Some Basic Theorems on Two-Component Quantum Systems

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ABSTRACT. A simple and self-contained treatment of the basic mathematics of two component quantum systems is presented. First, the main general properties of tensor product spaces and of linear operators on these spaces are analysed. Then, von Neumann's treatment of density operators on the tensor product of two Hilbert spaces is simplified and complemented with some additional theorems. A rigorous proof of the impossibility of superluminar transmissions by wave-packet-reduction is presented.

RESUME. On présente un développement simple et se suffisant à lui-même des mathématiques de base des systèmes quantiques à deux composantes. Premièrement, on analyse les principales propriétés générales des produits tensoriels de deux espaces et des opérateurs linéaires sur ces espaces. Ensuite, le développement de von Neumann des opérateurs densité sur le produit tensoriel de deux espaces de Hilbert est simplifié et complété avec des théorèmes supplémentaires. On présente une démonstration rigoureuse de l'impossibilité de transmissions supra-lumineuses au moyen de la réduction du paquet d'ondes.

1. Introduction

The aim of this paper is to present a rigorous, simple and sufficiently complete treatment of the basic mathematical theory of the tensor product of two Hilbert spaces and of linear operators on this space, with particular reference to density operators (self-adjoint, non-negative, unit-trace operators). The paper has been conceived to provide a clear and self-contained mathematical basis for a deep intrepretative analysis of some problems of the foundations of quantum mechanics which concern composite systems, such as the Einstein-Podolsky-Rosen (in short E.P.R.) paradox [1] and the theory of measurement [2].

A rigorous, self-contained, sufficiently complete and simple treatment of the mathematical formalism of two-component quantum systems does not seem to be available in the literature. Traditional textbooks on quantum mechanics either do not deal with the mathematical theory of tensor products of Hilbert spaces [3], or contain just some introductory remarks [4]. More recent books and papers deal only with particular aspects of this subject [5]. Some books of mathematics [6], or on the mathematical bases of quantum mechanics [7], present a rigorous approach to linear operators on the tensor product of Hilbert spaces, but do not discuss in detail density operators. The most complete treatment of the properties of density operators on the tensor product of two Hilbert spaces is still that presented by von Neumann [8] ; this, however, does not, contain the basic theory of tensor products of spaces and operators and, moreover, is rather involved.

In the present paper, von Neumann's treatment is simplified and complemented with a deeper insight into the structure of tensor product spaces and with some theorems which yield interesting results for a conceptual analysis of the foundations of quantum mechanics and thermodynamics. Among results not available in Ref.[8], we can point out rigorous proofs of some properties of linear operators on the tensor product of two Hilbert spaces (theorems 6, 7, 8 and 12 of section 3) and of the following statements.

a) The partial traces of any linear operator on the tensor product of two Hilbert spaces H_u and H_v are independent of the choices of the bases of H_u and H_v.

b) Two systems U and V, in states ρ_u and ρ_v, are uncorrelated if and only if the composite system W is in state $\rho_w = \rho_u \otimes \rho_v$;

c) Two non-interacting systems which are uncorrelated (or correlated) at the initial instant $t = 0$ remain uncorrelated (or correlated) at any instant t.

d) For every composite system $W = U + V$, the ideal measurement of an observable A of U does not modify the state of system V. This point shows that no superluminar transmission can be obtained as a consequence of von Neumann's wave-packet-reduction rule applied to composite systems. A different proof of this theorem, based on the "cyclic property of the partial trace", is presented in Ref.[9].

The paper is organized as follows. In section 2 we define the tensor product of two linear, euclidean and Hilbert spaces, and prove some properties of these spaces. In section 3 we prove some properties of linear operators on the tensor product of two Hilbert spaces and of their matrix representations, handling only continuous operators defined on the whole space. In section 4 we deal with the relations between density operators on the tensor product of two Hilbert spaces and their partial traces. In section 5 we analyse correlations between subsystems, and prove statements b) and c) listed above. In section 6 we prove statement d).

2. Tensor product of two Hilbert spaces

In this Section, we present a self contained introduction to the linear and metric properties of tensor product spaces, with reference to separable spaces of finite or infinite dimension.

2a. Tensor product of two linear spaces

Let L_u and L_v be two linear spaces. We will denote by $|u_1\rangle, \ldots,|u_i\rangle, \ldots,|u_k\rangle, \ldots$ the elements of L_u and by $|v_1\rangle, \ldots,|v_j\rangle, \ldots,|v_l\rangle, \ldots$ the elements of L_v. Let us consider the Cartesian product $L_u \times L_v$, which has as elements all pairs $\{|u_i\rangle, |v_j\rangle\}$, and define in $L_u \times L_v$ the following equivalence relation. **a)** $\{|u_i\rangle, |v_j\rangle\} \sim \{|u_k\rangle, |v_l\rangle\}$ if and only if $|u_k\rangle = c|u_i\rangle$ and $|v_l\rangle = 1/c|v_j\rangle$, for some $c \in C$.

It is easily proved that a) is indeed an equivalence relation and therefore defines equivalence classes in $L_u \times L_v$. We will denote by $|u_i\rangle \otimes |v\rangle$ the equivalence class which corresponds to $\{|u_i\rangle, |v_j\rangle\}$ and by W' the set of all equivalence classes so obtained. The elements of W' will be denoted also by the symbols $|w'_1\rangle, \ldots, |w'_n\rangle, \ldots$ or $|w_{1j}\rangle, \ldots, |w_{kl}\rangle, \ldots$, where $|w_{ij}\rangle = |u_i\rangle \otimes |v_j\rangle$, and will be called simple tensors [6,7]. On account of a), the following equality holds in W':

$$a|u_i\rangle \otimes |v_j\rangle = |u_i\rangle \otimes a|v_j\rangle \quad , \qquad \text{for every} \quad a \in C. \tag{2.1}$$

In fact : $\{a|u_i\rangle, |v_j\rangle\} \sim \{(1/a)a|u_i\rangle, a|v\rangle\} = \{|u_i\rangle, a|v\rangle\}$. Let us define the product by a scalar in W' as follows :

$$a(|u_i\rangle \otimes |v_j\rangle) = a|u_i\rangle \otimes |v_j\rangle = |u_i\rangle \otimes a|v\rangle \quad , \quad a \in C. \tag{2.2}$$

Let us call zero element of W' the following element :

$$|0\rangle = |0\rangle \otimes |v_j\rangle = |u_i\rangle \otimes |0\rangle \quad , \qquad \text{for every} \quad |u_i\rangle \quad \text{and} \quad |v_j\rangle. \quad (2.3)$$

On account of (2.2), $0(|u_i\rangle \otimes |v_j\rangle) = |0\rangle$ for every $|u_i\rangle$ and $|v_j\rangle$.

Let us now consider the family F of all ordered sets of elements of W', i.e., the family of all sets $f = (|w'_1\rangle, \ldots, |w'_n\rangle)$, with any n and any choice of $|w'_1\rangle, \ldots, |w'_n\rangle$. Let us define the following equivalence relations in F :

i) $f_1 \sim f_2$ if all non-zero simple tensors which belong to f_1 belong also to f_2 and vice-versa, independently of the order.

ii) $(|u_1\rangle \otimes |v_1\rangle, \ldots, |u_1\rangle \otimes |v_m\rangle) \sim (|u_1\rangle \otimes |v_{m+1}\rangle, \ldots, |u_1\rangle \otimes |v_n\rangle)$ if $|v_1\rangle + \ldots + |v_m\rangle = |v_{m+1}\rangle + \ldots + |v_n\rangle$;

iii) $(|u_1\rangle \otimes |v_1\rangle, \ldots, |u_m\rangle \otimes |v_1\rangle) \sim (|u_{m+1}\rangle \otimes |v_1\rangle, \ldots, |u_n\rangle \otimes |v_1\rangle)$ if $|u_1\rangle + \ldots + |u_m\rangle = |u_{m+1}\rangle + \ldots + |u_n\rangle$.

In particular, ii) yields $(|u_1\rangle \otimes |v_1\rangle, \ldots, |u_1\rangle \otimes |v_m\rangle) \sim |u_1\rangle \otimes |v_1 + \ldots + v_m\rangle$, and a similar result holds for iii).

It is easily verified that i)-iii) are reflexive, symmetric and transitive, i.e., are equivalence relations.

We can now define an equivalence relation R in F as follows : $f_1 \sim f_2$ if either one of i)-iii) holds between f_1 and f_2, or if f_1 and f_2 can be connected by a chain of equivalence relations of the set i)-iii). For instance, if $f_1 \sim f_3$ according to i) and $f_3 \sim f_2$ according to ii), then $f_1 \sim f_2$ according to R. Reflexivity of R is ensured by i), transitivity by the definition of R, symmetry by symmetry of i)-iii). Therefore R is an equivalence relation, which generalizes that between complex polynomials employed in elementary algebra.

We will denote by W the set of all equivalence classes so obtained, and by $|w\rangle = [|w'_1\rangle, \ldots, |w'_n\rangle]$ the equivalence class which corresponds to $f = (|w'_1\rangle, \ldots, |w'_n\rangle)$.

Let us define in F the following operations of sum and of product by a scalar. If $f_1 = (|w'_1\rangle, \ldots, |w'_m\rangle)$, $f_2 = (|w'_{m+1}\rangle, \ldots, |w'_n\rangle)$ and $c \in C$, then :

$$f_1 + f_2 = (|w'_1\rangle, \ldots, |w'_m\rangle, \ldots, |w'_n\rangle), \quad (2.4)$$

$$cf_1 = (c|w'_1\rangle, \ldots, c|w'_m\rangle). \quad (2.5)$$

It can be easily proved that (2.4) and (2.5) leave invariant the equivalence classes, i.e. : if $f_1 \sim g_1$ and $f_2 \sim g_2$ then $f_1 + f_2 \sim g_1 + g_2$ and $cf_1 \sim cg_1$.

Therefore, (2.4) and (2.5) yield the following operations of sum and of product by a scalar in W :

$$|w_1\rangle + |w_2\rangle = [|w'_1\rangle, \ldots, |w'_m\rangle, \ldots, |w'_n\rangle], \tag{2.6}$$

$$c|w_1\rangle = [c|w'_1\rangle, \ldots, c|w'_m\rangle]. \tag{2.7}$$

It is easily proved that (2.6) and (2.7) satisfy the axioms which define the operations of sum and product by a scalar in a linear space [7.10]. In fact, commutativity and associativity of (2.6) follow directly from i). Moreover, on account of i) there exists a zero element for the sum, which is given by (2.3). Finally, the following properties hold for (2.7) :

a) $$c(|w_1\rangle + |w_2\rangle) = [c|w'_1\rangle, \ldots, c|w'_n\rangle] = c|w_1\rangle + c|w_2\rangle.$$

b) $$\begin{aligned}(c_1 + c_2)|w_1\rangle &= [(c_1 + c_2)|w'_1\rangle, \ldots, (c_1 + c_2)|w'_m\rangle] \\ &= [(c_1 + c_2)|u_1\rangle \otimes |v_1\rangle, \ldots, (c_1 + c_2)|u_m\rangle \otimes |v_m\rangle] \\ &= [c_1|u_1\rangle \otimes |v_1\rangle, c_2|u_1\rangle \otimes |v_1\rangle, \ldots, c_1|u_m\rangle \otimes |v_m\rangle, c_2|u_m\rangle \otimes |v_m\rangle] \\ &= c_1|w_1\rangle + c_2|w_1\rangle.\end{aligned}$$

c) $$(c_1c_2)|w_1\rangle = [c_1c_2|w'_1\rangle, \ldots, c_1c_2|w'_m\rangle] = c_1(c_2|w_1\rangle).$$

d) $$1|w_1\rangle = |w_1\rangle.$$

As a consequence, the set W endowed with operations (2.6) and (2.7) is a linear space, which will be denoted by $L_w = L_u \otimes L_v$ and will be called tensor product of linear spaces L_u and L_v. To every element of $W', |w_{ij}\rangle$, there corresponds an element of $W, [|w_{ij}\rangle]$, which will be denoted by the same symbol $|w_{ij}\rangle = |u_1\rangle \otimes |v_j\rangle$ and will be called simple tensor of L_w. Every element of L_w can be written as a sum or a linear combination of simple tensors. Therefore, L_w is the linear space spanned by the simple tensors $|w_{ij}\rangle$.

Theorem 1. If $\{|\epsilon_i\rangle\}$ and $\{|\nu_j\rangle\}$ are linearly independent systems of two linear spaces L_u and L_v, then $\{|\zeta_{ij}\rangle = |\epsilon_i\rangle \otimes |\nu\rangle\}$ is a linearly independent system of $L_w = L_u \otimes L_v$.

Proof. The linear subspace $\overline{L_w}$ spanned by $\{|\zeta_{ij}\rangle\}$ is the tensor product of the linear subspaces $\overline{L_u}$ and $\overline{L_v}$ spanned by $\{|\epsilon_i\rangle\}$ and $\{|\nu_j\rangle\}$: $\overline{L_w} =$

$\overline{L_u} \otimes \overline{L_v}$. Let us consider a linear combination of the vectors $|\zeta_{ij}\rangle$, $|w\rangle = \sum_{ij} c_{ij}|\zeta_{ij}\rangle$, and prove that if $|w\rangle = |0\rangle$ then $c_{ij} = 0$ for every $\{i, j\}$. If $|w\rangle = |0\rangle$, then $|w\rangle$ is a simple tensor of $\overline{L_w}$. Therefore, there exist $|u\rangle \in \overline{L_u}$ and $|v\rangle \in \overline{L_v}$ such that :

$$|0\rangle = |w\rangle = |u\rangle \otimes |v\rangle = \sum_i a_i|\epsilon_i\rangle \otimes \sum_j b_j|\nu_j\rangle = \sum_{ij} a_i b_j|\zeta_{ij}\rangle = \sum_{ij} c_{ij}|\zeta_{ij}\rangle.$$

But $|u\rangle \otimes |v\rangle = |0\rangle$ if and only if either $|u\rangle = |0\rangle$ or $|v\rangle = |0\rangle$, i.e., since $\{|\epsilon_i\rangle\}$ and $\{|\nu_j\rangle\}$ are linearly independent systems, if and only if either every $a_i = 0$ or every $b_j = 0$, i.e., if and only if every $c_{ij} = 0$.

Theorem 2. If $\{|\epsilon_i\rangle\}$ and $\{|\nu_j\rangle\}$ are bases [11] of L_u and L_v, then $\{|\zeta_{ij}\rangle = |\epsilon_i\rangle \otimes |\nu_j\rangle\}$ is a basis of $L_w = L_u \otimes L_v$.

Proof. On account of Theorem 1, $\{|\zeta_{ij}\rangle\}$ is a linearly independent system. Moreover, every $|w\rangle \in L_w$ can be expressed as a finite linear combination of vectors of $\{|\zeta_{ij}\rangle\}$. In fact, by definition of basis of a linear space, every $|u\rangle \in L_u$ and every $|v\rangle \in L_v$ can be expressed as finite linear combinations of vectors of $\{|\epsilon_i\rangle\}$ and of $\{|\nu_j\rangle\}$ respectively. Therefore, for every simple tensor $|w_p\rangle \in L_w$: $|w_p\rangle = \sum_{i=1,m} a_i|\epsilon_i\rangle \otimes \sum_{j=1,n} b_j|\nu_j\rangle = \sum_{ij} a_i b_j|\zeta_{ij}\rangle$. Finally, every vector $|w\rangle \in L_w$ can be expressed as a finite linear combination of simple tensors and therefore of elements of $\{|\zeta_{ij}\rangle\}$.

Corollary of Theorem 2. If L_u has dimension m and L_v has dimension n, then $L_w = L_u \otimes L_v$ has dimension mn.

Proof. By definition of dimension, a basis $\{|\epsilon_i\rangle\}$ of L_u and a basis $\{|\nu_j\rangle\}$ of L_v have m and n elements respectively. On account of theorem 2, a basis of L_w is $\{|\epsilon_i\rangle \otimes |\nu_j\rangle\}$ and has, therefore, mn elements.

2b. Tensor product of two euclidean spaces

Let E_u and E_v be two euclidean spaces, in which distance is defined by means of the natural norm. We will define tensor product of E_u and E_v the euclidean space $E_w = E_u \otimes E_v$ given by the linear space $L_w = L_u \otimes L_v$ endowed with the following scalar product : if $\{|\epsilon_i\rangle\}$ and $\{|\nu_j\rangle\}$ are two orthonormal bases of E_u and E_v and if $|w_1\rangle = \sum_{ij} c_{ij}|\epsilon_i\rangle \otimes |\nu_j\rangle$, $|w_2\rangle = \sum_{ij} d_{ij}|\epsilon_i\rangle \otimes |\nu_j\rangle$ (where sums are finite), then we define :

$$\langle w_1|w_2\rangle = \sum_{ij}\sum_{kl} c^*_{ij} d_{kl}\langle\epsilon_j|\epsilon_k\rangle\langle\nu_j|\nu_l\rangle = \sum_{ij} c^*_{ij} d_{ij}. \tag{2.8}$$

The euclidean space E_w will be considered as endowed with the distance defined by means of the natural norm.

It is easily verified that (2.8) has the properties which define the scalar product of two vectors.

1) $\langle w_1|w_1\rangle \geq 0$, and $\langle w_1|w_1\rangle = 0$ if and only if $|w_1\rangle = 0\rangle$.

Proof. $\langle w_1|w_1\rangle = \sum_{ij} c_{ij}^* c_{ij} = \sum_{ij} |c_{ij}|^2 \geq 0$, and $\langle w_1|w_1\rangle = 0$ if and only if $c_{ij} = 0$ for every $\{i, j\}$.

2) $\langle w_1|w_2\rangle = \langle w_2|w_1\rangle^*$.

Proof. $\langle w_2|w_1\rangle^* = \sum_{ij} (d_{ij}^* c_{ij})^* = \sum_{ij} c_{ij}^* d_{ij} = \langle w_1|w_2\rangle$.

3) $\langle w_1|cw_2\rangle = c\langle w_1|w_2\rangle$.

Proof. $\langle w_1|cw_2\rangle = \sum_{ij} c_{ij}^* c d_{ij} = c\sum_{ij} c_{ij}^* d_{ij} = c\langle w_1|w_2\rangle$.

4) $\langle w_1|w_2 + w_3\rangle = \langle w_1|w_2\rangle + \langle w_1|w_3\rangle$.

Proof. If we denote by c_{ij}, d_{ij} and e_{ij} the coordinates of $|w_1\rangle$, $|w_2\rangle$ and $|w_3\rangle$ with respect to the basis $|\epsilon_1\rangle \otimes |\nu_j\rangle$, we have :

$$\begin{aligned}\langle w_1|w_2 + w_3\rangle &= \sum_{ij} c_{ij}^*(d_{ij} + e_{ij}) = \sum_{ij} c_{ij}^* d_{ij} + \sum_{ij} c_{ij}^* e_{ij} \\ &= \langle w_1|w_2\rangle + \langle w_1|w_3\rangle.\end{aligned}$$

On account of properties 1) - 4), (2.8) actually defines a scalar product in E_w. Therefore, it has all the other properties of scalar product. For instance, we have :

$$\begin{aligned}\langle w_1 + w_2|w_3\rangle &= \langle w_1|w_3\rangle + \langle w_2|w_3\rangle, \quad \text{(distributivity from the left)} \\ \langle cw_1|w_2\rangle &= c^*\langle w_1|w_2\rangle, \\ |\langle w_1|w_2\rangle|^2 &\geq \langle w_1|w_1\rangle\langle w_2|w_2\rangle, \quad \text{(Schwarz inequality)}\end{aligned}$$

where the equality holds if and only if $|w_2\rangle = c|w_1\rangle$.

Theorem 3. If $\{|\epsilon_i\rangle\}$ and $\{|\nu\rangle\}$ are orthonormal bases of E_u and E_v, then $\{|\zeta_{ij}\rangle = |\epsilon_1\rangle \otimes |\nu_j\rangle\}$ is an orthonormal basis of E_w.

Proof. The completeness of $\{|\zeta_{ij}\rangle\}$ in E_w is a direct consequence of its completeness in L_w. In fact, the linear space spanned by $\{\zeta_{ij}\}$ coincides with L_w and thus contains all the elements of E_w. The orthonormality of $\{|\zeta_{ij}\rangle\}$ follows from the definition of scalar product in E_w :

$$|\zeta_{ij}|\zeta_{kl}\rangle = \langle\epsilon_1| \otimes \langle\nu_j|\epsilon_k\rangle \otimes |\nu_l\rangle = \langle\epsilon_i|\epsilon_k\rangle\langle\nu_j|\nu_l\rangle = \delta_{ik}\delta_{jl}.$$

Theorem 4. If $\{|u_i\rangle\}$, $\{|\tilde{u}_k\rangle\}$ are sets of elements of E_u and $\{|v_j\rangle\}$, $\{|\tilde{v}_1\rangle\}$ are sets of elements of E_v, and if $w_1 = \sum_{ij} c_{ij}|u_i\rangle \otimes |v_j\rangle$, $w_2 = \sum_{kl} d_{kl}|\tilde{u}_k\rangle \otimes |\tilde{v}_l\rangle$, then :

$$\langle w_1|w_2\rangle = \sum_{ijkl} c_{ij}^* d_{kl} \langle u_i|\tilde{u}_k\rangle\langle v_j|\tilde{v}_1\rangle. \tag{2.9}$$

Proof. Let $\{|\epsilon_p\rangle\}$ and $\{|\nu_q\rangle\}$ be orthonormal bases of E_u and E_v. We have :

$$|u_i\rangle = \sum_p e_{ip}|\epsilon_p\rangle \quad , \quad |v_j\rangle = \sum_q h_{jq}|\nu_q\rangle$$

$$|u_i\rangle \otimes |v\rangle = \sum_{pq} e_{ip}h_{jq}|\epsilon_p\rangle \otimes |\nu_q\rangle$$

$$|w_1\rangle = \sum_{ij} c_{ij} \sum_{pq} e_{ip}h_{jq}|\epsilon_p\rangle \otimes |\nu_q\rangle = \sum_{pq}(\sum_{ij} c_{ij}e_{ip}h_{jq})|\epsilon_p\rangle \otimes |\nu_q\rangle$$

$$|w_2\rangle = \sum_{kl} d_{kl} \sum_{pq} \tilde{e}_{kp}\tilde{h}_{lq}|\epsilon_p\rangle \otimes |\nu_q\rangle = \sum_{pq}(\sum_{kl} d_{kl}\tilde{e}_{kp}\tilde{h}_{lq})|\epsilon_p\rangle \otimes |\nu_q\rangle$$

By applying definition (2.8) :

$$\langle w_1|w_2\rangle = \sum_{pqijkl} c_{ij}^* d_{kl} e_{ip}^* \tilde{e}_{kp} h_{jq}^* \tilde{h}_{lq}.$$

Let us now evaluate $\langle w_1|w_2\rangle$ by (2.9).

$$\langle u_i|\tilde{u}_k\rangle = \langle \sum_p e_{ip}\epsilon_p| \sum_r \tilde{e}_{kr}\epsilon_r\rangle = \sum_{pr} e_{ip}^*\tilde{e}_{kr}\langle\epsilon_p|\epsilon_r\rangle = \sum_p e_{ip}^*\tilde{e}_{kp}$$

$$\langle v_j|\tilde{v}_l\rangle = \langle \sum_q h_{jq}\nu_q| \sum_s \tilde{h}_{ls}\nu_s\rangle = \sum_{qs} h_{jq}^*\tilde{h}_{ls}\langle\nu_q|\nu_s\rangle = \sum_q h_{jq}^*\tilde{h}_{lq}$$

$$\langle w_1|w_2\rangle = \sum_{ijkl} c_{ij}^* d_{kl} \sum_p e_{ip}^*\tilde{e}_{kp} \sum_q h_{jq}^*\tilde{h}_{lq} = \sum_{pqijkl} c_{ij}^* d_{kl} e_{ip}^* \tilde{e}_{kp} h_{jq}^* \tilde{h}_{lq}.$$

2c. Tensor product of two Hilbert spaces.

Let H_u and H_v be complete euclidean spaces, i.e., Hilbert spaces, and let $E_w = H_u \otimes H_v$ be the tensor product space defined in 2b. The completion H_w of E_w will be called tensor product of the Hilbert spaces H_u and H_v. In symbols we will write $H_w = H_u \overline{\otimes} H_v$, where the bar means that the tensor product space has been completed.

Theorem 5. If $\{|\epsilon_i\rangle\}$ and $\{|\eta_j\rangle\}$ are orthonormal bases of the Hilbert spaces H_u and H_v, then $\{|\zeta_{ij}\rangle = |\epsilon_i\rangle \otimes |\eta_j\rangle\}$ is an orthonormal basis of $H_w = H_u \overline{\otimes} H_v$.

Proof. It is sufficient to prove that $\{|\zeta_{ij}\rangle\}$ is a complete system of H_w, i.e., that the closure of the linear space spanned by $\{|\zeta_{ij}\rangle\}$ is $H_w : [L\{|\zeta_{ij}\rangle\}] = H_w$. On account of theorem 3, $L\{|\zeta_{ij}\rangle\} = E_w = H_u \otimes H_v$. Since E_w is dense in H_w, i.e. $[E_w] = H_w$, then $[L\{|\zeta_{ij}\rangle\}] = [E_w] = H_w$.

3. Linear operators on the tensor product of two Hilbert spaces

In this section, we prove some basic properties of linear operators on the tensor product of two Hilbert spaces, $H_w = H_u \overline{\otimes} H_v$, and of their partial traces. For simplicity sake, we will assume that all the linear operators considered in this paper and their adjoints are continuous and defined on the whole spaces H_u, H_v and H_w respectively.

The linear operators on $H_w = H_u \overline{\otimes} H_v$ are defined in the same way as those on any Hilbert space H. This definition will not be repeated. Among the linear operators on H_w, there are the tensor products of pairs of linear operators on H_u and on H_v.

If A_u and A_v are linear operators on H_u and H_v respectively, then we define as tensor product of A_u and A_v the following linear operator on H_w :

$$(A_u \otimes A_v)|u\rangle \otimes |v\rangle = A_u|u\rangle \otimes A_v|v\rangle,$$

and, more generally :

$$(A_u \otimes A_v)\sum_{ij} c_{ij}|u_i\rangle \otimes |v_j\rangle = \sum_{ij} c_{ij}A_u|u_i\rangle \otimes A_v|v_j\rangle. \qquad (3.1)$$

The linearity of $A_u \otimes A_v$ follows directly from definition (3.1). In fact, if $\{|\epsilon_i\rangle \otimes |\eta_j\rangle\}$ is an orthonormal basis of H_w, $w_1 = \sum_{ij} c_{ij}|\epsilon_i\rangle \otimes |\eta_j\rangle$ and $w_2 = \sum_{ij} d_{ij}|\epsilon_i\rangle \otimes |\eta_j\rangle$ are two vectors of H_w and $a \in C$:

$$1) \qquad (A_u \otimes A_v)a|w_1\rangle = (A_u \otimes A_v)\sum_{ij}(ac_{ij})|\epsilon_i\rangle \otimes |\eta\rangle$$
$$= a\sum_{ij} c_{ij}A_u|\epsilon_i\rangle \otimes A_v|\eta_j\rangle = a(A_u \otimes A_v)|w_1\rangle;$$

$$2)\quad (A_u \otimes A_v)(|w_1\rangle + |w_2\rangle) = (A_u \otimes A_v) \sum_{ij} (c_{ij} + d_{ij}) |\epsilon_i\rangle \otimes |\eta_j\rangle$$
$$= \sum_{ij} (c_{ij} + d_{ij}) A_u |\epsilon_i\rangle \otimes A_v |\eta_j\rangle$$
$$= (A_u \otimes A_v)|w_1\rangle + (A_u \otimes A_v)|w_2\rangle.$$

Theorem 6.
$$(A_u \otimes A_v)^\dagger = A_u^\dagger \otimes A_v^\dagger.$$

Proof. If $|w_1\rangle = \sum_{ij} c_{ij} |\epsilon_i\rangle \otimes |\eta_j\rangle$ and $|w_2\rangle = \sum_{kl} d_{kl} |\epsilon_k\rangle \otimes |\eta_l\rangle$ are any two vectors of H_w :

$$\langle w_1 | (A_u \otimes A_v) w_2\rangle = \sum_{ij} c_{ij}^* \langle \epsilon_i | \otimes \langle \eta_j | \sum_{kl} d_{kl} |A_u \epsilon_k\rangle \otimes |A_v \eta_l\rangle$$
$$= \sum_{ijkl} c_{ij}^* d_{kl} \langle \epsilon_i | A_u \epsilon_k\rangle \langle \eta_j | A_v \eta_l\rangle = \sum_{ijkl} c_{ij}^* d_{kl} \langle A_u^\dagger \epsilon_i | \epsilon_k\rangle \langle A_v^\dagger \eta_j | \eta_l\rangle$$
$$= \sum_{ij} c_{ij}^* \langle A_u^\dagger \epsilon_i | \otimes \langle A_v^\dagger \eta_j | \sum_{kl} d_{kl} |\epsilon_k\rangle \otimes |\eta_l\rangle = \langle (A_u^\dagger \otimes A_v^\dagger) w_1 | w_2\rangle.$$

Corollary of theorem 6. If A_u and A_v are linear self-adjoint operators on H_u and H_v, then $A_u \otimes A_v$ is a linear self-adjoint operator on H_w.
Proof. On account of theorem 6 : $(A_u \otimes A_v)^\dagger = A_u^\dagger \otimes A_v^\dagger = A_u \otimes A_v$.

Theorem 7. If A_u and B_u are linear operators on H_u, and A_v and B_v are linear operators on H_v, then :

$$(A_u \otimes A_v)(B_u \otimes B_v) = A_u B_u \otimes A_v B_v,$$

and

$$(A_u + B_u) \otimes (A_v + B_v) = A_u \otimes A_v + A_u \otimes B_v + B_u \otimes A_v + B_u \otimes B_v.$$

Proof. If $|w\rangle = \sum_{ij} c_{ij} |u_i\rangle \otimes |v_j\rangle$ is any vector of H_w :

$$(A_u \otimes A_v)(B_u \otimes B_v) \sum_{ij} c_{ij} |u_i\rangle \otimes |v_j\rangle =$$
$$= (A_u \otimes A_v) \sum_{ij} c_{ij} B_u |u_i\rangle \otimes B_v |v_j\rangle$$
$$= \sum_{ij} c_{ij} A_u B_u |u_j\rangle \otimes A_v B_v |v_j\rangle = (A_u B_u \otimes A_v B_v) \sum_{ij} c_{ij} |u_i\rangle \otimes |v_j\rangle;$$

$$
\begin{aligned}
&(A_u + B_u) \otimes (A_v + B_v) \sum_{ij} c_{ij} |u_i\rangle \otimes |v_j\rangle = \\
&\quad = \sum_{ij} c_{ij} (A_u + B_u)|u_i\rangle \otimes (A_v + B_v)|v_j\rangle \\
&\quad = \sum_{ij} c_{ij} (A_u|u_i\rangle + B_u|u_i\rangle) \otimes (A_v|v_j\rangle + B_v|v_j\rangle) \\
&\quad = \sum_{ij} c_{ij} [(A_u|u_i\rangle) \otimes (A_v|v_j\rangle) + (A_u|u_i\rangle) \otimes (B_v|v_j\rangle) \\
&\quad + (B_u|u_i\rangle) \otimes (A_v|v_j\rangle) + (B_u|u_i\rangle) \otimes (B_v|v_j\rangle)] \\
&\quad = (A_u \otimes A_v + A_u \otimes B_v + B_u \otimes A_v + B_u \otimes B_v) \sum_{ij} c_{ij} |u_i\rangle \otimes |v_j\rangle.
\end{aligned}
$$

Corollary of theorem 7. If U_u and U_v are unitary operators on H_u and H_v, then $U_u \otimes U_v$ is a unitary operator on H_w.

Proof.

$$(U_u \otimes U_v)^\dagger (U_u \otimes U_v) = (U_u^\dagger \otimes U_v^\dagger)(U_u \otimes U_v) = U_u^\dagger U_u \otimes U_v^\dagger U_v = I_u \otimes I_v = I;$$

$$(U_u \otimes U_v)(U_u \otimes U_v)^\dagger = (U_u \otimes U_v)(U_u^\dagger \otimes U_v^\dagger) = U_u U_u^\dagger \otimes U_v U_v^\dagger = I_u \otimes I_v = I.$$

Theorem 8. If A_u and A_v are linear operators on H_u and H_v, and I_u and I_v are the identity operators on these spaces, then :

$$e^{Au \otimes Iv} e^{Iu \otimes Av} = e^{Au} \otimes e^{Av}.$$

Proof.

$$
\begin{aligned}
e^{Au \otimes Iv} e^{Iu \otimes Av} &= \sum_{n,m=0,\infty} 1/n!m!(A_u \otimes I_v)^n (I_u \otimes A_v)^m \\
= \sum_{n,m=0,\infty} 1/n!m! A_u^n \otimes A_v^m &= (\sum_{n=0,\infty} 1/n! A_u^n) \otimes (\sum_{m=0,\infty} 1/m! A_v^m) \\
= e^{Au} \otimes e^{Av}.
\end{aligned}
$$

Theorem 9. If A is a linear operator on H_w and $\{|\zeta_{ij}\rangle = |\epsilon_i\rangle \otimes |\eta_j\rangle\}$ is an orthonormal basis of H_w, then :

$$A = \sum_{ijkl} A_{ijkl} |\epsilon_i\rangle \otimes |\eta_j\rangle \langle \epsilon_k| \otimes \langle \eta_l|, \tag{3.2}$$

$$A = \sum_{ijkl} A_{ijkl} |\epsilon_i\rangle\langle\epsilon_k| \otimes |\eta_j\rangle\langle\eta_l|, \tag{3.3}$$

where $A_{ijkl} = \langle\epsilon_i| \otimes \langle\eta_j|A|\epsilon_k\rangle \otimes |\eta_l\rangle$.

The matrix A_{ijkl} will be called matrix of A with respect to the basis $\{|\zeta_{ij}\rangle\}$.

Proof. Expression (3.2) is simply the application to H_w of the usual matrix representation of an operator on a Hilbert space H. The direct proof in H_w is :

$$A = IAI = \sum_{ij} |\zeta_{ij}\rangle\langle\zeta_{ij}|A \sum_{kl} |\zeta_{kl}\rangle\langle\zeta_{kl}| = \sum_{ijkl} \langle\zeta_{ij}|A|\zeta_{kl}\rangle|\zeta_{ij}\rangle\langle\zeta_{kl}|$$
$$= \sum_{ijkl} \langle\epsilon_i| \otimes \langle\eta_j|A|\epsilon_k\rangle \otimes |\eta_l\rangle|\epsilon_i\rangle \otimes |\eta_j\rangle\langle\epsilon_k| \otimes \langle\eta_l|.$$

Expression (3.3) holds because $|\epsilon_i\rangle \otimes |\eta_j\rangle\langle\epsilon_k| \otimes \langle\eta_l| = |\epsilon_i\rangle\langle\epsilon_k| \otimes |\eta_j\rangle\langle\eta_l|$. Let us show that the matrix elements of these operators with respect to the basis $\{|\zeta_{ij}\rangle\}$ coincide.

$$\langle\epsilon_r| \otimes \langle\eta_s|\epsilon_i\rangle \otimes |\eta_j\rangle\langle\epsilon_k| \otimes \langle\eta_l|\epsilon_r\rangle \otimes |\eta_s\rangle = \langle\epsilon_r|\epsilon_i\rangle\langle\eta_s|\eta_j\rangle\langle\epsilon_k|\epsilon_r\rangle\langle\eta_l|\eta_s\rangle$$
$$= \delta_{ri}\delta_{sj}\delta_{rk}\delta_{sl}.$$

$$\langle\epsilon_r| \otimes \langle\eta_s|\epsilon_i\rangle\langle\epsilon_k| \otimes |\eta_j\rangle\langle\eta_l|\epsilon_r\rangle \otimes |\eta_s\rangle = \langle\epsilon_r|\epsilon_i\rangle\langle\eta_s|\eta_j\rangle\langle\epsilon_k|\epsilon_r\rangle\langle\eta_l|\eta_s\rangle$$
$$= \delta_{ri}\delta_{sj}\delta_{rk}\delta_{sl}.$$

Corollary of theorem 9. Every linear operator A on W can be expressed as a linear combination of tensor products of operators on H_u and on H_v :

$$A = \sum_{mn} a_{mn} A_{um} \otimes A_{vn}.$$

Proof. The proof is already contained in expression (3.3), because $|\epsilon_i\rangle\langle\epsilon_k|$ and $|\eta_j\rangle\langle\eta_l|$ are linear operators on H_u and H_v.

Matrix representation of a linear operator on H_w. As a consequence of theorem 9, every linear operator on H_w can be represented by a matrix which has as elements the coefficients A_{ijkl}. The pair $\{i, j\}$ is the row index and the pair $\{k, l\}$ is the column index. The matrix must be written by holding fixed in each row the row index and in each column the column index. Moreover, the order in the set $\{i, j\}$ must be the same as in the set $\{k, l\}$, so that the diagonal elements have $i = k$ and $j = l$, i.e. are of the kind A_{ijij}.

For instance, if $H_w = C^2 \otimes C^2$, the matrix of a linear operator A is 4×4 and the row and column indexes can be chosen either in the order $\{1, 1\}$, $\{1, 2\}$, $\{2, 1\}$, $\{2, 2\}$ or in the order $\{1, 1\}$, $\{2, 1\}$, $\{1, 2\}$, $\{2, 2\}$. With the first choice, we have :

$$[A] = \begin{pmatrix} A_{1111} & A_{1112} & A_{1121} & A_{1122} \\ A_{1211} & A_{1212} & A_{1221} & A_{1222} \\ A_{2111} & A_{2112} & A_{2121} & A_{2122} \\ A_{2211} & A_{2212} & A_{2221} & A_{2222} \end{pmatrix}$$

Matrix $[A]$ can be considered as the union of 4 blocks 2×2. In each block, the indexes i and k, which refer to basis $\{|\epsilon_i\rangle\}$, remain constant.

Theorem 10. If A and B are linear operators on H_u and H_v, then the matrix elements of $A \otimes B$ with respect to an orthonormal basis $\{|\epsilon_i\rangle \otimes |\eta_j\rangle\}$ of H_w are : $(A \otimes B)_{ijkl} = A_{ik}B_{jl}$.

Proof.

$$\begin{aligned} A \otimes B &= \sum_{ijkl} \langle\epsilon_i| \otimes \langle\eta_j|A \otimes B|\epsilon_k\rangle \otimes |\eta_l\rangle |\epsilon_i\rangle \otimes |\eta_j\rangle\langle\epsilon_k| \otimes \langle\eta_l| \\ &= \sum_{ijkl} \langle\epsilon_i| \otimes \langle\eta_j|A\epsilon_k\rangle \otimes |B\eta_l\rangle |\epsilon_i\rangle \otimes |\eta_j\rangle\langle\epsilon_k| \otimes \langle\eta_l| \\ &= \sum_{ijkl} \langle\epsilon_i|A|\epsilon_k\rangle\langle\eta_j|B|\eta_l\rangle |\epsilon_i\rangle \otimes |\eta_j\rangle\langle\epsilon_k| \otimes \langle\eta_l| \\ &= \sum_{ijkl} A_{ik}B_{jl} |\epsilon_i\rangle \otimes |\eta_j\rangle\langle\epsilon_k| \otimes \langle\eta_l|. \end{aligned}$$

Corollary of theorem 10. If A and B are linear operators on H_u and H_v, then : $Tr(A \otimes B) = TrATrB$.

Proof. On account of theorem 10, for any orthonormal basis of H_w :

$$Tr(A \otimes B) = \sum_{ij} (A \otimes B)_{ijij} = \sum_{ij} A_{ii}B_{jj} = TrATrB.$$

Partial traces of a linear operator on $H_w = H_u \overline{\otimes} H_v$.

Let A be a linear operator on $H_w = H_u \overline{\otimes} H_v$, and let $\{|\epsilon_i\rangle \otimes |\eta_j\rangle\}$ be an orthonormal basis of H_w. Then :

$$\begin{aligned} A &= \sum_{ijkl} A_{ijkl} |\epsilon_i\rangle \otimes |\eta_j\rangle\langle\epsilon_k| \otimes \langle\eta_l| \\ &= \sum_{ijkl} A_{ijkl} |\epsilon_i\rangle\langle\epsilon_k| \otimes |\eta_j\rangle\langle\eta_l|. \end{aligned}$$

We define partial trace of A in H_u (or, with respect to U) the linear operator on H_v :

$$\begin{aligned} Tr^u A &= \sum_{ijkl} A_{ijkl} Tr(|\epsilon_i\rangle\langle\epsilon_k|) |\eta_j\rangle\langle\eta_l| \\ &= \sum_{ijl} A_{ijil} |\eta_j\rangle\langle\eta_l| = \sum_{jl} (\sum_i A_{ijil}) |\eta_j\rangle\langle\eta_l|. \end{aligned} \tag{3.4}$$

Similarly, we define partial trace of A in H_v (or, with respect to V) the linear operator on H_u :

$$\begin{aligned} Tr^v A &= \sum_{ijkl} A_{ijkl} |\epsilon_i\rangle\langle\epsilon_k| Tr(|\eta_j\rangle\langle\eta_l|) \\ &= \sum_{ijk} A_{ijkj} |\epsilon_i\rangle\langle\epsilon_k| = \sum_{ik} (\sum_j A_{ijkj}) |\epsilon_i\rangle\langle\epsilon_k|. \end{aligned} \tag{3.5}$$

Theorem 11. The partial trace of A in H_u (or, in H_v) is independent of the choice of the bases $\{|\epsilon_i\rangle\}$ and $\{|\eta_j\rangle\}$.

Proof. Let $|\epsilon_i\rangle \otimes |\eta_j\rangle$ and $|\alpha_i\rangle \otimes |\beta_j\rangle$ be two bases of H_w. The second basis can be obtained from the first by means of a unitary operator : $|\alpha_i\rangle \otimes |\beta_j\rangle = (U_u \otimes U_v)|\epsilon_i\rangle \otimes |\eta_j\rangle = U_u|\epsilon_i\rangle \otimes U_v|\eta_j\rangle$. The linear operator A and its partial trace on H_u can be written as :

$$A = \sum_{ijkl} A_{ijkl} |\epsilon_i\rangle \otimes |\eta_j\rangle\langle\epsilon_k| \otimes \langle\eta_l|, \tag{3.6}$$

$$Tr^v A = \sum_{ijk} A_{ijkj} |\epsilon_i\rangle\langle\epsilon_k|, \tag{3.7}$$

$$A = \sum_{ijkl} A'_{ijkl} |\alpha_i\rangle \otimes |\beta_j\rangle\langle\alpha_k| \otimes \langle\beta_l|, \tag{3.8}$$

$$(Tr^v A)' = \sum_{ijk} A'_{ijkj} U_u |\epsilon_i\rangle\langle\epsilon_k| U_u^\dagger. \tag{3.9}$$

Let us prove that for every p and q :

$$\langle\epsilon_p| Tr^v A |\epsilon_q\rangle = \langle\epsilon_p|(Tr^v A)'|\epsilon_q\rangle.$$

$$\langle\epsilon_p| Tr^v A |\epsilon_q\rangle = \sum_{ijk} A_{ijkj} \langle\epsilon_p|\epsilon_i\rangle\langle\epsilon_k|\epsilon_q\rangle = \sum_j A_{pjqj}.$$

$$\begin{aligned}\langle\epsilon_p|(Tr^v A)'|\epsilon_q\rangle = &\sum_{ijk} \langle\epsilon_i| \otimes \langle\eta_j|(U_u^\dagger \otimes U_v^\dagger) A (U_u \otimes U_v)|\epsilon_k\rangle \\ &\otimes |\eta_j\rangle\langle\epsilon_p|U_u|\epsilon_i\rangle\langle\epsilon_k|U_u^\dagger|\epsilon_q\rangle.\end{aligned}$$

On account of (3.6) we have :

$$\begin{aligned}&\langle\epsilon_p|(Tr^v A)'|\epsilon_q\rangle = \\ &\quad \sum_{ijk} \sum_{rsmn} A_{rsmn} \langle\epsilon_i| \otimes \langle\eta_j|(U_u^\dagger \otimes U_v^\dagger)|\epsilon_r\rangle \otimes |\eta_s\rangle \\ &\quad \langle\epsilon_m| \otimes \langle\eta_n|(U_u \otimes U_v)|\epsilon_k\rangle \otimes |\eta_j\rangle\langle\epsilon_p|U_u|\epsilon_i\rangle\langle\epsilon_k|U_u^\dagger|\epsilon_q\rangle \\ &\quad = \sum_{ijk} \sum_{rsmn} A_{rsmn} (U_u^\dagger)_{ir} (U_v^\dagger)_{js} (U_u)_{mk} (U_v)_{nj} (U_u)_{pi} (U_u^\dagger)_{kq} \\ &\quad = \sum_{rsmn} A_{rsmn} [\sum_i (U_u)_{pi} (U_u^\dagger)_{ir}][\sum_j (U_v)_{nj} (U_v^\dagger)_{js}][\sum_k (U_u)_{mk} (U_u^\dagger)_{kq}] \\ &\quad = \sum_{rsmn} A_{rsmn} \delta_{pr} \delta_{ns} \delta_{mq} = \sum_s A_{psqs} = \langle\epsilon_p|Tr^v A|\epsilon_q\rangle.\end{aligned}$$

Since the operators $Tr^v A$ and $(Tr^v A)'$ have the same matrix elements with respect to the basis $\{|\epsilon_i\rangle\}$, they coincide. Obviously, the proof can be repeated for $Tr^u A$.

Theorem 12. If A_u and A_v are linear operators on H_u and H_v, and $A_w = A_u \otimes A_v$, then : $Tr^u A_w = (Tr A_u) A_v$ and $Tr^v A_w = (Tr A_v) A_u$.

Proof. Let $\{|\epsilon_i\rangle\}$ and $\{|\eta_j\rangle\}$ be orthonormal bases of H_u and H_v.

$$A_u = \sum_{ik}\langle\epsilon_i|A_u|\epsilon_k\rangle|\epsilon_i\rangle\langle\epsilon_k|$$

$$A_v = \sum_{jl}\langle\eta_j|A_v|\eta_l\rangle|\eta_j\rangle\langle\eta_l|$$

$$A_w = A_u\otimes A_v = \sum_{ijkl}\langle\epsilon_i|A_u|\epsilon_k\rangle\langle\eta_j|A_v|\eta_l\rangle|\epsilon_i\rangle\otimes|\eta_j\rangle\langle\epsilon_k|\otimes\langle\eta_l|$$

$$\begin{aligned} Tr^u A_w &= \sum_{ijl}\langle\epsilon_i|A_u|\epsilon_i\rangle\langle\eta_j|A_v|\eta_l\rangle|\eta_j\rangle\langle\eta_l| \\ &= \sum_{jl}(\sum_i\langle\epsilon_i|A_u|\epsilon_i\rangle)\langle\eta_j|A_v|\eta_l\rangle|\eta_j\rangle\langle\eta_l| = (TrA_u)A_v. \end{aligned}$$

$$\begin{aligned} Tr^v A_w &= \sum_{ijk}\langle\epsilon_i|A_u|\epsilon_k\rangle\langle\eta_j|A_v|\eta\rangle|\epsilon_i\rangle\langle\epsilon_k| \\ &= \sum_{ik}(\sum_j\langle\eta_j|A_v|\eta_j\rangle)\langle\epsilon_i|A_u|\epsilon_k\rangle|\epsilon_i\rangle\langle\epsilon_k| \\ &= (TrA_v)A_u. \end{aligned}$$

In particular, if A_u and A_v have trace 1, theorem 12 yields :

$$Tr^u A_w = A_v \quad \text{and} \quad Tr^v A_w = A_u.$$

4. Density operators on the tensor product of two Hilbert spaces and their partial traces.

In this Section, we deal with the basic relations between density operators on the tensor product of two Hilbert spaces and the partial traces of these operators. In particular, we point out the conditions in which a density operator on a tensor-product-space is determined, or is not determined, by its partial traces, i.e., under which conditions the state of a composite system is determined, or is not determined, by the states of its subsystems.

The concepts of state and of measurement will refer exclusively to ensembles [12] and the name state will be used both for pure and mixed states. We will assume as valid the following postulate :

If $\mathcal{A}$ is an observable of system U and A is the corresponding operator on H_u, then $\mathcal{A}$ is also an observable of the compound system $W = \{U, V\}$ and the corresponding operator on $H_w = H_u\overline{\otimes}H_v$ is $A\otimes I_v$, where I_v is the identity operator on H_v.

Theorem 13. If a compound system $W = \{U, V\}$ is in the state represented by the density operator ρ_w on $H_w = H_u \overline{\otimes} H_v$, then systems U and V are in the states represented by $\rho_u = Tr^v \rho_w$ and $\rho_v = Tr^u \rho_w$.

Proof. We will prove that $Tr[(A \otimes I_v)\rho_w] = Tr(A\rho_u)$ for every linear self-adjoint operator A on H_u, and that, as a consequence, ρ_u is a density operator on H_u and represents the state of system U. The proof can be repeated for system V.

Let $\{|\epsilon_i\rangle \otimes |\eta_j\rangle\}$ be an orthonormal basis of H_w, chosen so that $\{|\epsilon_i\rangle\}$ is the eigenbasis of A in H_u. This choice keeps the generality of the proof, because ρ_u is independent of the choice of the basis. Then, we have :

$$\rho_w = \sum_{ijkl} \rho_{ijkl} |\epsilon_i\rangle \otimes |\eta_j\rangle\langle\epsilon_k| \otimes \langle\eta_l|;$$

$$\begin{aligned}(A \otimes I_v)\rho_w &= \sum_{ijkl} \rho_{ijkl} |A\epsilon_i\rangle \otimes |\eta_j\rangle\langle\epsilon_k| \otimes \langle\eta_l| \\ &= \sum_{ijkl} \rho_{ijkl} a_i |\epsilon_i\rangle \otimes |\eta_j\rangle\langle\epsilon_k| \otimes \langle\eta_l|,\end{aligned}$$

where a_i is the eigenvalue of A which corresponds to $|\epsilon_i\rangle$;

$$Tr[(A \otimes I_v)\rho_w] = \sum_{ij} \rho_{ijij} a_i.$$

$$\rho_u = Tr^v \rho_w = \sum_{ijk} \rho_{ijkj} |\epsilon_i\rangle\langle\epsilon_k|;$$

$$A\rho_u = \sum_{ijk} \rho_{ijkj} a_i |\epsilon_i\rangle\langle\epsilon_k|;$$

$$Tr(A\rho_u) = \sum_{ij} \rho_{ijij} a_i = Tr[(A \otimes I_v)\rho_w]. \tag{4.1}$$

Let us now prove that ρ_u is a density operator, i.e., $Tr\rho_u = 1$ and ρ_u is positive definite.

$$Tr\rho_u = \sum_i \rho^u_{ii} = \sum_{ij} \rho_{ijij} = Tr\rho_w = 1. \tag{4.2}$$

Let $\{|\alpha_i\rangle\}$ be the eigenbasis of ρ_u ; then ρ_u can be written as : $\rho_u = \sum_k \lambda_k P_k$, where $P_k = |\alpha_k\rangle\langle\alpha_k|$. To every P_i there corresponds a projection operator $P_i \otimes I_v$ on H_w and an observable $\mathcal{P}_i$ of W [13]. The mean value of $\mathcal{P}_i$ is non-negative because $P_i \otimes I_v$ is positive definite. Therefore, on account of (4.1) we have :

$$\begin{aligned} 0 \leq \langle \mathcal{P}_i \rangle = Tr[(P_i \otimes I_v)\rho_w] = Tr(P_i \rho_u) = Tr(\sum_k \lambda_k P_i P_k) \\ = \sum_k \lambda_k Tr(P_i P_k) = \lambda_i \quad , \text{for every} \quad i. \end{aligned} \tag{4.3}$$

Equations (4.2) and (4.3) prove that ρ_u is a density operator.

Moreover, on account of (4.1) ρ_u is such that the mean value of any observable $\mathcal{A}$ of U in this state equals the mean value of $\mathcal{A}$ for system W in state ρ_w. Therefore, ρ_u represents the state of U.

On account of theorem 13, the state of a compound system always determines uniquely the states of the constituent subsystems.

Theorem 14. If ρ_u and ρ_v are density operators on H_u and H_v, and $\rho_u = \rho_u^2$ (i.e., it represents a pure state), then $\rho_w = \rho_u \otimes \rho_v$ is the unique density operator on H_w which has as partial traces ρ_u and ρ_v.

Proof. On account of theorem 12, $\rho_w = \rho_u \otimes \rho_v$ has as partial traces ρ_u and ρ_v. We must prove the uniqueness of ρ_w.

Let ρ_w be any density operator on H_w, and let $\{p_n\}$ and $\{|\psi_n\rangle\}$ be its eigenvalues and eigenvectors. Then :

$$\rho_w = \sum_n p_n |\psi_n\rangle\langle\psi_n|.$$

Let $\{|\epsilon_i\rangle\}$ and $\{|\eta_j\rangle\}$ be two orthonormal bases of H_u and H_v, chosen so that $\rho_u = |\epsilon_1\rangle\langle\epsilon_1|$. Then :

$$|\psi_n\rangle = \sum_{ij} c_{ij}^n |\epsilon_i\rangle \otimes |\eta_j\rangle,$$

$$\begin{aligned} \rho_w &= \sum_{nijkl} p_n c_{ij}^n (c_{kl}^n)^* |\epsilon_i\rangle \otimes |\eta_j\rangle\langle\epsilon_k| \otimes \langle\eta_l| \\ &= \sum_{ijkl} (\sum_n p_n c_{ij}^n (c_{kl}^n)^*) |\epsilon_i\rangle\langle\epsilon_k| \otimes |\eta_j\rangle\langle\eta_l|. \end{aligned} \tag{4.4}$$

Therefore, it will be :

$$Tr^v \rho_w = \sum_{ik} (\sum_{jn} p_n c^n_{ij} (c^n_{kj})^*) |\epsilon_i\rangle\langle\epsilon_k|. \tag{4.5}$$

However, by hypothesis :

$$Tr^v \rho_w = \rho_u = |\epsilon_1\rangle\langle\epsilon_1|. \tag{4.6}$$

By comparing expressions (4.5) and (4.6), we deduce that the internal sum in (4.5) must equal 1 for $i = k = 1$ and 0 for any other pair $\{i, k\}$. By considering the pairs $\{i, k\}$ with $k = i \neq 1$, we obtain : $\sum_{jn} p_n |C^n_{ij}|^2 = 0$. Since all coefficients are non-negative : $c^n_{ij} = 0$, for every $i \neq 1$.

Therefore, all the coefficients c^n_{ij} and c^n_{kl} which appear in (4.4) are zero, except those in which the first index equals 1. By setting in (4.4) $i = k = 1$ we obtain :

$$\begin{aligned}\rho_w &= \sum_{jl} (\sum_n p_n c^n_{1j} (c^n_{1l})^*) |\epsilon_1\rangle\langle\epsilon_1| \otimes |\eta_j\rangle\langle\eta_l| \\ &= |\epsilon_1\rangle\langle\epsilon_1| \otimes \sum_{jl} (\sum_n p_n c^n_{1j} (c^n_{1l})^*) |\eta_j\rangle\langle\eta_l| = \rho_u \otimes \rho_v.\end{aligned}$$

Theorem 15. If $\rho_u = \rho_u^2$ and $\rho_v \neq \rho_v^2$, then $\rho_w \neq \rho_w^2$.
Proof. On account of theorem 14, $\rho_w = \rho_u \otimes \rho_v$. Therefore :

$$\rho_w^2 = (\rho_u \otimes \rho_v)(\rho_u \otimes \rho_v) = \rho_u^2 \otimes \rho_v^2 = \rho_u \otimes \rho_v^2.$$

As a consequence of corollary of theorem 10 :

$$Tr\rho_w^2 = (Tr\rho_u)(Tr\rho_v^2) = Tr\rho_v^2 < 1 \quad , \qquad \text{hence} \quad \rho_w \neq \rho_w^2.$$

As a consequence of theorems 14 and 15, if ρ_w represents a pure state the following alternatives are possible :

a) both ρ_u and ρ_v represent pure states, and $\rho_w = \rho_u \otimes \rho_v$;
b) both ρ_u and ρ_v represent mixed states.

In case b), $\rho_w \neq \rho_u \otimes \rho_v$, as it can be proved by noting that :

$$Tr(\rho_u \otimes \rho_v)^2 = Tr(\rho_u^2 \otimes \rho_v^2) = Tr(\rho_u^2)Tr(\rho_v^2) < 1.$$

Theorem 16. If ρ_u and ρ_v are density operators on H_u and H_v, and none of them is a projector, then there exist infinite density operators on H_w which have as partial traces ρ_u and ρ_v.

Proof. Let us consider the operator $\tilde{\rho}_w = \rho_u \otimes \rho_v$, which has as partial traces ρ_u and ρ_v. Let $\{|\epsilon_i\rangle\}$ and $\{|\eta_j\rangle\}$ be the eigenbases of ρ_u and ρ_v. Then $\{|\epsilon_i\rangle \otimes |\eta_j\rangle\}$ is the eigenbasis of $\tilde{\rho}_w$, and the representation of ρ_w with respect to this basis is the diagonal matrix which has as elements : $\tilde{\rho}_{ijij} = \rho^u_{ii}\rho^v_{jj}$. Since neither ρ_u nor ρ_v is a projector, then each of them has at least two non-zero eigenvalues. By properly ordering the bases $\{|\epsilon_i\rangle\}$ and $\{|\eta_j\rangle\}$, the coefficients ρ^u_{11}, ρ^u_{22}, ρ^v_{11}, ρ^v_{22} will be non-zero and it will result :

$$\tilde{\rho}_{1111} = \rho^u_{11}\rho^v_{11} = \tilde{a}$$

$$\tilde{\rho}_{1212} = \rho^u_{11}\rho^v_{22} = \tilde{b}$$

$$\tilde{\rho}_{2121} = \rho^u_{22}\rho^v_{11} = \tilde{c}$$

$$\tilde{\rho}_{2222} = \rho^u_{22}\rho^v_{22} = \tilde{d}$$

where $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, $\tilde{d}$ belong to the real open interval $(0, 1)$.

By properly modifying the coefficients $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, $\tilde{d}$ and leaving the others unchanged it is possible to obtain infinite different density operators ρ_w on H_w which have as partial traces ρ_u and ρ_v and commute with $\rho_u \otimes \rho_v$. In fact, let us consider the diagonal matrix representation, $[\rho_w]_d$, of a density operator ρ_w on H_w, and let us suppose that it contains elements $\rho_{ijij} = \tilde{\rho}_{ijij}$ except :

$$\rho_{1111} = a \quad , \quad \rho_{1212} = b \quad , \quad \rho_{2121} = c \quad , \quad \rho_{2222} = d.$$

By considering the definitions of the partial traces,

$$Tr^v \rho_w = \sum_i (\sum_j \rho_{ijij})|\epsilon_i\rangle\langle\epsilon_i| \quad \text{and} \quad Tr^u \rho_w = \sum_j (\sum_i \rho_{ijij})|\eta_j\rangle\langle\eta_j|,$$

it is easily verified that ρ_w has the same partial traces as $\tilde{\rho}_w$ if :

$$\begin{aligned} a + b &= \tilde{a} + \tilde{b} = p \\ a + c &= \tilde{a} + \tilde{c} = q \\ b + d &= \tilde{b} + \tilde{d} = r \\ c + d &= \tilde{c} + \tilde{d} = s \end{aligned} \tag{4.7}$$

The given coefficients p, q, r, s belong to the open interval $(0,1)$ and the variables a, b, c, d must belong to the closed interval $[0,1]$. System (4.7) has infinite solutions $\{a,b,c,d\}$ in that interval. In fact, the determinant of the coefficients is zero and by substitution we obtain :

$$b = p - a \quad , \quad c = q - a \quad , \quad d = r - p + a \tag{4.8}$$

while the fourth equation is linearly dependent on the others and reduces to the identity : $q + r - p = s$. The variable a remains undetermined. Since a, b, c, d must belong to the interval $[0,1]$, the following conditions must hold :

$$0 \leq a \leq 1 \tag{I}$$

$$p - a \geq 0 \rightarrow a \leq p \tag{II}$$

$$p - a \leq 1 \rightarrow a \geq p - 1 \tag{III}$$

$$q - a \geq 0 \rightarrow a \leq q \tag{IV}$$

$$q - a \leq 1 \rightarrow a \geq q - 1 \tag{V}$$

$$r - p + a \geq 0 \rightarrow a \geq p - r \tag{VI}$$

$$r - p + a \leq 1 \rightarrow a \leq 1 + p - r \tag{VII}$$

Condition (I) implies (III) and (V), because $p-1$ and $q-1$ are negative. Condition (II) implies (VII), because $1 - r > 0$ and thus $1 + p - r > p$. Therefore, the relevant conditions are (I), (II), (IV), (VI), which can be rewritten as :

$$0 \leq a \leq 1 \quad , \quad p - r \leq a \leq p \quad , \quad p - r \leq a \leq q \tag{4.9}$$

Since $r > 0$, the second condition allows values of a contained in an interval of non-zero measure. The same holds for the third condition, because : $q - (p - r) = a + c - a - b + b + d = c + d = s > 0$. Moreover, the two intervals have the same first extreme and the second extreme contained in $(0,1)$. Therefore, the intersection of the three intervals defined by (4.9) is an interval I of non-zero measure. While a varies in I, we obtain the diagonal matrix representations $[\rho_w]_d$ of infinite different density operators ρ_w which have as partial traces ρ_u and ρ_v.

5. Correlations and separability

In this section, we prove that the necessary and sufficient condition for the statistical independence (non-correlation) of two systems U and

V at an instant t is $\rho_w(t) = \rho_u(t) \otimes \rho_v(t)$, and that two separable systems initially uncorrelated (or correlated) remain uncorrelated (or correlated).

Uncorrelated, perfectly correlated observables. Two commuting observables $\mathcal{A}$ and $\mathcal{B}$ of a system U will be called uncorrelated, at an instant t, if at that instant : the frequency of every pair of eigenvalues $\{a_i, b_j\}$ of $\mathcal{A}$ and $\mathcal{B}$ in simultaneous measurements equals the product of the frequencies of the eigenvalues a_i and b_j in separated measurements of $\mathcal{A}$ and $\mathcal{B}$. In symbols : $w(a_i, b_j) = w(a_i)w(b_j)$. On the contrary, $\mathcal{A}$ and $\mathcal{B}$ will be called perfectly correlated if for every eigenvalue a_i of $\mathcal{A}$ there exists an eigenvalue b_i of $\mathcal{B}$ such that : $w(a_i, b_l) = w(a_i) = w(b_l)$, $w(a_i, b_j) = 0$ if $j \neq l$.

Uncorrelated, or statistically independent, systems. Two systems U and V will be called uncorrelated, or statistically independent, at an instant t, if at that instant every observable $\mathcal{A}$ of U is uncorrelated with every observable $\mathcal{B}$ of V.

Theorem 17. Two systems U and V are statistically independent at an instant t if and only if, at that instant : for every pair of observables $\mathcal{A}$ of U and $\mathcal{B}$ of V the mean value of the observable $\mathcal{AB}$ equals the product of the mean values of $\mathcal{A}$ and $\mathcal{B}$. In symbols : $\langle \mathcal{AB} \rangle = \langle \mathcal{A} \rangle \langle \mathcal{B} \rangle$.

Proof. If U and V are statistically independent, then $\langle \mathcal{AB} \rangle = \langle \mathcal{A} \rangle \langle \mathcal{B} \rangle$ for every pair $\{\mathcal{A}, \mathcal{B}\}$. In fact : $\langle \mathcal{A} \rangle = \sum_i w(a_i) a_i$, $\langle \mathcal{B} \rangle = \sum_j w(b_j) b_j$, $\langle \mathcal{AB} \rangle = \sum_{ij} w(a_i, b_j) a_i b_j = \sum_{ij} w(a_i) w(b_j) a_i b_j = \langle \mathcal{A} \rangle \langle \mathcal{B} \rangle$.

If $\langle \mathcal{AB} \rangle = \langle \mathcal{A} \rangle \langle \mathcal{B} \rangle$ for every pair $\{\mathcal{A}, \mathcal{B}\}$, then U and V are statistically independent. Let us consider the following observables (question-observables) :

$Q(a_i)$: if a measurement of $\mathcal{A}$ yields the outcome a_i, then it yields the outcome 1 of $Q(a_i)$; otherwise it yields the outcome 0.

$Q(b_j)$: if a measurement of $\mathcal{B}$ yields the outcome b_j, then it yields the outcome 1 of $Q(b_j)$; otherwise it yields the outcome 0.

The product $Q(a_i)Q(b_j)$ has the measurement outcome 1 if the result of a simultaneous measurement of $\mathcal{A}$ and $\mathcal{B}$ is the pair $\{a_i, b_j\}$, and 0 in any other case. Therefore :

$$\langle Q(a_i)Q(b_j) \rangle = w(a_i, b_j) \quad , \quad \langle Q(a_i) \rangle = w(a_i) \quad , \quad \langle Q(b_j) \rangle = w(b_j),$$

and thus :

$$\langle Q(a_i)Q(b_j) \rangle = \langle Q(a_i) \rangle \langle Q(b_j) \rangle \quad \text{only if} \quad w(a_i, b_j) = w(a_i) w(b_j).$$

Theorem 18. Two systems U and V are uncorrelated, at an instant t, if and only if, at that instant, $\rho_w = \rho_u \otimes \rho_v$.

Proof. If $\mathcal{A}$ and $\mathcal{B}$ are observables of U and V respectively, with corresponding operators A and B, then the operator which corresponds to $\mathcal{AB}$ is $(A \otimes I_v)(I_u \otimes B) = A \otimes B$. Therefore, U and V are uncorrelated, at an instant of time t, if and only if for every pair of linear self-adjoint operators A on H_u and B on H_v which correspond to observables :

$$Tr[(A \otimes B)\rho_w] = Tr(A\rho_u)Tr(B\rho_v). \tag{5.1}$$

It is easily proved that $\rho_w = \rho_u \otimes \rho_v$ implies (5.1). In fact :

$$Tr[(A \otimes B)(\rho_u \otimes \rho_v)] = Tr(A\rho_u \otimes B\rho_v) = Tr(A\rho_u)Tr(B\rho_v).$$

as a consequence of theorem 7 and corollary of theorem 10.

Let us now prove that (5.1) implies $\rho_w = \rho_u \otimes \rho_v$ [13]. If $\{|\epsilon_i\rangle \otimes |\eta_j\rangle\}$ is an orthonormal basis of H_w, then, for every density operator ρ_w on H_w and pair $\{A, B\}$ of linear self-adjoint operators on H_u and H_v :

$$\rho_w = \sum_{ijkl} \rho_{ijkl} |\epsilon_i\rangle\langle\epsilon_k| \otimes |\eta_j\rangle\langle\eta_l|,$$

$$(A \otimes B)\rho_w = \sum_{ijkl} \rho_{ijkl} |A\epsilon_i\rangle\langle\epsilon_k| \otimes |B\eta_j\rangle\langle\eta_l|,$$

and, on account of the linearity of the trace and of corollary of theorem 10 :

$$Tr[(A \otimes B)\rho_w] = \sum_{ijkl} \rho_{ijkl} Tr(|A\epsilon_i\rangle\langle\epsilon_k|) Tr(|B\eta_j\rangle\langle\eta_l|).$$

Let us denote by ρ_u and ρ_v the partial traces of ρ_w. Then :

$$\rho_u = \sum_{ik} \rho^u_{ik} |\epsilon_i\rangle\langle\epsilon_k| \quad , \quad A\rho_u = \sum_{ik} \rho^u_{ik} |A\epsilon_i\rangle\langle\epsilon_k|,$$

$$Tr(A\rho_u) = \sum_{ik} \rho^u_{ik} Tr(|A\epsilon_i\rangle\langle\epsilon_k|).$$

$$\rho_v = \sum_{jl} \rho^v_{jl} |\eta_j\rangle\langle\eta_l| \quad , \quad B\rho_v = \sum_{jl} \rho^v_{jl} |B\eta_j\rangle\langle\eta_l|,$$

$$Tr(B\rho_v) = \sum_{jl} \rho^v_{jl} Tr(|B\eta_j\rangle\langle\eta_l|).$$

Condition (5.1) can be written as :

$$\sum_{ijkl} \rho_{ijkl} Tr(|A\epsilon_i\rangle\langle\epsilon_k|) Tr(|B\eta_j\rangle\langle\eta_l|)$$
$$= \sum_{ijkl} \rho^u_{ik}\rho^v_{jl} Tr(|A\epsilon_i\rangle\langle\epsilon_k|) Tr(|B\eta_j\rangle\langle\eta_j|),$$

i.e. :

$$\sum_{ijkl} (\rho_{ijkl} - \rho^u_{ik}\rho^v_{jl}) Tr(|A\epsilon_i\rangle\langle\epsilon_k|) Tr(|B\eta_j\rangle\langle\eta_l|) = 0. \tag{5.2}$$

By suitable choices of A and B it is possible to prove that (5.2) implies $\rho_{ijkl} = \rho^u_{ik}\rho^v_{jl}$ for every i, j, k, l, i.e., $\rho_w = \rho_u \otimes \rho_v$. In fact, let us consider the linear self-adjoint operators :

$$A_{mn} = |\epsilon_m\rangle\langle\epsilon_n| + |\epsilon_n\rangle\langle\epsilon_m| \quad , \quad B_{pq} = |\eta_p\rangle\langle\eta_q| + |\eta_q\rangle\langle\eta_p|,$$

$$A'_{mn} = i(|\epsilon_m\rangle\langle\epsilon_n| - |\epsilon_n\rangle\langle\epsilon_m|) \quad , \quad B'_{pq} = i(|\eta_p\rangle\langle\eta_q| - |\eta_q\rangle\langle\eta_p|).$$

By substituting the pair (A_{mn}, B_{pq}) in (5.2) we obtain :

$$\begin{aligned}
0 &= \sum_{ijkl} (\rho_{ijkl} - \rho^u_{ik}\rho^v_{jl}) Tr[(|\epsilon_m\rangle\langle\epsilon_n| + |\epsilon_n\rangle\langle\epsilon_m|)|\epsilon_i\rangle\langle\epsilon_k|] \\
&\qquad Tr[(|\eta_p\rangle\langle\eta_q| + |\eta_q\rangle\langle\eta_p|)|\eta_j\rangle\langle\eta_l|] \\
&= \sum_{ijkl} (\rho_{ijkl} - \rho^u_{ik}\rho^v_{jl}) Tr(\langle\epsilon_n|\epsilon_i\rangle|\epsilon_m\rangle\langle\epsilon_k| + \langle\epsilon_m|\epsilon_i\rangle|\epsilon_n\rangle\langle\epsilon_k|) \\
&\qquad Tr(\langle\eta_q|\eta_j\rangle|\eta_p\rangle\langle\eta_l| + |\eta_p|\eta_j\rangle|\eta_q\rangle\langle\eta_l|) \\
&= \sum_{ijkl} (\rho_{ijkl} - \rho^u_{ik}\rho^v_{jl})(\delta_{in}\delta_{km} + \delta_{im}\delta_{kn})(\delta_{jq}\delta_{lp} + \delta_{jp}\delta_{lq}) \\
&= \rho_{nqmp} - \rho^u_{nm}\rho^v_{qp} + \rho_{npmq} - \rho^u_{nm}\rho^v_{pq} \\
&\qquad + \rho_{mqnp} - \rho^u_{mn}\rho^v_{qp} + \rho_{mpnq} - \rho^u_{mn}\rho^v_{pq} \\
&= 2Re(\rho_{mpnq}) + 2Re(\rho_{mqnp}) - 4Re(\rho^u_{mn})Re(\rho^v_{pq}),
\end{aligned}$$

therefore :

$$Re(\rho_{mpnq}) + Re(\rho_{mqnp}) = 2Re(\rho^u_{mn})Re(\rho^v_{pq}). \tag{5.3}$$

Similarly, by substituting the pairs (A'_{mn}, B'_{pq}), (A_{mn}, B'_{pq}) and (A'_{mn}, B_{pq}) in (5.2) we obtain respectively :

$$Re(\rho_{mpnq}) - Re(\rho_{mqnp}) = -2Im(\rho^u_{mn})Im(\rho^v_{pq}), \tag{5.4}$$

$$Im(\rho_{mpnq}) - Im(\rho_{mqnp}) = 2Re(\rho^u_{mn})Im(\rho^v_{pq}), \tag{5.5}$$

$$Im(\rho_{mpnq}) + Im(\rho_{mqnp}) = 2Im(\rho^u_{mn})Re(\rho^v_{pq}). \tag{5.6}$$

By summing (5.3) and (5.4), (5.5) and (5.6) we get :

$$Re(\rho_{mpnq}) = Re(\rho^u_{mn})Re(\rho^v_{pq}) - Im(\rho^u_{mn})Im(\rho^v_{pq}), \tag{5.7}$$

$$Im(\rho_{mpnq}) = Re(\rho^u_{mn})Im(\rho^v_{pq}) + Im(\rho^u_{mn})Re(\rho^v_{pq}). \tag{5.8}$$

On account of identities :

$$Re(uv) = Re(u)Re(v) - Im(u)Im(v),$$
$$Im(uv) = Re(u)Im(v) + Im(u)Re(v),$$

which hold for every pair of complex numbers (u, v), (5.7) and (5.8) yield : $\rho_{mpnq} = \rho^u_{mn}\rho^v_{pq}$, for every m, n, p, q.

Separable system. A system will be called separable if its time evolution is uniquely determined by a hamiltonian which depends only on the generalized coordinates and momenta of the system.

The time evolution equation ensures that the energy of a separable system is conserved. Moreover, the following property holds : if $W = \{U, V\}$ is a compound system and U and V are separable, then W is separable and its hamiltonian is the sum of the hamiltonians of U and V. In quantum mechanics, the operator which corresponds to this hamiltonian is $H_w = H_u \otimes I_v + I_u \otimes H_v$.

If U and V are separable, they are also called non-interacting. In fact, their time evolutions are completely independent.

Theorem 19. Let U and V be non-interacting systems, and $W = \{U, V\}$. If, at the instant $t = 0$, $\rho_w(0) = \rho_u(0) \otimes \rho_v(0)$, then :

$$\rho_w(t) = \rho_u(t) \otimes \rho_v(t)$$

for every t.

Proof. $\rho_w(t) = U_w(t)\rho_w(0)U_w^\dagger(t)$, where :

$$U_w(t) = e^{-iH_w t} = e^{-i(H_u \otimes I_v + I_u \otimes H_v)t} = e^{-i(H_u \otimes I_v)t} e^{-i(I_u \otimes H_v)t}.$$

On account of theorem 8 :

$$U_w(t) = e^{-iH_u t} \otimes e^{-iH_v t} = U_u(t) \otimes U_v(t).$$

Therefore :

$$\rho_w(t) = [U_u(t) \otimes U_v(t)][\rho_u(0) \otimes \rho_v(0)][U_u(t) \otimes U_v(t)]^\dagger.$$

On account of theorems 6 and 7 :

$$\begin{aligned}\rho_w(t) &= [U_u(t) \otimes U_v(t)][\rho_u(0) \otimes \rho_v(0)][U_u^\dagger(t) \otimes U_v^\dagger(t)] \\ &= [U_u(t)\rho_u(0)U_u^\dagger(t)] \otimes [U_u(t)\rho_v(0)U_v^\dagger(t)] = \rho_u(t) \otimes \rho_v(t).\end{aligned}$$

Corollary of theorem 19. If U and V are non-interacting systems and $\rho_w(0) \neq \rho_u(0) \otimes \rho_v(0)$, then $\rho_w(t) \neq \rho_u(t) \otimes \rho_v(t)$ for every t.

Proof. Let us suppose that $\rho_w(t) = \rho_u(t) \otimes \rho_v(t)$. On account of the time-reversibility of the equation of motion, there would exist a time evolution of $W = \{U, V\}$ from the state $\rho_w(t) = \rho_u(t) \otimes \rho_v(t)$ to the state $\rho_w(0) \neq \rho_u(0) \otimes \rho_v(0)$, in contrast with theorem 19.

6. Wave-packet-reduction for a compound system, due to a measurement on a subsystem

In this section we study the changes of the states of a compound system W and of its subsystems U and V which are due to an ideal measurement of an observable of U according to von Neumann's wave-packet-reduction rule. In particular, we prove that for any compound system W the measurement of an observable of U does not change the state of V.

Theorem 20. For any compound system $W = U + V$ in any state ρ_w, the wave-packet-reduction due to the ideal measurement of an observable of U never changes the state of V. The post-measurement state of U can be determined by applying the wave-packet-reduction rule to U only, independently of the presence of V.

Proof. Let $\mathcal{A}$ be an observable of U and A be the linear self-adjoint operator on H_u which corresponds to $\mathcal{A}$. We will assume that A has a discrete spectrum, but can have degenerate eigenvalues a_i. If $\{|a_{ip}\rangle\}$ is the orthonormal eigenbasis of A in H_u, where p is a degeneracy index ranging from 1 to r_i, and $\{|\eta_j\rangle\}$ is any orthonormal basis of H_v, then $\{|a_{ip}\rangle \otimes |\eta_j\rangle\}$ is an orthonormal eigenbasis of $A \otimes I_v$. In fact :

$$(A \otimes I_v)|a_{ip}\rangle \otimes |\eta_j\rangle = a_i|a_{ip}\rangle \otimes |\eta_j\rangle.$$

Every eigenvalue a_i of $A \otimes I_v$ is degenerate, and its degeneracy equals the dimension of H_v times the degeneracy of a_i as eigenvalue of A.

If ρ_w is any state of W, we have, with respect to the basis $\{|a_{ip}\rangle \otimes |\eta_j\rangle\}$:

$$\rho_w = \sum_{ijkl} \sum_{\substack{p=1,r_i \\ q=1,r_k}} \alpha_{ip,j,kq,l}|a_{ip}\rangle\langle a_{kq}| \otimes |\eta_j\rangle\langle\eta_l| \tag{6.1}$$

$$\rho_v = Tr^u \rho_w = \sum_{jl}(\sum_i \sum_{p=1,r_i} \alpha_{ip,j,ip,l})|\eta_j\rangle\langle\eta_l| \tag{6.2}$$

$$\rho_u = Tr^v \rho_w = \sum_{ik} \sum_{\substack{p=1,r_i \\ q=1,r_k}} (\sum_j \alpha_{ip,j,kq,j})|a_{ip}\rangle\langle a_{kq}| \tag{6.3}$$

According to von Neumann's wave-packet-reduction rule [8], the state of W after measurements is $\hat{\rho}_w = \sum_i P_i \rho_w P_i$, where P_i is the projector on the eigenspace which corresponds to eigenvalue a_i. Therefore, we have :

$$P_m = \sum_{s=1,r_m} |a_{ms}\rangle\langle a_{ms}| \otimes I_v,$$

$$\begin{aligned}
P_m \rho_w P_m &= (\sum_{s=1,r_m} |a_{ms}\rangle\langle a_{ms}| \otimes I_v) \\
&\quad (\sum_{ijkl} \sum_{\substack{p=1,r_i \\ q=1,r_k}} \alpha_{ip,j,kq,l}|a_{ip}\rangle\langle a_{kq}| \otimes |\eta_j\rangle\langle\eta_l|)(\sum_{t=1,r_t} |a_{mt}\rangle\langle a_{mt}| \otimes I_v) \\
&= \sum_{ijkl} \sum_{\substack{s=1,r_m;p=1,r_i \\ q=1,r_k;t=1,r_m}} (\alpha_{ip,j,kq,l}|a_{ms}\rangle\langle a_{ms}|a_{ip}\rangle\langle a_{kq}|a_{mt}\rangle\langle a_{mt}|) \otimes (|\eta_j\rangle\langle\eta_l|) \\
&= \sum_{jl} \sum_{\substack{p=1,r_m \\ q=1,r_m}} \alpha_{mp,j,mq,l}|a_{mp}\rangle\langle a_{mq}| \otimes |\eta_j\rangle\langle\eta_l|.
\end{aligned}$$

$$\hat{\rho}_w = \sum_i P_i \rho_w P_i = \sum_{ijl} \sum_{\substack{p=1,r_i \\ q=1,r_i}} \alpha_{ip,j,iq,l}|a_{ip}\rangle\langle a_{iq}| \otimes |\eta_j\rangle\langle\eta_l|, \tag{6.4}$$

$$\hat{\rho}_v = Tr^u \hat{\rho}_w = \sum_{jl} (\sum_i \sum_{p=1,r_i} \alpha_{ip,j,ip,l}) |\eta_j\rangle\langle\eta_l|. \tag{6.5}$$

$$\hat{\rho}_u = Tr^v \hat{\rho}_w = \sum_i \sum_{\substack{p=1,r_i \\ q=1,r_i}} (\sum_j \alpha_{ip,j,iq,j}) |a_{ip}\rangle\langle a_{iq}|. \tag{6.6}$$

Equations (6.1) – (6.6) prove that the wave-packet-reduction of ρ_w due to the measurement of A does not change ρ_v, but normally changes ρ_w and ρ_u. We must still prove that $\hat{\rho}_u$ can be determined also by applying the wave-packet-reduction rule to ρ_u alone, i.e., by the expression:

$$\hat{\rho}_u = \sum_i P_{ui} \rho_u P_{ui},$$

where P_{ui} is the projector on the subspace of H_u spanned by the eigenvectors of A which correspond to a_i. By this method, we obtain :

$$\begin{aligned} P_{um} \rho_u P_{um} &= (\sum_{s=1,r_m} |a_{ms}\rangle\langle a_{ms}|)(\sum_{ik} \sum_{\substack{p=1,r_i \\ q=1,r_k}} (\sum_j \alpha_{ip,j,kq,j}) |a_{ip}\rangle\langle a_{kq}|) \\ &\quad (\sum_{t=1,r_m} |a_{mt}\rangle\langle a_{mt}|) \\ &= \sum_{ik} \sum_{\substack{s=1,r_m; p=1,r_i \\ q=1,r_k; t=1,r_m}} (\sum_j \alpha_{ip,j,kq,j}) |a_{ms}\rangle\langle a_{ms}|a_{ip}\rangle\langle a_{kq}|a_{mt}\rangle\langle a_{mt}| \\ &= \sum_{\substack{p=1,r_m \\ q=1,r_m}} (\sum_j \alpha_{mp,j,mq,j}) |a_{mp}\rangle\langle a_{mq}|. \end{aligned}$$

Therefore :

$$\hat{\rho}_u = \sum_i P_{ui} \rho_u P_{ui} = \sum_i \sum_{\substack{p=1,r_i \\ q=1,r_i}} (\sum_j \alpha_{ip,j,iq,j}) |a_{ip}\rangle\langle a_{iq}|,$$

in agreement with (6.6).

Restriction of theorem 20 to the case of no degeneracy of the eigenvalues of A.

If A has no degenerate eigenvalues, the proof of theorem 20 is formally simpler, as it can be easily checked. Obviously, the results of the wave-packet-reduction can be obtained by reducing (6.1) – (6.6) to the particular case of no degeneracy, and are as follows.

$$\rho_w = \sum_{ijkl} \alpha_{ijkl} |a_i\rangle\langle a_k| \otimes |\eta_j\rangle\langle\eta_l|, \tag{6.7}$$

$$\rho_v = \sum_{jl} (\sum_i \alpha_{ijil}) |\eta_j\rangle\langle\eta_l|, \tag{6.8}$$

$$\rho_u = \sum_{ik} (\sum_j \alpha_{ijkj}) |a_i\rangle\langle a_k|, \tag{6.9}$$

$$\hat{\rho}_w = \sum_{ijl} \alpha_{ijil} |a_i\rangle\langle a_i| \otimes |\eta_j\rangle\langle\eta_l|, \tag{6.10}$$

$$\hat{\rho}_v = \sum_{jl} (\sum_i \alpha_{ijil}) |\eta_j\rangle\langle\eta_l|, \tag{6.11}$$

$$\hat{\rho}_u = \sum_{i} (\sum_j \alpha_{ijij}) |a_i\rangle\langle a_i|. \tag{6.12}$$

7. Conclusions

We have presented a rigorous treatment of the main general properties of the tensor product of two Hilbert spaces and of linear operators thereon. Then, we have simplified von Neumann's treatment of density operators on such a space, and we have complemented it with some additional theorems. In particular, we have proved the following statements. The condition $\rho_w = \rho_u \otimes \rho_v$ is necessary and sufficient for the statistical independence of U and V ; the time evolution of non-interacting systems cannot create or destroy correlations ; the wave-packet-reduction due to an ideal measurement of an observable of system U does not change the state of system V.

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[11] We call basis of a linear space L a linearly independent system S of L such that every element of L can be expressed as a finite linear combination of elements of S. In Ref.[10], S is called "Hamel basis". In our treatment, the bases of L_u and of L_v will be assumed as either finite or countable.

[12] In particular, by the phrase "measurement of an observable of a system" we will mean the measurement of that observable on each member system of an ensemble.

[13] In this part of the proof we will assume that every linear self-adjoint operator with a complete spectrum corresponds to an observable.

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