De Broglie's hypothesis and splitting of energy level for a relativistic material particle

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ABSTRACT. A material particle having inherent rest mass m_0 is moving with uniform linear velocity v with respect to a stationary observer B. According to Einstein's special theory of relativity the rest energy of the particle is $E_0 = m_0c^2$. While in motion the particle's external kinetic energy is increased to $m_0c^2(1-\beta^2)^{-1/2}$ with respect to the stationary observer, where $\beta = v/c$. Following Planck's quanta in the radiation energy $E = h\nu$, de Broglie introduced quanta into relativistic dynamics with frequency ν_0 associated with the eigen mass m_0. The fragment of quantum energy is thus associated with Einstein's energy equation by $E_0 = m_0c^2 = h\nu_0 = E_1$. Einstein relativistic increase of kinetic energy for the moving material particle means that the stationary observer B observes an increased frequency $\nu_2 = \nu_0(1-\beta^2)^{-1/2}$ for the moving particle. On the other hand, Lorentz-Einstein time transformation envisages slowing down of the periodic phenomenon in a moving particle in the ratio of 1 to $(1-\beta^2)^{1/2}$ for the stationary observer B also. Thus the reduced frequency is $\nu_3 = \nu_0(1-\beta^2)^{1/2}$. So following Planck, Lorentz, Einstein and de Broglie the stationary observer B observes two different frequencies ν_2 and ν_3 for the moving material particle. Considering de Broglie's idea of associating quanta to a moving material particle we can reconstruct the two energy levels for the particle by Planck $h\nu$ concept. The two energy levels for the moving material particle are : $E_2 = h\nu_2 = m_0c^2(1-\beta^2)^{-1/2}$ and $E_3 = h\nu_3 = m_0c^2(1-\beta^2)^{1/2}$. The rest energy $E_0 = m_0c^2$ may be conceived as the inherent potential energy of the material particle. Thus, E_2 is the increased kinetic energy and E_3 is the reduced potential energy of the particle. The experimental proof for the existence of energy level E_2 is well known in the charged particle accelerated by the machine. The existence of the energy level E_3 is also forthcoming. Calculations show that the decrease of "potential energy" correctly gives the "fine structure terms" of hydrogen atomic spectra which were also calculated by Sommerfeld considering relativistic motion of electron in the elliptic orbit. The above potential energy difference $(E_0 - E_3) = m_0c^2 - m_0c^2(1-\beta^2)^{1/2}$ seems to be a hidden real physical quantity. It may possibly help us in understanding some basic problems in physics.

RESUME. L. de Broglie a introduit, en suivant Planck et Einstein, une fréquence quantique pour une particule matérielle de masse au repos m_0 par $\nu_0 = m_0c^2/h = \nu_1$. Quand cette particule se déplace à la vitesse v, l'accroissement bien connu de la masse donne une fréquence accrue $\nu_2 = m_0c^2(1-\beta^2)^{-1/2}/h$ pour un observateur immobile B, avec $\beta = v/c$. La transformation de Lorentz prédit le ralentissement d'un phénomène périodique dans le rapport 1 à $(1-\beta^2)^{1/2}$. Donc B observe une fréquence réduite $\nu_3 = m_0c^2(1-\beta^2)^{1/2}/h$. En remontant à l'énergie, nous avons donc trois niveaux d'énergie, $h\nu_1, h\nu_2, h\nu_3$. Nous appellerons le troisième "énergie potentielle diminuée" $E_3 = m_0c^2(1-\beta^2)^{1/2}$. L'existence d'une "différence d'énergie potentielle", comme une "différence d'énergie cinétique", est corroborée par "les termes de structure fine du spectre de l'atome d'hydrogène". Cette variation d'énergie potentielle inhérente à la particule en mouvement existe aussi bien pour la mécanique de Newton que pour celle d'Einstein. Elle paraît être une quantité physique réelle mais cachée. Elle peut nous aider à comprendre d'importants problèmes de physique.

1. Introduction

The most outstanding result of the special theory of relativity of Einstein [1,2] is that the mass of a body is a measure of its energy content. The first person to give the correct relationship was Poincaré [3] who suggested that the electromagnetic energy should possess a mass density to be given by $(1/c^2)$ times the energy density. Einstein [2] was first to prove that if the energy changes by L the mass will change by (L/c^2). The variation of dimensions of a moving body and the concept of local time for a moving system travelling with uniform velocity v was first given by Lorentz (4) in his transformation equations. Lorentz also gave the concept of longitudinal and transverse mass of electron. Einstein [1] gave the presently accepted relation between mass variation with linear velocity v.

In this paper we reexamine the well known work and the concept of Planck [5,6], Einstein [1] and de Broglie [7]. In Section 2 we give the proof that within the frame work of Planck's quantum hypothesis, Einstein's special theory of relativity and de Broglie's wave mechanics there exists another energy level dependent on the velocity of the particle, in addition to the rest energy and increased kinetic energy, to a stationary observer. We combine the concepts of Planck, Einstein and de Broglie for understanding de Broglie's two wave phenomena in a moving material particle. In Section 3 we show the splitting of original rest energy $E_0 = E_1$ into two levels E_2 and E_3 when the particle moves with uniform velocity v. Section 4 deals with energy difference and brings out some interesting results. In Section 5 we give the experimental support for the existence of above third energy level through splitting of energy level for a relativistic electron. The same splitting was explained earlier by Sommerfeld [8] through the concept of elliptic orbit and relativistic motion of electron for hydrogen atom fine structure terms.

2. Combining the concepts of quantum hypothesis, relativity theory and wave mechanics

A material particle having inherent rest mass m_0 has rest Energy E_0 according to Einstein's relativity theory [1,2]

$$E_0 = m_0 c^2 = E_1 \tag{2.1}$$

Let us call this –first energy level E_1. This can be conceived as potential energy. When the above particle moves with uniform velocity v with respect to a stationary observer the rest energy is increased. This increased energy E_{in} is only given in all text-books Eddington [9], Synge [10], Goldstein [11] and Goldenberg [12] as

$$E_{in} = m_0 c^2 (1 - v^2/c^2)^{-1/2} = m_0 c^2 (1 - \beta^2)^{-1/2} = E_2 \tag{2.2}$$

Let us call this –second energy level E_2. Here, $\beta = v/c$. The difference between the energy levels E_2 and E_1 is called the relativistic kinetic energy T.

$$T = m_0 c^2 (1 - \beta^2)^{-1/2} - m_0 c^2 \tag{2.3}$$

No text-book or scientific literature gives the decrease of rest or potential energy for a relativistic material particle. The purpose of this paper is to prove the existence of a decreased energy E_{de} or a third energy E_3, in addition to above two, for a relativistic material particle. This will be done following the arguments of de Broglie [7].

We shall briefly recollect de Broglie's arguments [7] as they will help us directly to develop the concept of the third energy level for a relativistic material particle.

The energy E of a harmonic oscillator was given by Planck [3,4] as

$$E = h\nu \tag{2.4}$$

where $h =$ Planck constant, ν frequency.

According to de Broglie [7], we may extend Planck's idea of associating a frequency ν with an isolated quantum of energy E, to associating a frequency ν_0 with an isolated fragment of energy E_0 which is inherent in an eigen mass m_0 of a stationary particle. Thus the relation (2.1) is extended to

$$E_0 = m_0 c^2 = h\nu_0 \tag{2.1}$$

As we have called it earlier the first energy level E_1, the relation becomes

$$E_1 = m_0 c^2 = h\nu_1 \tag{2.5}$$

Following (2.1) we call this –the first frequency ν_1.

$$\nu_1 = m_0 c^2 / h \tag{2.6}$$

Here, the frequency ν_1 is measured in the system fixed to the particle. The frequency ν_1 is particle's inherent body frequency. The above frequency may be conceived as the frequency of particle's personal clock. If the particle moves with uniform velocity v we would like to know what will happen to the above energy and the frequency. Let us take a closer look at the above phenomenon from the point of view of relativistic kinematic, with two observers B and B' stationary at two origins O and O' of two cartesian coordinate systems $S(x, y, z, ict)$ and $S'(x', y', z', ict')$ respectively. The particle with inherent mass m_0 is lying at rest at the origin O' of the system S'. Following Einstein [1] we further assume that at time $t = t' = 0$ both the systems are stationary, O and O' are coincident and similar coordinate axes are coincident. Equations (2.5) and (2.6) are true for both the observers B and B'. Now the system S' with the observer B' and the particle at rest at the origin O' is raised to a final uniform velocity v along the x-direction. The x-axis and x'-axis remain coincident, the z-axis and z'-axis remain parallel. We may have to apply a Newtonian force [13] for a limited time Δt to the particle lying at rest at O' in the system S'. We intend to study the system only after the system is stabilized and attend a uniform velocity v along the x-direction. It is quite obvious that the observer B' will not find any change in the material particle as both of them are stationary with respect to each other. But the observer B is stationary in S-system. He will observe the increase of mass and increase of energy according to Einstein [1] for the moving material particle eqn. (2.2). Following Planck [6] quantum hypothesis the stationary observer B will also see an increased frequency ν_{in}. As we have called it earlier second energy level E_2 the relations become

$$\begin{aligned} E_2 &= m_0c^2(1-\beta^2)^{-1/2} = h\nu_{in} = h\nu_2 \\ \nu_{in} &= m_0c^2(1-\beta^2)^{-1/2}/h \end{aligned} \tag{2.7}$$

$$\text{or } \nu_2 = m_0c^2(1-\beta^2)^{-1/2}/h \tag{2.8}$$

Let us call this –second frequency ν_2. This is an increased frequency.

The Lorentz-Einstein time transformation tells us that a periodic phenomenon associated with the moving body slows down, i.e. there is a time dilatation. A striking verification of the increase of time period is provided by the observed increase in the apparent life time of high energy μ-mesons from cosmic rays by Rossi and Hall [14]. So the stationary observer will see the frequency ν_1 of the above first periodic phenomenon reduced in the ratio $1 : (1-\beta^2)^{1/2}$ as shown by de Broglie [7]. Thus the inherent body frequency ν_1 will decrease to a value ν_{de}. Thus the relation is

$$\nu_{de} = \nu_0(1-\beta^2)^{1/2} = m_0c^2(1-\beta^2)^{1/2}/h$$

or

$$\nu_3 = m_0c^2(1-\beta^2)^{1/2}/h \tag{2.9}$$

Let us call this –third frequency ν_3. This is a decreased frequency and was correctly predicted by de Broglie [7]. Now, we can reconstruct from eqn. (2.9) logically a decreased energy E_{de} as

$$E_{de} = h\nu_{de} = m_0c^2(1-\beta^2)^{1/2}$$

or

$$E_3 = m_0c^2(1-\beta^2)^{1/2} = h\nu_3 \tag{2.10}$$

Naturally, we call this –the third energy level E_3. We reaffirm, this is a decreased energy with respect to the rest energy E_1 of the particle given by eqn. (2.5).

Thus, the stationary observer B according to Planck-Lorentz-Einstein-de Broglie will observe two different periodic phenomena in the uniformly moving material particle having two different frequencies ν_2 and ν_3 given by eqns. (2.8) and (2.9) respectively. De Broglie [7] solved this paradox of two different observable periodic phenomena and frequencies observed by the stationary observer B for a moving material particle by assuming that the above two periodic phenomena exist and they are always in same phase for the stationary observer B. There is truth in it. He then, we all know, went on to deduce the phase velocity V, group velocity U and the wavelength λ associated with the material particle moving with uniform velocity v as

$$V = c/\beta = c^2/v \tag{2.11}$$

$$U = v \tag{2.12}$$

$$\lambda = h/[m_0 v^2 (1-\beta^2)^{-1/2}] \tag{2.13}$$

3. Splitting of energy levels for a uniformly moving material particle

If the quantum hypothesis, relativity theory and wave mechanics assign three frequencies ν_1, ν_2 and ν_3 to a material particle for a stationary observer B, it is obvious that there will be three energy E_1, E_2 and E_3 as given by eqns. (2.5), (2.7) and (2.10) respectively. When the particle is stationary with respect to the stationary observer B the energy is E_1. When the particle is moving with uniform velocity v along x-direction with respect to the stationary observer B, the energy is split into two levels E_2 and E_3. The three energy levels, L_1, L_2 and L_3 are shown in Fig.1. The three corresponding frequencies are : ν_1 = the particle's inherent clock frequency noted by the stationary observer B when the particle is stationary in the system of the observer B. ν_2 = the increased frequency noted by the stationary observer B due to the well established phenomenon of increase of mass and energy of the moving particle. ν_3 = the decreased frequency noted by the stationary observer B due to the well known phenomenon of slowing down of clock in the moving system of the particle.

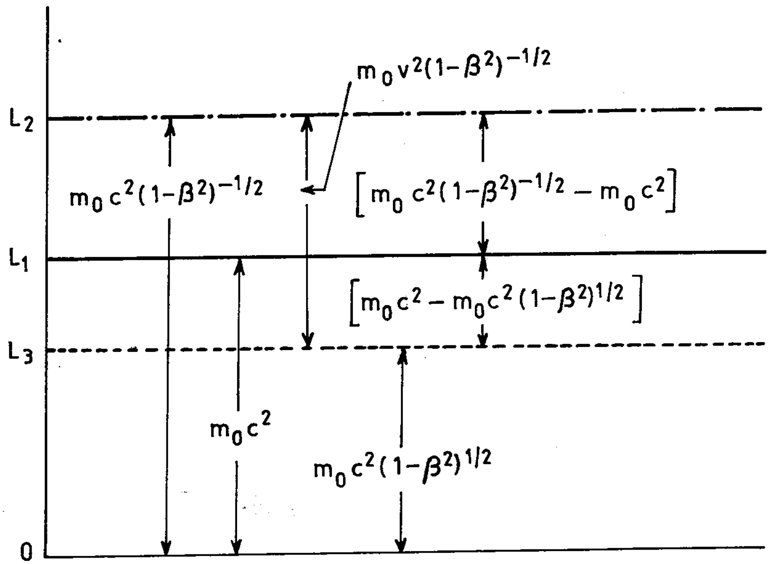

Figure 1. Energy level diagram for a relativistic material particle.

4. Difference of energy

From Sections 2 and 3 it is clear that the stationary observer B observes two energy changes for a uniformly moving particle with respect to its stationary state :

(A) *First Change*: E_2 = Increased total relativistic energy of the particle when in motion; E_1 = Inherent energy of the particle at rest. So, from the stationary state the total energy is increased, and the kinetic energy difference is

$$T_{diff} = [m_0c^2(1-\beta^2)^{-1/2} - m_0c^2] = E_2 - E_1 \qquad (4.1)$$

(B) *Second Change*: E_1 = Inherent or "Potential" energy of the particle at rest; E_3 = Decreased "Relativistic Potential" energy of the particle when in motion. So, from the stationary state the potential energy is decreased, and the potential energy difference is

$$P_{diff} = [m_0c^2 - m_0c^2(1-\beta^2)^{1/2}] = E_1 - E_3 \tag{4.2}$$

Hence, logically the total change of energy associated with kinetic motion of the particle is "Sum of eqn. (4.1) and (4.2)"

$$\begin{aligned} E_{ch} &= [m_0c^2(1-\beta^2)^{-1/2} - m_0c^2] + [m_0c^2 - m_0c^2(1-\beta^2)^{1/2}] \\ &= m_0v^2(1-\beta^2)^{-1/2} \end{aligned} \tag{4.3}$$

Equations (4.1) and (4.2) may be expanded respectively as follows :

$$\begin{aligned} \text{Increase of kinetic energy } &= [m_0c^2(1-\beta^2)^{-1/2} - m_0c^2] \\ &= [\frac{1}{2}m_0v^2 + \frac{3}{8}m_0\frac{v^4}{c^2} + \frac{5}{16}m_0\frac{v^6}{c^4} + \frac{35}{128}m_0\frac{v^8}{c^6} + \ldots] \end{aligned} \tag{4.4}$$

$$\begin{aligned} \text{Decrease of potential energy } &= [m_0c^2 - m_0c^2(1-\beta^2)^{1/2}] \\ &= [\frac{1}{2}m_0v^2 + \frac{1}{8}m_0\cdot\frac{v^4}{c^2} + \frac{1}{16}m_0\frac{v^6}{c^4} + \frac{5}{128}m_0\frac{v^8}{c^6} + \ldots] \end{aligned} \tag{4.5}$$

It can be shown that the series represented by eqn. (4.4) is divergent and that by (4.5) is convergent. In the nonrelativistic approximation when $v \ll c$, both eqns. (4.4) and (4.5) respectively representing change of kinetic energy and change of potential energy approach $(1/2)m_0v^2$. i.e.

$$T_{diff} \simeq P_{diff} = \frac{1}{2}m_0v^2 \tag{4.6}$$

Hence, logically in the nonrelativistic sense the total change of energy for a moving material particle is

$$E_{ch} = \frac{1}{2}m_0v^2 + \frac{1}{2}m_0v^2 = m_0v^2 \tag{4.7}$$

This is an interesting result, Kundu [15]. This may possibly help us in understanding some fundamental problems of physics including wave-particle duality aspect correctly predicted by de Broglie [7].

5. Fine structure terms of hydrogen atom explained through decrease of potential energy

The kinetic energy difference eqn. (4.1) for relativistic particle has been verified in accelerating machines. The potential energy difference eqn. (4.2) for a moving electron can be verified from fine structure terms in spectroscopy. We know that Sommerfeld [8] considered relativistic Kepler motion of an electron around a nucleus and explained the fine structure terms of hydrogen atomic spectra. We know further that Rojansky [16] stated the remarkable and strange fact that Sommerfeld's formula gave same results as given by the newer quantum theories where both relativity and spin were simultaneously taken into account. Further, if the relativity or spin is taken into account only individually the results do not tally with those of Sommerfeld. Sommerfeld gave the energy equation considering kinetic energy, electrostatic potential energy of the electron and balanced them as

$$E_{kin} + E_{pot} = W \tag{5.1}$$

or

$$[m_0c^2(1-\beta^2)^{-1/2} - m_0c^2] - Ze^2/r = W \tag{5.2}$$

We identify the above energy change same as the decrease of relativistic potential energy eqn. (4.2)

$$W = -[m_0c^2 - m_0c^2(1-\beta^2)^{1/2}] \tag{5.3}$$

We now have correspondance on the following two accounts :

5.1 Term value of hydrogen atomic spectra

Sommerfeld [17] obtained the solution for energy or term value from eqn. (5.2) as

$$m_0c^2/[1+\alpha^2Z^2/\{n_r+(n_\Phi^2+\alpha^2Z^2)^{1/2}\}^2]^{-1/2} - m_0c^2 = W \tag{5.1.1}$$

Taking $n_\Phi = k$, $n_\Phi + n_r = n$, and expanding LHS of eqn. (5.1.1) upto two terms we obtain

$$[-(1/2)m_0c^2(\alpha Z)^2/n^2 - \{(1/2)m_0c^2(\alpha Z)^4/n^4\}\{n/k - 3/4 + \ldots\}] = W \tag{5.1.2}$$

where α-Sommerfeld fine structure constant, n, k-principal and auxiliary quantum number, Z-atomic number etc. Similarly we can expand RHS of eqn. (5.3) upto two terms and we obtain

$$W = [-(1/2)m_0v^2 - (1/8)m_0v^2(v^2/c^2)] \tag{5.1.3}$$

The term value $\overline{\nu}$ cm^{-1} can be found dividing LHS of eqn. (5.1.2) and RHS of (5.1.3) by hc.

For $n = k$, the term values $\overline{\nu_1}$, $\overline{\nu_2}$ cm^{-1} for hydrogen atomic spectra have been calculated by Sommerfeld [8,17] and our method are shown in Table-1.

Table-1. Term Values $\bar{\nu}_1$, $\bar{\nu}_2$ for Hydrogen Atomic Spectra, unit cm^{-1}

Quantum No.	By Sommerfeld's method (Term value)		By our method (Term value)	
	$\bar{\nu}_1$	$\bar{\nu}_2$	$\bar{\nu}_1$	$\bar{\nu}_2$
n = k = 1	109737.310	1.46091440	109737.300	1.46091430
n = k = 2	27434.325	0.09130714	27434.320	0.09130710
n = k = 3	12193.034	0.01803590	12193.033	0.01803590
n = k = 4	6858.582	0.00570669	6858.580	0.00570669

There is complete agreement between term values calculated by LHS of Sommerfeld method eqn. (5.1.2) and RHS of our method eqn. (5.1.3). This proves the correctness of our approach. The above two sides of two equations are equal, i.e.

$$\begin{aligned}&[-(1/2)m_0c^2(\alpha Z)^2/n^2 - \{(1/2)m_0c^2(\alpha Z)^4/n^4\}\{n/k - 3/4\}] \\ &\qquad = W = [-(1/2)m_0v^2 - (1/8)m_0v^2(v^2/c^2)]\end{aligned} \tag{5.1.4}$$

5.2 Extended term values

Agreement is obtained even if more terms are included in the Sommefeld's equation (5.2). Expanding Sommerfeld's equation to four term values in (5.1.1) we obtain

$$\begin{aligned}\overline{\nu} =&[-RZ^2\{1/n^2\} - (R\alpha^2 Z^4/n^4)\{n/k - 3/4\}\\&- (R\alpha^4 Z^6/n^6)\{(1/4)(n/k)^3 + (3/4)(n/k)^2 - (3/2)(n/k) + (5/8)\}\\&- \{(R\alpha^6 Z^8/n^8)(1/8)(n/k)^5 + (3/8)(n/k)^4 + (1/8)(n/k)^3\\&- (15/8)(n/k)^2 + (15/8)(n/k) - (35/64)\}]\end{aligned} \tag{5.2.1}$$

with $n = k$

$$\begin{aligned}\overline{\nu} =&[-(1/2)\{m_0v^2/(hc)\} - (1/8)\{m_0v^4/(hc^3)\}\\&- (1/16)\{m_0v^6/(hc^5)\} - (5/128)\{m_0v^8/(hc^7)\}]\end{aligned} \tag{5.2.2}$$

Our decrease of relativistic potential energy eqn.(5.3) when expanded to four terms, the term values become

$$\begin{aligned}\overline{\nu} =&[-(1/2)\{m_0v^2/(hc)\} - (1/8)\{m_0v^4/(hc^3)\\&- (1/16)\{m_0v^6/(hc^5)\} - (5/128)\{m_0v^8/(hc^7)\}]\end{aligned} \tag{5.2.3}$$

There is remarkable agreement between eqns. (5.2.2) and (5.2.3). The former has been obtained by Sommerfeld [8,17] from the relativity theory and illiptic orbit with the help of integration through complex variables. Whereas the latter has been obtained from the relativity theory and the simple concept of decrease of relativistic potential energy difference.

6. Discussion and conclusion

For a stationary observer B the rest energy for a material particle is $E_0 = E_1 = m_0c^2$. The preceding arguments, deductions and experimental evidences show that when the material particle is uniformly moving with velocity v, for the above stationary observer the kinetic energy level of the particle is not only increased to $E_2 = m_0c^2(1 - \beta^2)^{-1/2}$ but also the internal rest or potential energy level is decreased to $E_3 = m_0c^2(1 - \beta^2)^{1/2}$, Fig.1. The paper thus presents a broader view on the energy and energy level concept of a relativistic material particle. Further, the familiar result of release of rest energy, say from the electron positron annihilation, may be explained from another approach i.e. the potential energy difference equation (4.2)

$$[P_{diff} = m_0c^2 - m_0c^2\big(1 - \frac{v^2}{c^2}\big)^{1/2}] \tag{4.2}$$

At $v = c$, the second term on the RHS of the above equation giving the variable part of relativistic potential energy comes to zero. Thus the difference of potential energy now represented by the inherent potential energy m_0c^2 is released for the stationary observer B.

We have seen that Sommerfeld's relativistic energy equation (5.2) is same as our decrease of relativistic potential energy equation (5.3) for an electron moving with high velocity around the nucleus. The term values $\overline{\nu_1}$ and $\overline{\nu}_2$ cm^{-1} given by both methods are exactly same as given by eqn. (5.1.4) and in Table-1. The equivalence of above two equations (5.2) and (5.3) is complete when we extend them to four term values and find they are also exactly same.

Another interesting result of this study is that we already know, in the nonrelativistic Newtonian approximation the increase of kinetic energy eqn. (4.4) gives $\frac{1}{2}m_0v^2$. The decrease of potential energy eqn. (4.5) also gives $\frac{1}{2}m_0v^2$. Thus the total change of energy is m_0v^2 for a particle of rest mass m_0 moving with a small uniform velocity, $v \ll c$, eqn. (4.7). This may possibly help us in understanding and solving some basic problems in physics Kundu [15].

Finally, we feel that the change in internal rest or inherent potential energy is a real and a physical quantity. It is a hidden quantity. This concept for a moving material particle, and even for a material body, from Einstein relativistic mechanics to Newton classical mechanics, will probably have far reaching consequences in physics.

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