About magnetic monopoles (without a string) and the Clifford bundle formalism*

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ABSTRACT. By adopting the Clifford Bundle language, we recently put forth a satisfactory lagrangian formalism for electromagnetism with magnetic monopoles without a string. In our approach, charges and monopoles do interact with one another —without violating the required gauge invariances— via a *single* type of field and of photons. Here, by taking advantage of the wellcome opportunity of some recents comments by E.Comay (and while answering them), we "complete" that formalism. In particular, we show how the Lorentz forces and the motion equations, for both electric and magnetic charges, can be *derived* from the generalized Maxwell equations: without any further recourse to a variational principle.

RESUME. En utilisant le langage des fibrés de Clifford, nous avons récemment proposé un formalisme lagrangien de l'électromagnétisme avec monopôles magnétiques, sans cordes. Dans notre approche, les charges et les monopôles interagissent bien les uns avec les autres – sans violer les invariances de jauge requises – via un unique *type de champ et de photons. Ici, saisissant l'occasion offerte par les récents commentaires de E. Comay (et en guise de réponse), nous "complétons" ce formalisme. En particulier, nous montrons comment on peut* déduire *les forces de Lorentz et les équations du mouvement, à la fois pour des charges électriques et magnétiques, des équations généralisées de Maxwell: sans recours supplémentaire à un principe variationnel.*

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Introduction

In a recent paper Comay [1] criticized our use of a Clifford bundle formalism [2] in the formulation of classical electromagnetism for charges and monopoles. The main points of his comments, according to his own words were: (i) "... if no corrections are introduced, the theory cannot include the respective Lorentz law of force. Thus, in its original form, this theory lacks a vital element required for energy-momentum conservation", (ii) "... the assumption saying that the electromagnetic fields of charges are identical to those of monopoles is non covariant in nature".

We seize the opportunity of those wellcome comments for showing how the previous points are tackled by our approach: In particular, how the Lorentz forces (for both electric and magnetic charges) can be *derived* from the (generalized) Maxwell equations. To attain clarity (for the readers that are more familiar with differential forms) we first present our generalized Maxwell equations in the Clifford bundle of differential forms (called the Kähler-Atyah-Clifford bundle and denoted by $\mathcal{K}(\tau^*M, \widehat{g})$), at variance with the presentation in ref.[2], that used the Clifford bundle of multivectors, $\mathcal{C}(\tau M, g)$. We then show that the formulation of the classical electromagnetism for charges and monopoles within $\mathcal{K}(\tau^*M, \widehat{g})$ predicts the following results:

(a) the correct Lorentz force law for charges and monopoles;

(b) the correct energy-momemtum conservation law for the system consisting of electromagnetic field plus charges and monopoles, wherefrom even the correct motion equations for both charges and monopoles are derived.

We show moreover that our formalism, which makes recourse to geometrical objects that are sums of tensors of different ranks, is Lorentz covariant, as it must be.

Let us take advantage of the present opportunity also for stressing the motivations for the adoption of our formalism. They are based on the following remarks [3,4,5]:

(I) It is wellknown that, when describing the electromagnetic field $F_{\mu\nu}$ produced by a Dirac monopole [6] in terms of one single potential A_μ only, such a potential has to be singular along an arbitrary line starting from the monopole and going to infinity. This "string" has been considered – since long – as *unphysical* [7], since the singularity in A_μ does not correspond to any singularity in $F_{\mu\nu}$;

(II) It is also wellknown that, in the $U(1)$ gauge theory of electromagnetism which has as mathematical model a Principal Fiber Bundle (PFB) $\pi : P \to M$ with group $U(1)$, magnetic monopoles appear *only* when we consider a *non trivial* base. M is in general a four dimensional Lorentzian manifold, modelling the spacetime. The standard model is obtained by taking $M = \mathbf{R}^{1,3}$ (Minkowski spacetime) and deleting from $\mathbf{R}^{1,3}$ the world line of the monopole. We then have as model the PFB $\pi : P \to \mathbf{R}^2 \times S^2$ with group $U(1)$ and the monopole charges appear as the Chern-numbers characterizing the PFB. *That is to say*, even the ordinary topological theory does *not* put on equal footing the electric charge and the monopole, since the former is introduced through the electric current and the latter is a hole moving in space time [3,8,9]. It is to be noted that the topology of spacetime becomes even more exotic when generalized monopoles are present [10].

A way out has been looked for by many authors [7,11] via the introduction of a *second potential* B_μ. But they did not completely succeed in dispensing with an exotic spacetime, whenever they wanted to stick to the ordinary vector-tensor algebra. *However* (just on the basis of both a vector potential $A \in \Lambda^1 \tau M \subset \sec \mathcal{C}(\tau M, g)$ and a pseudovector potential $\gamma_5 B \in \sec \Lambda^3 \tau M \subset \sec \mathcal{C}(\tau M, g)$), we recently constructed [2-5] a satisfactory formalism for magnetic monopoles without string (i.e., living in the ordinary Minkowski spacetime $\mathbf{R}^{1,3}$), by making recourse to the Clifford algebra $\mathbf{R}_{1,3}$, or more precisely to the Clifford-bundle $\mathcal{C}(\tau M, g)$ [where$(T_x M, g) = \mathbf{R}^{1,3}$]. Let us stress that $\mathbf{R}_{1,3}$ is an algebra sufficiently powerful to allow adding together tensors of different ranks (grades). In ref. [12], for example, both the electric and the magnetic current are vectorial, whilst in our approach they are represented by a vectorial and a pseudovectorial current, respectively (and nevertheless we can add them together [2-5]).

From Clifford to Kähler

We now pass from the $\mathcal{C}(\tau M, g)$-language, used in ref. [2], to the $\mathcal{K}(\tau^* M, \widehat{g})$-language, i.e., to the language of the differential forms in $\tau^* M$ (equipped with the Kähler-Clifford algebra [3,13,14])[1]. This paves

[1] Let us notice that the metric tensor $g \in \sec(\tau^* M \times \tau^* M)$ induces the "dual metric" $\widehat{g}$ in the space $\Lambda^k(\tau^* M)$: [3]

$$\widehat{g}(\varphi_1, \varphi_2)\gamma^5 = \varphi_1 \wedge * \varphi_2,$$

the way, incidentally, for a generalization of our "monopoles without string" to non abelian gauge groups. The new language will allow us to approach the question of a suitable formalism for interacting charges and monopoles without string from a *geometrical* point of view in the spacetime manifold [15,16].

We remember that $\mathcal{K}(T_x^*M,\widehat{g}) \simeq \mathcal{C}(T_xM,g) = \mathbf{R}_{1,3}$, which is the so-called spacetime algebra[2]. Now $\mathcal{K}(T_x^*M,\widehat{g})$, as a linear space over the real field, can be written:

$$\Lambda^0(T_x^*M) + \Lambda^1(T_x^*M) + \Lambda^0(T_x^*M) + \Lambda^3(T_x^*M) + \Lambda^4(T_x^*M) \qquad (1)$$

where $\Lambda^k(T_x^*M)$ is the $\binom{4}{k}$-dimensional space of the k-forms. Quantity $\Lambda(T_x^*M) = \sum \Lambda^k(T_x^*M)$ is called the Cartan algebra, and the pair $[\Lambda(T_x^*M), \widehat{g}_x]$ is called the Hodge algebra. An analogous terminology exists for the vector bundles associated with these algebras [3,17].

In $\mathcal{K}(\tau^*M,\widehat{g})$ there exists a particular differential operator ∂ odd in the $\mathbf{Z}_2$-gradation of the algebra [3]. To introduce ∂, consider first, for any $t^* \in \sec\tau^*M \subset \sec\mathcal{K}(\tau^*M,\widehat{g})$ and any $t \in \sec\tau M$, dual to t^*, the bilinear tensorial map of type $(1,1)$ given by

$$\Psi \to t^*\nabla_t\Psi \; , \qquad (2)$$

where Ψ is any element of $\sec\mathcal{K}(\tau^*M,\widehat{g})$ and ∇_t is the covaraint derivative of Ψ (considered as an element of the tensor bundle). Then ∂ is defined as the tensorial trace of the map:

$$\partial = Tr(\tau^*\nabla_t). \qquad (3)$$

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where $\varphi_1, \varphi_2 \in \sec(\Lambda^k\tau^*M)$. The pair $(\Lambda\tau^*M,\widehat{g})$ is the so called Hodge bundle. For future reference, note that, in the particular case where $\varphi_1 = \varphi_2 = \varphi \in \sec(\Lambda^1\tau^*M,\widehat{g})$, then

$$\widehat{g}(\varphi,\varphi) = -\widehat{g}(^*\varphi, *\varphi).$$

[2] By adopting Hestenes' notations (cf. the second one of refs. [18]), we call *spacetime algebra* the Clifford algebra $\mathbf{R}_{1,3}$ that we called "Dirac algebra" in ref. [2]. More correctly we shall reserve the name Dirac algebra for $\mathbf{R}_{4,1} = \mathbf{C}(4)$. Notice, incidentally, that the *Majorana algebra* $\mathbf{R}_{3,1}$ is quite different from $\mathbf{R}_{1,3}$, so that two algebras $[\mathbf{R}_{1,3} = \mathbf{H}(2)$ and $\mathbf{R}_{3,1} = \mathbf{R}(4)]$ can be naturally associated with Minkowski spacetime, and this can have a bearing on physics (even for the mathematical problems connected with tachyons, for instance). At last, the *Pauli algebra* is $\mathbf{R}_{3,0} = \mathbf{C}(2)$.

[3] Recall that we denote the Clifford product in $\mathcal{C}(\tau M,g)$, as well as in $\mathcal{K}(\tau^*M,\widehat{g})$, by mere *juxtaposition* of symbols.

In terms of a local basis $\{\gamma^\mu\}$ of 1-form fields and its dual basis $\{e_\mu\}$ of vector fields, we can write

$$\partial = \gamma^\mu \nabla_\mu. \tag{3'}$$

In particular, taking any local neighbourhood $U \subset M$ with a local basis $\{dx^\mu\}, \partial = \gamma^\mu \nabla_\mu$, we can show [3,14] that for any $\Psi \in \sec(\Lambda\tau^*, \widehat{g}) \subset \sec \mathcal{K}(\tau^*, \widehat{g})$:

$$\partial\Psi = dx^\mu \wedge (\nabla_\mu \Psi) + \partial_\mu \rfloor (\nabla_\mu \Psi), \tag{4}$$

where $\rfloor$ is the usual contraction operator of the theory of differential forms. We have:

$$dx^\mu \wedge (\nabla_\mu \Psi) = d\Psi \tag{5}$$

$$\partial_\mu \rfloor (\nabla_\mu \Psi) = -\delta\Psi, \tag{6}$$

where d is the usual differential, and δ is the Hodge coderivative operator here defined as:

$$\delta\Psi_k = (-1)^k *^{-1} d * \Psi_k \tag{7}$$

where $*$ is the Hodge star operator and $\Psi_k \in \sec(\Lambda^k\tau^*M, \widehat{g}) \subset \sec \mathcal{K}(\tau^*M, \widehat{g})$.

The power of the Kälher bundle formalism appears clearly once we add to the fundamental formula (a consequence [3] of eq.(4) and eqs.(5),(6))

$$\partial\Psi = (d - \delta)\Psi \tag{8}$$

the result [3,12]

$$\gamma^5 \Psi_k = (-1)^t * \Psi_k, \tag{9}$$

where $\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3$ is the volume element,[4] and $t = 1$ for $k = 1, 2$ and $t = 2$ for $k = 0, 3, 4$, in the particular case of the spacetime algebra $\mathbf{R}_{1,3}$ and with the conventions here used. We have also that $\partial^2 = (d - \delta)^2$ is the D'Alambertian operator. The use of $\mathcal{K}(\tau^*M, \widehat{g})$ is essential for the "unified" treatment of various questions in theoretical physics [17].

Observe, in fact, that the "completed" Maxwell equations, $\delta F = J_e; \;\; dF = - * J_m$, where $F \in \sec(\Lambda^2\tau^*M, \widehat{g}) \subset \sec \mathcal{K}((\tau^*M, \widehat{g})$ is the

[4] Recall that, whereas γ^5 is the volume element in $\mathcal{K}(\tau^*M, \widehat{g})$, in ref. [2] we defined $\gamma_5 = e_0e_1e_2e_3 \in \mathcal{C}(\tau M, g)$, where $\{e_\mu\}$ is an orthonormal basis of $\mathbf{R}^{1,3}$.

electromagnetic field and $J_e, J_m \in \sec(\Lambda^1\tau^*M,\widehat{g}) \subset \sec\mathcal{K}(\tau^*M,\widehat{g})$ are the electric and magnetic currents, respectively, can then be written [3,4] as a single equation:

$$\partial F = J_e - *J_m = J_e + \gamma^5 J_m \equiv \overline{J}. \tag{10}$$

With the introduction of the *generalized potential* $\overline{A} \equiv A + \gamma^5 B$, where $A, B \in \sec(\Lambda^1\tau^*M,\widehat{g}) \subset \sec\mathcal{K}(\tau^*M,\widehat{g})$, we get $F = \partial\overline{A} = \partial\wedge A + \partial(\gamma^5 B)$, if we impose the Lorentz gauge $\partial\circ\overline{A} = 0$.[5] Then we can write eq.(1), as:

$$\partial^2 A = J_e \;\; ; \;\; \partial^2 B = J_m. \tag{11}$$

In our previous work [2], we wrote eqs.(1) and (11) in $\mathcal{C}(\tau M, g)$, instead of $\mathcal{K}(\tau^*M,\widehat{g})$. There we succeeded in introducing a non conventional lagrangian which yields the correct field equations when varied with respect to the generalized potential (and when we take care of the order of the factors in the products). Our approach, however, cannot overcome the "no - go theorems" by Rosenbaum et al. [12]; for instance Rohrich [12] showed that a single Lagrangian can yield both the field equations and the charge and pole motion-equations *only* in the trivial case when $J_m = kJ_e$, where k is a constant. Here we shall show that (once the field equations are known) one can derive the motion equations, without any further recourse to a variational principle. In so doing, we shall adopt a new technique, instead of our previous hamiltonian formalism [2] (which would require some further mathematical clarification). [Let us remind, incidentally, that the essence of the use of the canonical formalism in classical (as well as in quantum) physics is the derivation of the conservation laws: actually, lagrangians are mathematical objects without physical meaning, whose only purpose is permitting the derivation of the equations of motion of the theory through the use of a variational principle. But in ref.[3] we showed, with some mathematical rigor, that no such variational principle exists for a system consisting of electromagnetic field plus charges and monopoles].

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[5] Note that the scalar product between $\Psi_r \in \Lambda^r(T_x^*M)$ and $\Psi_k \in \Lambda^k(T_x^*M)$ is defined by $\Psi_r \cdot \Psi_k = \langle \Psi_r \Psi_k \rangle_{|r-s|}$, i.e., it is the component in $\Lambda^{|r-s|}(T_x^*M)$ of the Clifford product of Ψ_r and Ψ_k.
Sometimes we make recourse also to the *ball product* ($\circ$) which, in terms of the Clifford product, is defined as follows: $A\circ B = \frac{1}{2}(AB^+ + BA^+)$. The $+$ operation, called reversion (represented by a tildle in ref. [2]), in its turn is defined as follows: $D = d_1 d_2, \ldots, d_r; D^+ = d_r, \ldots, d_2 d_1$, where the $d_i[i = 1, 2, \ldots, d_r]$ are vectors in $\mathbf{R}^{1,3}$.

Conservation Laws and Lorentz Forces

To obtain the results that follow one needs knowing how to do calculations with the Clifford algebra $\mathbf{R}_{1,3}$. The reader is referred to [18–20] for details (in what follows, incidentally, we generalize for our case results already obtained by Hestenes [18] for the ordinary Maxwell theory). Let us come, then, to our new formalism and observe that from eq.(10), by applying the antiautomorphism $+$ (reversion) [3], we get the equation

$$F^{+}\partial^{\wedge} = J_e + J_m\gamma^5, \tag{12}$$

where the symbol $\partial^{\wedge}$ means that the Dirac operator acts on the right, i.e., $F^{+}\partial^{\wedge} = -\partial_{\alpha}(F_{\mu\nu})\gamma^{\mu}\gamma^{\nu}\gamma^{\alpha}$.

Multiplying eq.(10) by F^{+} on the left and eq.(12) by F on the right, and summing, we get:

$$\frac{1}{2}(F^{+}\partial F + F^{+}\partial^{\wedge}F) = \frac{1}{2}(J_eF - FJ_e) + \frac{1}{2}(J_m\gamma^5F - \gamma^5FJ_m). \tag{13}$$

Defining moreover

$$S^{\mu} \equiv -\frac{1}{2}F^{+}\gamma^{\mu}F, \tag{14}$$

eq.(12) can be written as[6]

$$\partial_{\mu}S^{\mu} = F\cdot J_e - \ast F\cdot J_m. \tag{15}$$

Now, from eq.(14), we get immediately that $S^{\mu+} = S^{\mu}$ and $\overline{S}^{\mu} = -S^{\mu}$, where the *bar* indicates the inversion, i.e., the main antiautomorphism of the Clifford algebra [3]. The unique objects in the Clifford algebra of the differential forms that satisfy those equations are the 1-forms. We call the quantities S^{μ} the energy-momentum 1-forms. The reason for such a name is that $E^{\mu\nu} = S^{\mu}\cdot\gamma^{\nu}$ are the components of the symmetric energy-momentum tensor of the electromagnetic field, as we show below.

In particular $S^0 = -\frac{1}{2}F\,F\gamma^0,\ F = \gamma^0F\gamma^0$ and, writing $F = \vec{E} + \gamma^5\vec{B}$, we get (by projecting into the Pauli-algebra [2,3,17]) the following splitting:

$$S^0\gamma^0 = U + \vec{S}^0 \ ; \ U = \frac{1}{2}(\vec{E}^2 + \vec{B}^2) \ ; \ \vec{S}^0 = \vec{E}\times\vec{B}\ , \tag{16}$$

[6] Cf footnote 5.

which we recognize as the energy-density and the Poynting vector of the electromagnetic field, respectively. More generally we have:

$$\begin{aligned} E^{\mu\nu} &= -\langle \frac{1}{2} F\gamma^{\mu} F\gamma^{\nu}\rangle = -\langle (F\cdot\gamma^{\mu})F\gamma^{\nu}\rangle - \frac{1}{2}\langle\gamma^{\mu}F^2\gamma^{\nu}\rangle \\ &= (F\cdot\gamma^{\mu})(F\cdot\gamma^{\nu}) - \frac{1}{2}(F\cdot F)\gamma^{\mu}\cdot\gamma^{\nu} \\ &= F^{\mu\alpha}F^{\lambda\nu}\eta_{\alpha\lambda} + \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}. \end{aligned} \tag{17}$$

By writing

$$\begin{aligned} K_e = F\cdot J_e K_m = \; -*F\cdot J_m, \\ (K_e)_{\mu} = F_{\mu\nu}J_e^{\nu}(K_m)_{\mu} = \; -*F_{\mu\nu}J_m^{\nu} \end{aligned} \tag{18}$$

and projecting K_e and K_m on the Pauli-Algebra, we get:

$$\begin{aligned} K_e\gamma^0 &= \vec{j}_e\vec{E} + (\rho_e\vec{E} + \vec{j}_e\times\vec{B}) \;\; ; \qquad \text{(a)} \\ K_m\gamma^0 &= -\vec{j}_m\vec{B} + (-\rho_m\vec{B} + \vec{j}_m\times\vec{E}) \;\; . \qquad \text{(b)} \end{aligned} \tag{19}$$

We see, then, that K_e and K_m represent the Lorentz forces that act on the electric charges and the magnetic monopoles, respectively. As this result has been derived only from the Maxwell equations, we arrive at the conclusion that the (electric and magnetic) Lorentz forces don't have to be postulated, as on the contrary done, e.g., in paper [1].

We note that, due to the symmetry $E^{\mu\nu} = E^{\nu\mu}$, we can write $\partial_{\nu}E^{\mu\nu} = \partial_{\mu}E^{\nu\mu} = \partial_{\mu}(S^{\nu}.\gamma^{\mu}) = \partial S^{\nu}$. Then eq.(15) can be written:

$$\partial\cdot S^{\nu} = Q^{\nu} \quad ; \quad Q^{\nu} = (F\cdot J_e)\gamma^{\nu} - (*F\cdot J_m)\gamma^{\nu} \tag{15'}$$

The interpretation of eq.(15) is now clear. The equation

$$\partial_{\mu}E^{\mu\nu} = F^{\mu\nu}J_{e\nu} - \; *F^{\mu\nu}J_{m\nu} \tag{15''}$$

expresses the fact that the energy-momentum of the field is not conserved, $\partial_{\mu}E^{\mu\nu} \neq 0$, when matter (described by J_e, J_m) is present. Actually, one expects that only the *total* energy-momentum of field *and* currents be conserved. Actually, if we write the r.h.s. of eq.(15') as $-\partial_{\mu}M^{\mu\nu}$,

$$\partial_{\mu}E^{\mu\nu} = \gamma^{\nu}\cdot K_e + \gamma^{\nu}\cdot K_m, \tag{20}$$

then eq.(15) assumes the structure of a global conservation equation:

$$\partial_\mu (E^{\mu\nu} + M^{\mu\nu}) = 0, \tag{21}$$

where $M^{\mu\nu}$ plays the role of the symmetric energy-momentum tensor of matter (i.e. of the currents).

The Motion Equations derived from Maxwell Equations [3,4]

In analogy to what happens in general relativity, the identification of $M^{\mu\nu}$ with the actual energy-momentum tensor of the matter currents leads directly to the motion equations.

Let us show this in the simple, but easily generalizable, case in which the field F is generated by a single electric charge e and a single magnetic charge g. Be, in fact, the electric and magnetic matter represented by the triples (m_e, e, γ) and (m_g, g, σ), where m_e (m_g) are the masses of the charge (monopole), $e(g)$ are the electric (magnetic) charge and γ, σ are future-pointing timelike curves in $\mathbf{R}^{1,3}$, representing the world-lines of the charge and the monopole, respectively. The most general symmetric tensor that we can write to represent matter is then [19]:

$$\begin{aligned} M = M_{\mu\nu}\gamma^\mu \otimes^s \gamma^\nu = &- m_e \int ds\delta(x - \gamma(s))\gamma^* \otimes^s \gamma^* \\ &- m_g \int ds'\delta(x - \sigma(s'))\sigma^* \otimes^s \sigma^*. \end{aligned} \tag{22}$$

In eq.(22) $\otimes^s$ represents the symmetrical tensor product and $\gamma^* = g(\gamma_*, \ ); (\sigma^* = g(\sigma_*, \ ))$; where $\gamma_*(\sigma_* \ )$ is the tangent vector field to the curve $\gamma(\sigma)$. In components, writing $x^\mu(g(s)) = z^\mu(s); x^\mu(\sigma(s')) = y^\mu(s')$, eq.(22) reads:

$$M_{\mu\nu} = -m_e \int ds\delta(x^\alpha - z^\alpha)\frac{dz^\mu}{ds}\frac{dz^\nu}{ds} - m_g \int ds\delta(x^\alpha - y^\alpha)\frac{dy^\mu}{ds'}\frac{dyz^\nu}{ds'}. \tag{22'}$$

The classical currents are $J_e = \int ds\delta(x-\gamma(s))\gamma^*$; $J_m = \int ds\delta(x-\sigma(s'))\sigma^*$; which satisfy $\partial \cdot J_e = \partial \cdot J_m = 0$. Now, comparing eq.(19) with eq.(20) and recalling eqs.(18), it is immediately seen that:

$$m_e\ddot{z}_i = \rho_e E_i + (\vec{z} \times \vec{B})_i \quad ; \quad m_g\ddot{y}_i = \rho_g B_i + (\vec{y} \times \vec{E})_i, \tag{23}$$

which are the *correct* equations of the motion of electric and magnetic charges.

Before going on, let us take advantage of the present opportunity for pointing out some misprints appeared in the previous paper [2], that might make difficult for the interested reader to rederive those results of ours: (1) at page 234, column 2, line 18; the two expressions $\partial \cdot J$ ought rather to read $\partial \circ J$; (2) at page 235, eqs.(14) and (15): all the three expressions $\overline{J} \cdot \overline{A}$ should be written $\overline{J} \circ \overline{A}$; (3) at page 235: the last term in the r.h.s. of eq.(17) ought to be eliminated; (4) at page 236, column 1, line 22: "pseudoscalars" should be corrected into "pseudovectors". Let us stress that the "ball product"$(\circ)$ is *not* a new fundamental product, since in terms of the Clifford product we have, for $A, B \in \sec \mathcal{C}(\tau M, g)$, that [7] $A \circ B = \frac{1}{2}(AB^+ + BA^+)$.

We finally comment on the question of the Lorentz covariance of our formalism. We remember [17] that $\mathcal{K}(\tau^* M, \widehat{g})$ is a vector bundle associated to the principal bundle of orthonormal frames, i.e., $\mathcal{K}(\tau^* M, \widehat{g}) = P_{SO_+(1,3)} \times_{Ad} \mathbf{R}^{1,3}$, where $Ad : SO_+(1,3) \rightarrow \ Aut(\mathbf{R}^{1,3})$, given by: $Ad_u \phi = u \phi u^{-1}, u \in Spin_+(1,3) = Sl(2, C)$ and $\phi \in \mathbf{R}^{1,3}$ [of course, Ad means adjoint representation, and Aut means automorphisms]. This implies that all the geometrical objects of $\mathcal{K}(\tau^* M, \widehat{g})$ transform in the same way under a change of the Lorentz frame [17] and shows that the use of the Clifford bundle formalism is consistent with Lorentz invariance. It is clear that our generalized theory violates parity, but it is also clear that parity is not violated for a system consisting only of electric charges; the parity violation is predicted only when monopoles are actually present.

The results (a) and (b), derived above, show —incidentally— the elegance of the Clifford bundle formalism when applied to the classical electromagnetism of charges and monopoles.

Note added in Proofs: Recently, our attention has been called to the interesting paper by C. Daviau, [Ann. Fond. Louis de Broglie **14** (1989) 273], which also deals with magnetic monopoles by using the spacetime algebra $\mathbf{R}_{1,3}$. In that paper Daviau arrives at the conclusion that the generalized electromagnetic field can be the sum of a 0-form plus a 2-form plus a 4-form. We had reached the same conclusion in References [4,15].

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[7] Cf footnote 5.

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References

[1] E. Comay: *Phys. Letters* B**207** (1988) 56.

[2] M.A. Faria-Rosa, E. Recami and W.A. Rodrigues Jr.: *Phys. Letters* B**173** (1986) 233; B**188** (1987) 511.

[3] R. Mignani and E. Recami: *Nuovo Cimento* A**30** (1975) 533; W.A. Rodrigues Jr., M.A. Faria-Rosa, A. Maia Jr. and E. Recami: *Hadronic Journal*, vol.**12** (1989) 187.

[4] A. Maia Jr., E. Recami, W.A. Rodrigues Jr. and M.A. Faria-Rosa: *J. Math. Phys.* **31** (1990) 502; and *Mod. Phys. Letters* A**5** (1990) 543.

[5] W.A. Rodrigues Jr., E. Recami, A. Maia Jr., and M.A. Faria-Rosa: *Phys. Letters* B**220** (1989) 195.

[6] P.A.M. Dirac: *Proc. R. Soc. London* A**133** (1931) 60; *Phys. Rev.* **74** (1948) 817.

[7] N. Cabibbo and E. Ferrari: *Nuovo Cimento* **23** (1964) 1147.

[8] Y. Choquet-Bruhat, C. de Witt-Morette and M. Dillard-Bleick: *Analysis, Manifolds and Physics* (North Holland, Amsterdam, 1982), pp. 409-410. See Also T. Eguchi, P.B. Gilkey and A.J. Hanson, *Phys. Reports* **66** (1980) 213.

[9] C.N. Yang: *N.Y. Acad. of Sc.* **294** (1977) 86; E. Lubkin: *Ann. of Phys.* **23** (1963) 233.

[10] G. t'Hooft: *Nuc. Phys.* B**79** (1974) 276; A.M. Polyakov: *Sov. Phys. JEPT Letters* **20** (1974) 194.

[11] E. Ferrari: in *Tachyons, Monopoles and Related Topics*, ed. by E. Recami (North-Holland, Amsterdam, 1978), p. 203, and references therein; R.Mignani and E.Recami: *Nuovo Cimento* A**30** (1975) 533, and references therein.

[12] F. Rohrlich: *Phys. Rev.* **150** (1966) 1104; D. Rosenbaum: *Phys. Rev.* **147** (1965) 804; E.F. Carter and H.A. Comen: *Am. J. Phys.* **451** (1973) 994.

[13] E. Kähler: *Abd. Dt. Akad. Wiss. Berlin, K. fur Math. Phys. and Techn.* (1960), no. 1 and no. 4.

[14] W. Graf: *Ann. Inst. Henri Poincaré* **29** (1978) 85.

[15] A. Maia Jr. and W.A. Rodrigues Jr.: *Rev. Bras. Física* **19** (1989), 292.

[16] M.A. Faria Rosa and W.A. Rodrigues Jr.: *Mod. Phys. Letters* A**4** (1989) 175.

[17] V.L. Figueiredo, E.C. de Oliveira and W.A. Rodrigues Jr.: *Algebras, Groups and Geometries*, **7** (1990) 153; *Int. J. Theor. Phys.* **29** (1990) 411; W.A. Rodrigues Jr. and V.L. Figueiredo: *Int. J. Theor. Phys.* **29** (1990) 451; W.A. Rodrigues Jr. and E.C. de Oliveira: *Int.*

J. Theor. Phys. **29** (1990) 435; W.A. Rodrigues Jr. and Q.A.G. de Souza: in *Proc. SILARG–VII*, ed. by M. Ryan and M. Rosembaum (World Scientific, Singapore, 1991).
[18] D. Hestenes and G. Sobczyk: *Clifford Algebras to Geometrical Calculus*, (Reidel, Dordrecht, 1989).
[19] A.O. Barut: "Electromagnetic Fields on Manifolds: Betti numbers, Monopoles and Strings, and Minimal Couplings", in *Differential Geometric Methods in Mathematical Physics*, ed. by A. Dold and E. Eckmann (Springer, Berlin, 1978).
[20] See e.g. D. Hestenes: "A Unified Language for Mathematics and Physics", in *Clifford Algebras and their Applications in Mathematical Physics*, ed. by J.S.R. Chisholm and A. Common (Reidel, Dordrecht, 1986); E. Tonti: *Rend. Sem. Mat. Fis. Milano* **46** (1976) 163; P. Lounesto: *Ann. Inst. H. Poincaré* A**33** (1980) 53; I. Porteous, *Topological Geometry* (Cambridge Univ. Press, Cambridge, 1981).

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