Proper and coordinate times: a non-closed one-form and its integral?

S. Bergia, G. Morandi

Dip. di Fisica, Univ. di Bologna and I.N.F.N., Sez. di Bologna

ABSTRACT. We analyse the question whether and in what sense a 1-form may be associated with the proper time interval $d\tau$ on a Minkowski and on a curved space-time, and the conditions for its Frobenius integrability. It is stressed that the latter is an alternative way of seeing the possibility of introducing a global time coordinate in the case of a stationary space-time and a cosmic time in standard cosmology. The discussion is carried out at an intuitive level, and its substance may be considered in the teaching of relativistic theories. A comparison is made with a previous, distinct approach by Sachs and Wu.

RESUME. Nous étudions la question de savoir si, et dans quel sens, on peut associer une 1-forme aux intervalles de temps propre d'un espace-temps de Minkowski ou d'un espace-temps courbe, et les conditions dans lesquelles il est intégrable au sens de Frobenius. On souligne que cette dernière propriété est une autre façon de voir la possibilité d'introduire une coordonnée de temps globale dans le cas d'un espace-temps stationnaire et d'un temps cosmique en cosmologie standard. La discussion est faite à un niveau intuitif et sa substance peut être envisagée dans un enseignement des théories relativistes. Une comparaison est faite avec une autre approche précédemment proposée par Sachs et Wu.

1.Introduction

Proper time is characterized, in Special Relativity, by the two properties of being invariant under Lorentz transformations, as the measure of the norm of the space-time distance between two events, and of being "non-integrable", in the sense that the integral $\int_P^Q d\tau$ depends on the path chosen between events P and Q. The two properties lead one to think that its nature could be that of a non-closed 1-form; more specifically, the very fact that the value of the integral of the proper time between two events is coordinate- and parameter- independent seems to qualify it as a 1-form, while the path dependence of the integral testifies that, *if* it is a 1-form, then it cannot be closed.

If it is indeed so, it can apparently be concluded that, in Special Relativity, this 1-form is always integrable in the sense of Frobenius, that the relativistic factor γ, $\gamma = (1-\frac{v^2}{c^2})^{-1/2}$, is its integrating factor, and that the coordinate time in any Lorentz frame is the integral of the form. To reach this conclusion, one could argue as follows: read the formula relating the time coordinate t in a Lorentz frame and the proper time τ along any world line,

$$d\tau = \frac{1}{\gamma}dt, \tag{1.1}$$

as a relation between the 1-forms $\tilde{d\tau}$ (the notation is improper, since it suggests that $\tilde{d\tau}$ is an exact 1-form, and is kept here for convenience) and $\tilde{dt}$. Since t, as a coordinate, is integrable in the ordinary sense by definition, or, stated in other terms, $\tilde{dt}$ is by definition an exact 1-form, γ must be the integrating factor.

This way or arguing, however, oversimplifies the matter, as a moment's reflection will make evident. To begin with, Eq.(1.1) can be read as a relation between 1-forms on the space time manifold only if γ is seen as a function of the space-time coordinates $\gamma[v(x, y, z, t)]$ on Minkowski space-time. This requirement implies its one-valuedness. This in turn implies that Eq.(1.1) can be read as a relation between forms only if a family of non intersecting world lines is arbitrarily selected. At this stage, one is rather at a loss as to the physical meaning (if any) of such a selection. Another way of phrasing the same thing is by reading Eq.(1.1) as a relation between 1-forms for a fluid, γ being a function of its velocity field. But again, why should one be obliged to select a fluid, and what would be the meaning of this selection?

From what we have just said, it is apparent that the question must be set on altogether firmer grounds. It is also evident enough that a similar issue arises in General Relativity, where also the proper time interval is the essential metric ingredient and a global time coordinate may be introduced in special cases.

Before entering into any detail, however, let us mention a few points that should be kept in mind for a proper understanding of the discussion that follows. From our point of view there are, in particular, two basic questions that must be answered: firstly, whether, on a general space-time manifold, a 1-form field can indeed be *associated* with the proper time interval $d\tau$, and what is the nature of this association; secondly, what is the manifold where the 1-form field belongs naturally (this manifold, as it turns out, need not coincide with the space-time manifold).

Both questions have been addressed, either directly or indirectly, by Sachs and Wu (1977), whose work provides also a clue as to the meaning of the selection of a family of world lines that appeared necessary in our initial naive approach. It should also be mentioned that these authors deal from the beginning with the general case of a connected differentiable manifold rather than with the flat space-time manifold of Special Relativity.

However, our interest concentrates on the *conditions* for the possibility of *defining* a 1-form $\tilde{\omega}$ associated with the proper time interval $d\tau$ and on the *conditions for its Frobenius integrability*, which turn out to depend on the existence of a complete time-like vector field $\bar{T}$ on the space-time manifold, while the analysis of Sachs and Wu concentrates on which properties of $\bar{T}$ are related to properties of $\tilde{\omega}$. Moreover, as we will argue, alternative answers can be given to the second of the basic questions listed above, namely the one concerning the nature of the manifold where $\tilde{\omega}$ naturally belongs. There are also further aspects of the problem that do not emerge from Sachs and Wu's analysis and are worth emphasising: for instance, the peculiarities of the special-relativistic case, the comparison with thermodynamics and its non-closed Pfaffian forms; and what is it that makes $\tilde{\omega}$ in general non-closed. Finally, we are interested in reformulating the conditions for the introduction of a global time coordinate in the case of a stationary space-time and a cosmic time in standard cosmology from the general viewpoint expressed here. Due to the overall distinct viewpoints, and for pedagogical reasons (we have in mind the teaching of relativistic theories at various levels), we will proceed by gradually expanding our initial naive approach, hence altogether independently from Sachs and Wu, and make a comparison with their approach at the end.

The plan of the paper is as follows: in section 2, we try to answer the question whether a Pfaffian form, in the classical sense, may be associated with the proper time interval of Special Relativity; the answer is adfirmative (if some conventions are stipulated), provided Minkowski space-time is replaced by its tangent bundle; in section 3, a similar question is analysed, with the classical notion of Pfaffian form, however, replaced by the modern notion, requiring the invariance under any diffeomorphism of the manifold; the connection between diffeomorphisms and coordinate transformations is discussed at some length, since it is relevant for the matter dealt with; in section 4, the physical feature preventing $\tilde{\omega}$ from being closed is analysed; in section 5, the problem whether a 1-form may be associated with the proper time interval in Special Relativity is discussed from other points of view; peculiarities of the special-relativistic case are discussed; in section 6, aspects of the extension to curved manifolds are analysed; in section 7, a comparison with Sachs and Wu's approach is outlined; finally, in section 7, the conditions for the introduction of a cosmic time in standard cosmology are re-analysed from the general viewpoint followed here.

We denote vectors (1-forms) with a bar (a tilde); tensors other than vectors and 1-forms are in boldface.

2.The proper time interval as a Pfaffian form

The modern notion of differential form has acquired some distinctive features, as we will comment, with respect to the classical notion of Pfaffian form, which was not necessarily intended as a linear functional on the elements of a linear vector space. Let us first see if the proper time interval on the tangent bundle passes the test as a Pfaffian form. In order to analyse this point, we briefly recall the discussion of Pfaffian forms as given by Falk and Jung (1959).

The analysis of these authors moves from the basic notion of Process-Covering (*Prozeß-Belegung*): consider a physical quantity which, for any piece of an oriented curve on a manifold, possesses a definite value. Such a quantity determines then an association between curves S and numbers $\Omega(S)$, a *Process-Covering.* A process-covering is called linear if it has the following properties:

1. If with a curve S is associated the number Ω, then with the curve oriented in the opposite way is associated the number $-\Omega$;

2. If the curve S consists of two pieces, S' and S'', in such a way that the end-point of S' is the initial point of S'', then it holds, between the corresponding numbers: $\Omega = \Omega(S') + \Omega(S'')$;

3. Ω depends continuously and differentiably on the curve parameter;

4. by approximating a curve in terms of an *arbitrary* polygonal, one obtains at the same time an approximation of the Ω values.

The mathematical notion adequate for the description of linear process-coverings is that of Pfaffian form: indeed, Falk and Jung demonstrate the following theorem: Every linear process-covering possesses a generating Pfaffian form, and every Pfaffian form defines a linear process-covering.

It can however be concluded that the Euclidean length of a curve is not a linear process-covering, since it does not fulfill the fourth requirement (actually, it does not fulfill the first requirement either). This is illustrated in Fig.1, where the approximating polygonals have always the length 2, while the basic line has length 1.

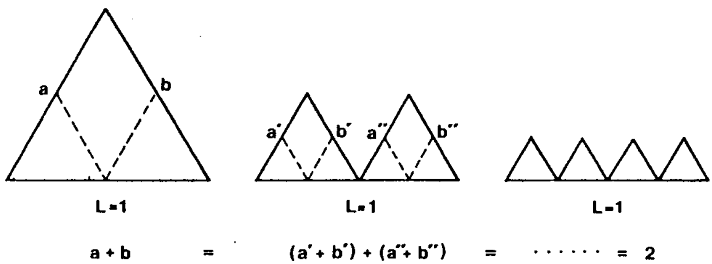

Figure 1. Approximating a straight line in terms of a sequence of polygonals not exhibiting tendency to the line's length.

In fact the approximation of the arc-length of a curve can only be achieved in terms of cord-polygonals. From this point of view, things appear at first sight to be even worse for the pseudo-Euclidean length of a curve in Minkowski space time, as illustrated in Fig.2, where the first element of a sequence of approximating polygonals to the world line of a particle at rest is taken along light lines: indeed in this case the approximation would be in terms of polygonals of vanishing lenghts! A polygonal like the one considered in Fig.2 cannot in fact be the first element of a sequence of polygonals *approximating* the world line of a particle at rest: indeed, *every* element of a sequence built from it as in Fig.1 would have, at each point of contact with the world line, fiber coordinates differing by a fixed amount from those of the latter, and the sequence would show no tendency to the world line.

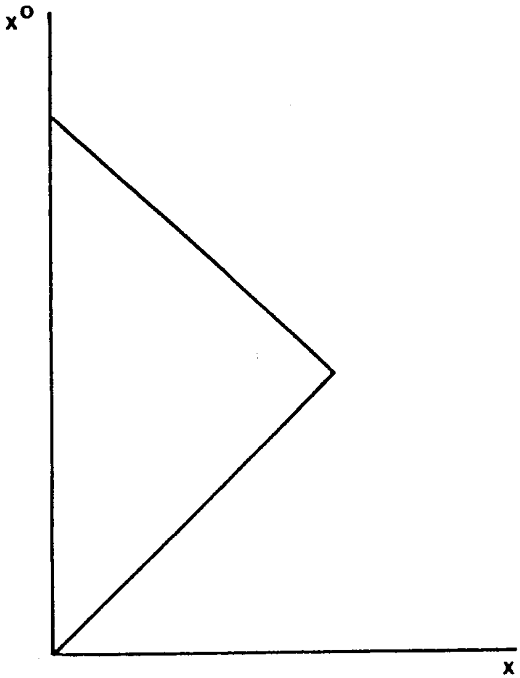

Figure 2. Approximating the world line of a particle at rest in terms of a sequence of polygonals corresponding forth and back bouncing at the speed of light.

The conclusion extends to *any* sequence of polygonals corresponding to forth and back bouncing from the world line with a fixed scalar velocity. This difficulty may however be overcome by considering the canonical lifting of the curve from the space time manifold to its tangent bundle: in this case sequences of cord polygonals may indeed produce an approximation to the curve's length. This gives a first hint for the idea that it is the tangent bundle to the space-time manifold where $\tilde{d\tau}$ could be most naturally considered a 1-form. Note that the pseudo-Euclidean length, like the Euclidean one, does not even fulfill the first of the requirements above; this is, however, a minor difficulty, since it can always be stipulated, in agreement with the fact that the square of the proper time interval is a quadratic form in the coordinates, that a negative sign must always be assigned to a world line oriented backwards in time. Up to this point one does not see therefore any obstacle to the possibility of considering $\tilde{d\tau}$ as a 1-form on the tangent bundle TM to the space time manifold. Indeed, locally on TM, any 1-form $\tilde{\omega}$ on TM can be expanded on the basis $(\{\tilde{dx}^{\mu}, \tilde{du}^{\mu}\}), \mu = 0, \ldots, 3$, with $x^0 = t$ and the u^{μ}'s coordinates along the fibres, with the coefficients that are functions on TM. In this respect, $\tilde{d\tau}$ as given by Eq. (1.1) seems to fulfill the requirements to qualify it as a 1-form, though not on the whole of TM, but on the open sets defined by $v < c$ (note that this condition is a Poincaré-invariant one). The only residual difficulty (which is shared by the Euclidean case) arises in connection with the problem of approximating straight lines (or straight world lines): in this case, indeed, the problem degenerates, since no non-trivial sequence of approximating cord polygonals can be envisaged. It seems therefore that the idea of associating a Pfaffian form with the proper time interval can be rescued if it is stipulated that approximations in terms of sequences of cord polygonals should include trivial sequences as well.

3.The proper time interval of Special Relativity as a 1-form

As we have anticipated, the modern notion of differential 1-form has acquired some distinctive features with respect to the classical notion of Pfaffian form. It is generally agreed that one cannot consider an object as a 1-form on a manifold unless it is invariant under any diffeomorphism (active viewpoint) of the manifold. It is tempting to identify at once diffeomorphisms with coordinate transformations: an identification that would pave the road toward an adfirmative answer to the question as to whether the proper time interval can indeed be interpreted as a 1-form in the modern sense.

A warning should however be given against an immediate identification of active diffeomorphisms of a manifold and coordinate transformations. Roughly speaking, coordinate transformations *are* the counterpart, from the passive viewpoint, of the (active) diffeomorphisms, much in the same way as the rotation of a Cartesian triple of axes is the counterpart of the rotation of the Euclidean space as a whole in the opposite direction. However, as stressed, in particular, by Göckeler and Schücker (1987), "coordinate transformations are not diffeomorphisms. The latter are globally defined and form a non-Abelian group. The former are in general not globally defined and do not form a group"[1]. In order to understand how this objection can be overcome in particular cases, it is perhaps worthwhile to recall wherefrom it arises. First and basic point is

[1] Göckeler (1987), p.76

that the coordinatization of a manifold can in general be achieved only locally, namely n-tuples of real numbers are associated by a map ϕ, in general a homeomorphism, with the points of an open domain of the manifold. Extension to the manifold as a whole is prevented, in general, by the fact that the global topology of the manifold is not that of an R^n (endowed with its natural topology). The very definition of a manifold requires that every point in it belongs to the domain of at least one map ϕ; coordinate transformations arise from the different representations that a neighbourhood receives from different maps. One can then endow the set of coordinate transformations with a binary composition law and the other group properties: what prevents them to form a group is, however, the fact that the elements of the set do not have the same domain (the domains can actually be disjoint). This short discussion shows that the first and basic obstacle toward the identification of diffeomorphisms and coordinate transformations is removed whenever the global topology of the space-time manifold does not differ from that of R^n. This is indeed the case for Minkowski space-time.

Apart from the difficulty just mentioned, the identification between the active and passive viewpoints is achieved as follows (for a general discussion of the relationship between diffeomorphisms and coordinate transformations, see J. Norton (1989)): with the diffeomorphism

$$f : M \longrightarrow M \tag{3.1a}$$

by

$$p \longmapsto f(p) \tag{3.1b}$$

is associated a mapping between the coordinates $\{x^i\}$ and $\{\bar{x}^i\}$ of the original and image points: the "new" coordinates, $\{x^{i'}\}$, of p under a coordinate transformation should be numerically identified with the coordinates $\{\bar{x}^i\}$ of the image point $f(p)$ of p under the diffeomorphisms.

As far as the behaviour of a geometrical object under diffeomorphisms and coordinate transformations is concerned, we shall limit ourselves, as an example, to a $\binom{0}{2}$ tensor **g**. In coordinate notation, one has the transformation laws

$$g_{ij} \longmapsto g_{\bar{i}\bar{j}} = \frac{\partial x^i}{\partial \bar{x}^i}\frac{\partial x^j}{\partial \bar{x}^j} g_{ij}, \tag{3.2a}$$

from the active viewpoint; and

$$g_{ij} \longmapsto g_{i'j'} = \frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^j}{\partial x^{j'}} g_{ij}, \tag{3.2b}$$

from the passive viewpoint. The two viewpoints describe the same state of affairs once the mentioned numerical identification between the coordinates $\{\bar{x}^i\}$ and $\{x^{i'}\}$ is made. If it is symmetric and non-degenerate, the tensor $\mathbf{g}$ is a metric.

Isometries are equivalently described in Special Relativity from the active and passive viewpoints as follows. Consider two events labeled by the coordinate sets $\{x_1^\mu\}$ and $\{x_2^\mu\}$ ($\{x_{1,2}^i\}, i = 1, 2, 3$ are *orthogonal Cartesian coordinates*) and the "vector" (the terminology is improper, and tied to the fact that R^4 can be naturally identified with its tangent space at any of its points, but harmless in this context) $\bar{X} = \{x_2^\mu - x_1^\mu\}$; its norm Δs^2 is given, in terms of the Minkowski metric η, by:

$$\Delta s^2 = \eta(\bar{X}, \bar{X}) \tag{3.3}$$

From the passive viewpoint, an isometry is a coordinate transformation $\{x^\mu\} \mapsto \{x^{\mu\prime}\}$ (in Special Relativity *only* a transformation of the Poincaré group) that leaves expression (3.3) invariant numerically and in form. From the active viewpoint, the events labeled by the coordinate sets $\{x_1^\mu\}$ and $\{x_2^\mu\}$ are mapped into the events $f(\{x_1^\mu\}) = \bar{x}_1^\mu$ and $f(\{x_2^\mu\}) = \bar{x}_2^\mu$ and the vector $\bar{X}$ into the vector $f'(\bar{X})$. The mapping is an isometry if

$$\eta(f'(\bar{X}), f'(\bar{X})) = \eta(\bar{X}, \bar{X}). \tag{3.4}$$

Once again, the two viewpoints describe the same state of affairs once the mentioned numerical identification between the coordinates $\{\bar{x}^i\}$ and $\{x^{i'}\}$ is made.

It should be noted that, notwithstanding the duality between active and passive viewpoints that can be established in the terms we have just recalled, they are employed, in Special Relativity and as far as isometries are concerned, to deal with different physical problems and in logically distinct set-ups. The basic physical assumption of Special Relativity that "observers" in relative rectilinear motions measure the same Δs^2 between two events is naturally phrased in terms of the passive viewpoint. The physical requirement is that there should be agreement between mesaurements, and the logics is: we know that a certain norm is invariant: what is then the form of the isometries? An isometry seen from the active viewpoint, on the other hand, is, for example, the mapping of the space and time unit intervals from a Lorentz frame S to a Lorentz frame S', suitably visualized by the calibration hyperbolae in a Minkowski diagram. The physical requirement is that the units be related by Lorentz boosts as Lorentz invariants, and the logics is: given the form of the isometry (i.e. the Lorentz boost), what are the image units?

Since, however, agreement between measurements performed by different observers and relation between unit measures are logically interdependent, invariance under active or passive transformations should be simultaneously required, and we can equivalently ask our candidate 1-form to be invariant under either of the two.

In Special Relativity, as a matter of fact, only isometries are considered (transformations of the Poincaré group *are* isometries). The necessity of limiting oneself to consider isometric diffeomorphisms causes a difficulty with respect to the primary requirement that a 1-form should be invariant under *any* diffeomorphism of the manifold. More generally, and stated in slightly different terms, any tensor on a manifold should by definition be invariant under any coordinate transformation (passive view-point), since its components transform contravariantly with respect to the basis. Since, in Special Relativity, the proper time interval is invariant only under a special class of diffeomorphisms, or coordinate transformations, namely the transformations of the Poincaré group, it could not, therefore, strictly speaking, be considered as a 1-form. One should however consider that tensors in Special Relativity are also such only with respect to Lorentz transformations; nevertheless, by general convention, they are referred to as *tensors.* By the same convention, one may agree to refer to $\tilde{d\tau}$ as a 1-form. .

4. What it is that prevents $\tilde{\omega}$ from being closed

We have seen in what sense and to what extent one can associate a 1-form field on the tangent bundle with Minkowski space-time, let us call it $\tilde{\omega}$, to the proper time interval $d\tau$ in Special Relativity. As we have already recalled, since its integral is path-dependent, the form must be non-closed.

One should then analyse what it is in general that prevents $\tilde{\omega}$ from being closed. It is somewhat instructive to examine the question thinking the forms given on the space time manifold. It is then compulsory to take the view (Sec.1) that we are dealing with a fluid, γ being a function of its velocity field v. Taking the external derivative of

$$\tilde{\omega} = \frac{1}{\gamma}\tilde{dt}, \tag{4.1}$$

one gets:

$$\tilde{d}\tilde{\omega} = \tilde{d}(\frac{1}{\gamma}) \wedge \tilde{d}t. \tag{4.2}$$

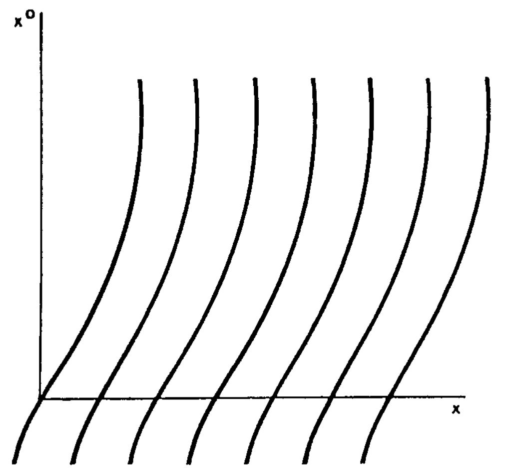

Figure 3. x-congruence of world lines.

With no loss of generality, as far as this specific problem is concerned, we can consider a manifold with one time and one space dimension; let us refer it to coordinates x and t. Then

$$\tilde{d}\tilde{\omega} = -\frac{v}{c^2}(1 - \frac{v^2}{c^2})^{-\frac{1}{2}} \frac{\partial v}{\partial x} \tilde{d}x \wedge \tilde{d}t, \tag{4.3}$$

since $\tilde{d}t \wedge \tilde{d}t$ vanishes. In order the 1-form $\tilde{\omega}$ to be closed, it must be either $v = 0$ or $\frac{\partial v}{\partial x} = 0$. The latter condition says that $\tilde{d}\tilde{\omega}$ vanishes only on classes of x-congruent world lines in the Minkowski diagram for the Lorentz frame endowed with the coordinates x and t (the former condition is then only a particular case, namely the x-congruence is realized as parallelism to the t axis); in other terms the motions of the fluid particles must share the same speed with respect to the reference frame at any given instant of time (Fig.3). General fluids, of course, do not comply with this condition. The non-closedness of the 1-form $\tilde{\omega}$ is then due to a relative velocity effect, as one would have guessed.

5. Other viewpoints

We will now analyse the possibility of associating 1-form fields with the proper time interval $d\tau$ in Special Relativity following two others lines of reasoning, slightly differing among themselves, which point out the existence of further difficulties and suggest further developments.

Consider the world line of a particle on the four dimensional space-time metric manifold of Special Relativity, and let λ parametrize it. The length of the curve's line element has the expression

$$ds^2 = \eta(\bar{T}, \bar{T}) d\lambda^2, \tag{5.1}$$

where $\bar{T} = \frac{d}{d\lambda}$ denotes the tangent vector to the world line. Then, for the element of proper time along the world line, one has;

$$d\tau = \frac{1}{c}[\eta(\bar{T}, \bar{T})]^{1/2} d\lambda. \tag{5.2}$$

Clearly, we would like to associate 1-forms, say $\tilde{\omega}$ and $\tilde{d}\lambda$, with $d\tau$ and $d\lambda$. In this respect, one would be naively tempted to formulate the following statement: on the space-time manifold one can assign a tensor field η, such that, given a differentiable curve on the manifold, parametrized by λ, hence with tangent vector $\bar{T} = \frac{d}{d\lambda}$, at any point the 1-form field $\tilde{\omega}_{(1)} = \frac{1}{c}[\eta(\bar{T}, \bar{T})]^{1/2} d\lambda$ is specified. This statement is evidently erroneous: since $\bar{T} = \frac{d}{d\lambda}$ is only defined along a world line, neither a scalar *field* $\frac{1}{c}[\eta(\bar{T}, \bar{T})]^{1/2}$ nor a 1-form *field* $\tilde{d}\lambda$ are assigned on the manifold.

There are two ways out from this difficulty. The first one, which we will now examine at some length, consists in assigning $\bar{T}$ as a field, and in associating with it the 1-form field $\tilde{\omega}_{(1)}$; $\frac{1}{c}[\eta(\bar{T}, \bar{T})]^{1/2}$ is then a scalar field on M.

The manifold on which $\tilde{\omega}_{(1)}$ is defined is in this case $M \times R$, with R the space of the parameters λ. According to Sachs and Wu (1977), the normalization condition $\eta(\bar{T}, \bar{T}) = 1$ fixes the parametrization along the integral curves of $\bar{T}$, and the parameter should then be identified with the proper time along the latter. An integral curve $\gamma : R \longrightarrow M$ of $\bar{T}$ can be lifted trivially to a curve $\tilde{\gamma} : R \longrightarrow M \times R$ by identifying the last coordinate (i.e.: $\tilde{\gamma} : \lambda \mapsto (\gamma(\lambda), \lambda)$). Then, if $\eta(\bar{T}, \bar{T}) = 1$, one easily shows that $\tilde{\gamma}^* \omega_{(1)} = \tilde{d}\lambda$, where λ is now the proper time time along the integral curves of $\bar{T}$. In this sense, $\omega_{(1)}$ is associated with proper time, but only along the family of the integral curves of $\bar{T}$. In the preceding section, it was clarified in what sense the group of diffeomorphisms under which the would-be 1-form $\tilde{\omega}_{(1)}$ is invariant is restricted, in Special Relativity, to the Poincaré group. In the framework outlined here, it is apparent that one should formally consider, as admittable diffeomorphisms (active viewpoint):

i) the group G_L of the tangent lifts of the transformations of the Poincaré group:

$$G_L : TM \longrightarrow TM \tag{5.3a}$$

$$\bar{T} \longmapsto f'(\bar{T}) \tag{5.3b}$$

ii) the reparametrizations:

$$G_R : R \longrightarrow R \tag{5.4a}$$

$$\lambda \longmapsto \lambda' = f(\lambda), f' > 0 \tag{5.4b}$$

The latter extend trivially to:

$$\tilde{G}_R : TM \times R \longrightarrow TM \times R \tag{5.5a}$$

$$(\bar{T} = \frac{d}{d\lambda}, \lambda) \longmapsto (f'(\bar{T}) = \frac{d}{d\lambda'}, \lambda' = f(\lambda)) \tag{5.5b}$$

Since

$$d\lambda \longmapsto (f')^{-1} d\lambda', \tag{5.6}$$

it is:

$$d\tau \longmapsto d\tau \tag{5.7}$$

(Note that, if $f' < 0, d\tau \longmapsto -d\tau$; this would correspond to a change of sign of the "time coordinate" -see below- and should not be considered as a drawback). Formally, the overall group is therefore the semi-direct product:

$$\tilde{G} = \tilde{G}_R \otimes G_L. \tag{5.8}$$

Looked at from the passive viewpoint, on the other hand, the question as to the invariances of the would-be 1-form $\tilde{\omega}_{(1)}$ does not introduce new elements. Indeed, invariance under reparametrizations, due to the 1-1 correspondence between $\bar{T}$ and $\tilde{d\lambda}$ established by the metric, is automatically guaranteed by the very structure of (5.2), as reflected in the semi-direct product structure of $\tilde{G}$, Eq. (5.8). As to the question of the invariance under G_L, the vector $\bar{T}$ *is* invariant (coordinate independent) under transformations of the Poincaré group like the "vector" $\bar{X}$ of section 3.

One further point that must be brought to focus is the following: even with the above specifications, $\tilde{\omega}_{(1)} = \frac{1}{c}[\eta(\bar{T},\bar{T})]^{1/2}\tilde{d}\lambda$ can be a 1-form over $M \times R$ only if its domain is restricted, at every point of M, to time-like tangent vectors, i.e. vectors such that $\eta(\bar{T},\bar{T}) > 0$; the 1-form lives therefore in the interior of the light-cone of each event.

We must now investigate what physical meaning may be attributed to the assignment, on the space-time manifold, of the vector field $\bar{T}$, a complete time like vector field generating a one-parameter group of diffeomorphisms of the manifold. Note that particles with the world lines coinciding with the orbits of the diffeomorphism would correspond to a class of "observers" sharing the time coordinate t. One can thus appreciate that the difficulty mentioned in section 1, apparently arising from an inadequate choice of the manifold, has an origin which is independent of that choice and can be understood on a physical basis. Indeed, the selection of a class of motions corresponding to the choice of a time coordinate is by no means surprising, to begin with, in Special Relativity, where the specification of a time coordinate goes with the specification of a single Lorentz frame. The parameter λ can be identified with the time coordinate t of a given Lorentz frame if and only if the family of world lines is chosen as that of the particles at rest in the frame.

These considerations deserve a little expansion. Before going into further details, however, some general remarks are in order. First of all, the conclusion has been reached that the existence of a time like complete vector field $\bar{T}$ is a necessary and sufficient condition in order that 1-form field $\tilde{\omega}_{(1)}$ on $M \times R$ may be associated with the proper time interval. The question whether this form is integrable in the sense of Frobenius is a priori a distinct one. But in this case its structure, corresponding to Eq. (5.6), with $\tilde{d}\lambda$ now an exact 1-form, automatically guarantees its Frobenius integrability. Hence the existence of a time like complete vector field $\bar{T}$ and of a global time coordinate is altogether a necessary and sufficient condition for the Frobenius integrability of $\tilde{\omega}_{(1)}$. Note that in this case the signs of τ and t are 1-1 correlated. A change of sign of τ corresponds to a change of sign of the time coordinate. The possibility that such a complete vector field may be assigned on a manifold depends in turn on its global properties, in fact on its global hyperbolicity[2]. A

[2] see, for instance, Wald (1984), Theorem 8.3.14, p. 209 and 255

comparison with thermodynamics might be profitable at this stage of the discussion. There, the object of interest is the δQ 1-form, evidently a non-closed one. In thermodynamics two cases must be distinguished: the case of a one-component fluid and the more general case of more such fluids in thermal equilibrium. In the first case, δQ, as a Pfaffian form in a two-dimensional state space, is always integrable in the sense of Frobenius: the existence of an absolute temperature T (the integrating factor) and of state-function entropy S (the 1-form integral) is then a purely mathematical fact. In the second case, the Frobenius integrability may only be guaranteed by a physical principle, which may be identified with the second principle of thermodynamics or with Caratheodory's axiom. In each case, as the consequence of Frobenius integrability, one has a foliation of the state manifold in hypersurfaces of constant S, on whose tangent spaces (its annihilators) the 1-form δQ vanishes.

In our case, the Frobenius integrability can never be a purely mathematical fact, since we deal only with four-dimensional manifolds. The integrability can therefore be guaranteed only by a physical principle, which is identified with the requirement that the space-time manifold is endowed with (at least) one time like complete vector fields.

It is to be stressed that the existence of a time like complete *Killing* vector field is a sufficient but not a necessary condition for the Frobenius integrability of $\omega_{(1)}$. With this in mind, we can now go back to a more detailed discussion of the special relativistic case.

Note that being the Minkowski space time flat, it admits four linearly independent Killing vectors, which can be chosen as the generators of the translations along the space and time axes of a given Lorentz frame, $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial t}$. The Killing vector field $\frac{\partial}{\partial t}$ can be taken as the generator of the one-parameter group of diffeomorphisms mentioned above. The orbits of the diffeomorphism are the world lines of the particles which share the state of motion corresponding to the diffeomorphism, i.e. of the particles at rest in the Lorentz frame with time coordinate t. Note that in this case the diffeomorphism is by definition an isometry. On the other hand, any linear combination of the Killing vector fields, for example with constant coefficients, will also be a Killing vector; hence, any translation in the Minkowski "plane" will be an isometry, in particular, time-like translations (boosts), to each of which a time coordinate may be associated; these time coordinates are immediately visualized in a Minkowski diagram as those of the Lorentz frames reached by the boost.

These results can be presented in the following way. Given the Minkowski space time as a metric manifold, the 1-form field $\tilde{\omega}_{(1)}$ admits an integrating factor in correspondence of any given Lorentz frame; the coordinate time t' pertaining to the Lorentz frame is its integral. Pretending that the existence of a time coordinate in any Lorentz frame was not an a priori assumption in Special Relativity, it appears that it is guaranteed by the Frobenius integrability of $\tilde{\omega}_{(1)}$, which is in turn guaranteed by the existence of a one-parameter family of time-oriented world lines mapping isometrically the manifold into itself: actually, the world lines of the particles at rest in the frame. The existence of any such time coordinate can be formulated as the possibility of foliating the space-time manifold, in any Lorentz frame, in terms of spacelike hypersurfaces labeled by t', or, in still other terms, by the fact that there is given on the manifold a time-like Killing vector field $\bar{T} = \frac{d}{dt'}$. The 1-form $\tilde{dt}'$ is exact on the entire space time manifold since the Killing vector field is complete.

In conclusion, the Frobenius integrability of the 1-form associated with the proper time interval in each Lorentz frame is guaranteed by the fact that the space time manifold admits an uniform Killing vector field $\frac{d}{dt'}$ for any space-like direction, producing a foliation of the manifold according to the values of t'.

Note that *any* integral λ of the proper time 1-form $\tilde{\omega}_{(1)}$ would provide a foliation of the space-time manifold in space-like hypersurfaces, hence a time coordinate. As has already been observed, it is not required that the field $\bar{T}$ be a Killing vector field. Of course, in the general case, the time coordinate corresponding to $\bar{T}$ would have little physical meaning from the point of view of Special Relativity: not being attached to a Lorentz frame, il would not correspond to anything measurable by its privileged observers; in particular it would not be associated with the standard operative procedure for synchronizing, and calibrating the rates of, physical clocks.

As an even more general remark, one may note that overall time coordinates are here selected by observers (here identified with frames), hence associated with states of motion; in particular, the plurality of time coordinatizations in Special Relativity is related to the absence of *one* priviledged state of motion. Whatever the residual meaning of a time coordinate *not* associated with a Killing vector field, it would necessarily have to be associated with a class of observers.

Let us now discuss a second way out of the difficulty mentioned at the beginning of this section, arising, as we may recall, from the fact that

$\bar{T} = \frac{d}{d\lambda}$ in Eq. (1) is only defined along a world line. As an alternative, one can consider the assignment, on the space-time manifold, of any time like vector field $\bar{V}$, bearing no relation with λ. A 1-form field $\tilde{\omega}_{(2)} = \sqrt{\eta(\bar{V},\bar{V})}\tilde{d}\lambda$ can nevertheless be assigned on $F \times R$, where F denotes the open submanifold of TM obtained attaching to each point of M the interior of the corresponding light cone (Sachs and Wu 1977, Section 1.2), with the same transformation group as in the previous case. Invariance under reparametrizations in then guaranteed by the group structure.

6. Extension to curved manifolds

The above remarks open the road for an extension of the observations made about proper and coordinate times to the general case of curved manifolds.

In the general-relativistic view, the element of proper time along the world line of a particle, parametrized by λ, has the expression

$$d\tau = \frac{1}{c}[\mathbf{g}(\bar{T},\bar{T})]^{1/2}d\lambda, \tag{6.1}$$

in terms of the metric tensor of Lorentzian signature $\mathbf{g}$ and of the tangent vector $\bar{T}$ to the world line. Again, one would like to associate 1-forms $\tilde{\omega}$ and $\tilde{d}\lambda$ with $d\tau$ and $d\lambda$. Hence, one needs again considering $\frac{1}{c}[\mathbf{g}(\bar{T},\bar{T})]^{1/2}$ as a scalar field on the manifold involved; in this respect the situation is exactly the same as in the special-relativistic case. Just as in that case, one is allowed to consider 1-form fields of the types $\tilde{\omega}_{(1)}$ or $\tilde{\omega}_{(2)}$ associated with the proper time interval $d\tau$, according to the two schemes outlined, with a corresponding enlargement of the manifold to $M \times R$ or $TM \times R$.

The group of diffeomorphisms under which the would-be 1-form $\tilde{d}\tau$ is invariant is no more restricted to the isometries of Special Relativity. Things are best seen from the passive viewpoint, where nothing changes with respect to the special relativistic case as far as the invariance under reparametrizations is concerned. The vector $\bar{T}$, on the other hand, is a geometric object on the manifold, and therefore coordinate independent by definition.

No new problem arises in connection with the necessity of restricting the domain of the $\tilde{\omega}'s$ on TM to time like vectors, since the causal structure of space-time is preserved locally.

As in the special-relativistic case, the only way to achieve the complete view, i.e. to introduce a time coordinate as an integral of whichever 1-form $\tilde{\omega}$, is again to specify on the space-time manifold a complete time like vector field $\bar{T}$ generating a one-parameter group of diffeomorphisms of the manifold, either in 1-1 correspondence with, or independent of, λ. Again, particles with world lines coinciding with the orbits of the diffeomorphism would correspond to a class of observers, sharing the time coordinate t. We shall have to say something more on the subject in the section devoted to cosmic time.

Curved manifolds may admit Killing vector fields. The closest contact with the situation arising in Special Relativity, and the case on which we initially focus our attention, is that of a space-time which admits a time-like complete *Killing* vector field. Such a space time, as well known, is called stationary, since its metric properties do not depend on the values of the real number parametrizing the integral curves of the field, which can be taken as a time coordinate, since the curves are time-like. That a stationary space time admits a global time coordinate is of course well known, as well as the fact that this result is a consequence of the existence of a complete time-like Killing vector field. To this we add the viewpoint that things can be read as a consequence of the Frobenius integrability (guaranteed by the Killing vector field) of the proper-time 1-form and the subsequent foliation of the spacetime manifold in terms of space-like hypersurfaces. Note that the proper-time 1-form, $\tilde{\omega}$, vanishes on the tangent spaces to the space-like hypersurfaces of the foliation (its annihilator). The diffeomorphism generated by $\bar{T}$ maps isometrically these hypersurfaces into one another: the selected "motion" is the displacement in time of the "particles" of one such section.

A particularly interesting case is that of a static spacetime, which is a stationary space time satisfying the further condition that the motions of test-bodies in it should be time-reversal invariant; this requirement forces upon the metric the further constraint that its mixed terms, g_{0j}, should vanish. In this case, the tangent vector to the integral curves of the Killing vector field is always orthogonal to the space-like surfaces of the foliation. We see here that the selection of a *unique* time coordinate t is associated with that of a *privileged* class of "observers", namely those which have as world lines the integral curves ot the time-like field (orthogonal geodesics). Space coordinates may be assigned on the space-like manifolds at constant t in such a way that points on orthogonal geodesics have fixed space coordinates (co-moving coordinates). As well known, standard clocks at different space points desynchronize even in a static field; however, a recalibration of their rates may in principle be envisaged[3], as well as a synchronization procedure[4]. Note that if the space manifold is further requested to possess spherical symmetry (as in the case of the Solar field), co-moving coordinates can be chosen as the spherical coordinates θ and ϕ and the 'area coordinate' r; since metrical properties do not depend on θ and ϕ, the class of privileged observers is in this case identified as that of the observers at constant r.

As we have repeatedly stressed, it is not necessary that $\bar{T}$ be a Killing vector field. A time coordinate on a curved space-time can be globally defined if a time like complete vector field can be assigned on the space-time manifold.

Global hyperbolicity of the manifold, as previously recalled, is a sufficient condition for this purpose. The corresponding foliation of the space-time manifold by Cauchy surfaces, parametrized by a global time function, is essential for the initial value formulation of General Relativity[5] and its Hamiltonian formulation, since the latter provides perhaps the best motivation for the viewpoint that Einstein's equations describe the evolution of the spatial metric with time[6]. What is relevant in this context is *the possibility* of defining a global time coordinate. However, there is no *preferred* time coordinate, in correspondence with the fact that there is no privileged class of physical observers. However, there is a unique time variable in Hamiltonian quantum mechanics. In constructing the quantum mechanics of General Relativity as a classical field theory, a first task is to identify this preferred time, and there is no general consensus as to wherefrom the choice should arise[7].

7.A comparison with previous analyses

The mathematical features of the association between proper time and a 1-form field have been analysed from a distinct standpoint by Sachs and Wu (1977)[8]. These authors consider a space-time connected, time-oriented, metric manifold $(M, \mathbf{g}, D)$ ($\mathbf{g}$ is the metric tensor of Lorentzian

[3] see, for instance, Rindler (1969), p. 119 ff.
[4] see, for instance, Landau (1975), p. 234 ff.
[5] see, for instance, Wald (1984), chapter 10
[6] Wald (1984), p. 450
[7] Hartle (1989), p. 7
[8] p. 52 ff.; see also Rodrigues and Rosa (1989).

signature, D the Levi Civita connection). The tangent vector to a curve γ (world line)

$$\gamma : E \longrightarrow M, E \subseteq R \tag{7.1}$$

at a point p is denoted as $\gamma_*(u)$, its norm as $|\gamma_*|$. A world line can always be parametrized in terms of the proper time. Formally, this parametrization is characterized by the condition

$$|\gamma_*(u)| = 1, \forall u \in E; \tag{7.2}$$

a time like future-oriented curve satisfying (7.2) is called an observer; γ_* is the the observer's four-velocity. A reference frame on a space-time manifold is a vector field $\bar{T}$, whose integral curves are observers. The metric tensor $\mathbf{g}$ determines a bijection

$$\mathbf{g} : TM \longrightarrow T^*M \tag{7.3a}$$

by

$$\bar{X} \longmapsto \tilde{\omega}_{\bar{X}} = \mathbf{g}(\bar{X}, \cdot). \tag{7.3b}$$

Denote, in particular, with $\tilde{\omega}_{(3)}$ the 1-form field associated with a reference system $\bar{T}$,

$$\tilde{\omega}_{(3)} = \mathbf{g}(\bar{T}, \cdot). \tag{7.4}$$

$\tilde{\omega}_{(3)}$ is the quantity that corresponds to our 1-form fields $\tilde{\omega}_{(1)}$. Note that $\tilde{\omega}_{(3)}$ is defined, by Eq. (7.4), on the manifold M.

The authors' approach shows clearly that the proper time is just a real number u parametrizing a curve, in no way belonging in the manifold. To find a contact with it, one must go through the pull-back γ^* of $\tilde{\omega}$ associated with the mapping (7.1); indeed, as the authors show, it is:

$$\gamma^* \tilde{\omega}_{(3)} = du \tag{7.5}$$

(our metric). Note that Eq. (7.5) can be extended to our $\tilde{\omega}_{(1,2)}$. The discussion up to this point shows how our intuitive approach to the idea of a proper time 1-form field can be anchored on firmer ground. We now continue the comparison between Sachs and Wu's analysis and our discussion by summarising the latter authors' presentation of the integrability problem. Whereas our emphasis has been on the *conditions* for the possibility of *defining* a 1-form field $\tilde{d\tau}$ and on the *conditions for its Frobenius integrability*, which turns out to depend on the existence of a complete time like vector field $\bar{T}$, their interest is on which properties of $\bar{T}$ are related to properties of $\tilde{\omega}_{(3)}$. $\bar{T}$ is *defined* as

a) locally synchronizable iff

$$\tilde{\omega} \wedge d\tilde{\omega} = 0 \tag{7.6a}$$

b) locally proper time synchronizable iff

$$d\tilde{\omega} = 0 \tag{7.6b}$$

c)synchronizable iff there exist functions h and t on M such that

$$\tilde{\omega} = h\tilde{d}t \tag{7.6c}$$

d) proper time synchronizable iff there exists a function t such that

$$\tilde{\omega} = \tilde{d}t \tag{7.6d}$$

An immediate contact with our discussion is established for case c), corresponding to Frobenius integrability; d) is evidently the case of exactness; a) and b) are the local versions of c) and d); in particular, a) is the condition for the validity of Frobenius theorem, and implies c) only locally; similarly, b) implies d) only locally (the converse Poincaré lemma).

Note that we are not interested in local conditions, nor, in general, in the case of exactness, which is clearly non-physical.

8.Cosmic time

As well known, the cosmological principle of standard cosmology implies that the space-time continuum is foliated by a one-parameter family of space like hypersurfaces of homogeneity[9]. The time coordinate is not yet univocally determined at this stage. However, isotropic observers must have world lines perpendicular to their hypersurfaces of homogeneity[10] (perpendicular geodesics). This requirement identifies a privileged time coordinate as the arc-length along perpendicular geodesics.

[9] see, for instance, Wald (1984), p. 92
[10] ibidem, p. 93

In this case, the existence of a privileged time coordinate corresponds to the selection of a privileged class of observers, the isotropic observers of the substratum.

The association of the emergence of a cosmic time with the existence of a privileged class of observers is a constant in the history of modern cosmology, ever since the formulation of Einstein's static model, where cosmic time is the time that would be measured by clocks associated with a reference system in which matter is on the average at rest. Its necessity became apparent after the formulation of de Sitter's model, whose physical implications were hard to decipher until Weyl supposed that the stars (galaxies in later contexts) lie on a pencil of geodesics diverging from a common event in the past[11], thus providing, at a time, a family of preferred motions and a universal time (the so-called Weyl's principle).

The association is to such extent a necessary one that nowadays a cosmological model is defined as the union of a space-time manifold and a class of observers[12].

This necessity is made more evident in the framework we have outlined, in which the existence of *one* complete time like vector field specifies both a global unique time coordinate and a privileged class of observers.

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[11] see, for instance, North (1965), p. 100 ff.
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