.

Acknowledgement

I'd like to thank the physics departments at Harvard University and Williams College for the use of the facilities.

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(Manuscrit reçu le 28 août 1989)