Unified field theory revisited

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ABSTRACT. After reviewing some of the unified field theories of the contemporary period, the author's approach is outlined, demonstrating a unification of the gravitational and electromagnetic field theories, in the sense of Einstein, with an underlying 16-component quaternion field formalism. The electromagnetic part of the formalism is shown to reveal a current density that conforms with that of wave mechanics, in the local limit of the generally covariant unified field theory.

RESUME. Après avoir passé en revue quelques unes des théories du champ unifié contemporaines, les grandes lignes de l'approche de l'auteur sont rappelées. Elles montrent une unification des théories des champs gravitationnel et électromagnétique, au sens d'Einstein, grâce à un formalisme de champ de quaternions à 16 composantes. On montre que la partie électromagnétique de ce formalisme fournit une densité de courant qui, par passage à la limite locale de la théorie du champ unifié généralement covariant, s'accorde avec celle de la mécanique quantique.

1. Introduction.

I will start out with a brief history of unified field theories, thence to explain the basic differences between Einstein's concept of a unified field theory and that of the present day elementary particle approaches. Finally, I will outline my own resolution of this problem in general relativity. The latter is based on a theoretical development shown in detail in my two books, *General Relativity and Matter* (Reidel, 1982) and *Quantum Mechanics from General Relativity* (Reidel, 1986).[1]

There has been a great deal of discussion in this century and the last one on the subject of a unified field theory. In the 19th century the idea was proposed by Michael Faraday that the fundamental variables of matter are continuous functions of the independent space and time variables, rather than the discrete variables of singular things of matter. To Faraday, the fields are to represent the continuously distributed 'potentialities' for matter to act on other matter (the 'test body'). This led Faraday to the unification of the electric and magnetic fields of potential force, as two of the manifestations of a general field of potential force.

In this century, Einstein found that a unified field theory is a consequence of the theory of general relativity, except that the field, in his theory, represents an 'actuality' rather than a 'potentiality', incorporating the 'test body' as a fundamental part of a closed system. His attempt was to demonstrate a generalization of his field equations, that had already revealed the nature of gravitation, superseding Newton's theory of universal gravitation. He wished to generalize the formalism in such a way so as to unify the gravitational manifestation of interacting matter with the electromagnetic field theory, as expressed with the Maxwell formalism. His hope was that such a unification may also eventually explain the origin of the formalism of quantum mechanics –a problem that was uppermost in his mind during the greater part of his professional career– as well unifying all of the other fundamental forces [2].

The unified field theory was pursued by other 20th century physicists, primary among them was Schrödinger [3]. Other important investigators of this subject were Eddington, Weyl, Lanczos, Kaluza and Klein [4]. They were all trying to unify the electromagnetic formalism with Einstein's field equations in general relativity, though none of these studies were successful. It was their hope, as it was Einstein's, that a successful unification would lead to further unification with other theoretical explanations, including the quantum theory.

In recent years a different sort of unified theory has been proposed in the domain of elementary particle physics ; this is a theory that generalizes the gauge of electrodynamics in the context of quantum field theory. The idea is to thereby generate a generalization of the electromagnetic force in the quantum domain, so that all forces may be included under a single umbrella, including the force of gravitation. Thus far, this approach has demonstrated a unification of the electromagnetic force with the weak interaction –the electroweak gauge quantum field theory of Glashow, Salam and Weinberg. The fusion with the other forces –the strong (nuclear) force and the gravitational force– are under investigation, but have not yet been demonstrated [5].

It is important to emphasize that this unified theory is in an entirely different context than the unified field concept of Faraday and Einstein. It is in the context of a quantum field theory –a theory of measurement expressed in terms of a linear Hilbert space, describing an open system in terms of fundamental probabilities that incorporate the different sorts of force fields of elementary particles (as observed by an external, macroapparatus). The Einstein/Faraday theory is in the context of a nonlinear field theory for a closed system. This is characterized by a single force field whose different manifestations are a function of the nature of the observation –e.g. the observation of a purely electric field if the observer is at rest with respect to an electrical body or the observation of the magnetic and the electric field if the observer is in motion with respect to the electrical body. The philosophy of this approach, to Faraday as well as Einstein, was to extend from the unification of electricity and magnetism to include all other possible force manifestations of interacting matter in this way, in terms of a single, general field of force, for a closed system at the outset, manifesting itself in one way or another, depending on the conditions of observation.

2. A unified field theory. Gravitation and electromagnetism.

I now wish to discuss the unified field theory that I have investigated. It follows the philosophy of a unified field theory according to Faraday's and Einstein's approach. I will focus here only on the unification of gravitation and electromagnetism, though other extensions were accomplished in this investigation, particularly to include the fusion with the formalism of quantum mechanics [QMGR][1].

If such a unification is indeed achievable, what did the other authors not do to achieve it ? I believe that the answer to this question lies in advice that Einstein gave but did not follow, himself, in his study of a unified field theory, nor did any of the other authors of this approach to physics. In an article that he published in 1945, in the Annals of Mathematics [6], Einstein said in the Introduction that toward the goal of such a theory in general relativity one should not only exploit the geometrical logic that is implicit but also the algebraic logic. This refers to fully exploiting the algebraic symmetry group of general relativity –the 'Einstein group'. This group is the set of spacetime transformations that specifies the transferral from one spacetime coordinate frame in which the laws are expressed to an infinitesimally displaced spacetime frame, such that the laws expressed in the different frames remain in one-to-one correspondence. This is the requirement of the 'principle of covariance' –which is the axiomatic basis of the theory of general relativity.

A salient feature of the Einstein group is that it is a continuous group of transformations. Thus this group lacks any reflections in space and time. On the other hand, when we examine the mathematical structure of Einstein's second-rank tensor formulation of general relativity, we see that it not only is covariant with respect to the continuous transformations of the spacetime coordinates, as it should be, it is also covariant with respect to the reflections in space and time. Indeed, it is this extra symmetry of the Einstein formalism that yields it as 10 independent relations at each spacetime point, rather than 16 relations.

The reason that there should be 16 independent field relations at each spacetime point in general relativity theory is that the Einstein group is a 16-parameter group –it entails 16 essential parameters [7]. Just as the rotation group in three-dimensional space entails three essential parameters (the three Eulerian angles to define a general rotation) and the Poincaré group of special relativity entails 10 essential parameters (three rotations is space, three components of the relative velocity between inertial frames and the four infinitesimal translations in space and time), so the Einstein group entails 16 essential parameters, characterized by the derivatives of four space and time coordinates of one reference frame with respect to any other,

$$\partial x'^{\mu}/\partial x^{\nu} \quad , \quad (\mu, \nu = 0, 1, 2, 3).$$

Thus we see that Einstein's field formalism in general relativity is too symmetric, because it is covariant with respect to the reflections, not contained in its underlying group, as well as the required continuous transformations that define the covariance of the theory. If we should then remove the reflections from the underlying covariance group of the Einstein symmetric, second-rank tensor formalism, it would factorize, yielding a new formalism with extra degrees of freedom, leading to the proper number of component equations –16 independent relations at each spacetime point.

The latter is entirely analogous to the factorization of the Klein-Gordon equation in special relativity by Dirac, when the reflection transformations are removed, yielding the factorized pair of two-component spinor equations :

$$(\Box + \kappa^2)\Phi = 0 \to \sigma^{\mu}\partial_{\mu}\eta = -\kappa\chi \quad , \quad \tilde{\sigma}^{\mu}\partial_{\mu}\chi = -\kappa\eta$$

where

$$\square \equiv \sigma^\mu \partial_\mu \tilde{\sigma}^\mu \partial_\mu = \sigma^0(\nabla^2 - \partial_0^2).$$

In general relativity one starts out with the invariant Riemannian differential metric. This may be factorized as follows :

$$ds^2 = g^{\mu\nu} dx_\mu dx_\nu = ds d\tilde{s} \rightarrow \begin{cases} ds = q^\mu dx_\mu \\ d\tilde{s} = \tilde{q}^\mu dx_\mu \end{cases} \tag{1}$$

where q^μ is, geometrically, a four-vector, but each of its vector components is, algebraically, a quaternion-valued field rather than a real number-valued field [8]. Thus, q^μ has $4 \times 4 = 16$ independent components. The hint is then present that this field could entail both gravitation and electromagnetism because the gravitational field, according to Einstein's general relativity, entails 10 component fields and the electromagnetic field, according to Maxwell's formalism, entails 6 component fields (3 components of electric field and 3 components of magnetic field).

We proceed to formulate the unified field theory by starting out as one does in Einstein's formulation, except that here the metrical field components are the quaternion variables q^μ, rather than the metric tensor variables $g^{\mu\nu}$. The quaternion algebraic form is chosen because the irreducible representations of the Einstein group (of general relativity) –a global extension of those of the Poincaré group (of special relativity)– obey the algebra of quaternions. One then proceeds as one does in the formulation of general relativity theory, except that here the variational parameters are the quaternion fields q^μ, rather than the metric tensor of the standard Riemannian spacetime.

From $q^\mu q_\mu =$ constant, it follows that the covariant derivatives of the quaternion fields vanish. Taking account of the fact that the quaternion q^μ is equivalent to a second-rank spinor of the form, $\eta \times \eta^*$, as well as a four-vector, where η is a first-rank two-component spinor, it follows from $q^\mu_{;\rho} = 0$ that the spin-affine connection field has the form

$$\Omega_\mu = \frac{1}{4}(\partial_\mu q^\rho + \Gamma^\rho_{\tau\mu} q^\tau)\tilde{q}_\rho = -\frac{1}{4} q_\rho(\partial_\mu \tilde{q}^\rho + \Gamma^\rho_{\tau\mu}\tilde{q}^\tau) \tag{2}$$

where Ω_μ is defined in terms of the covariant derivatives of a first-rank spinor, as follows :

$$\eta_{;\mu} = \partial_\mu \eta + \Omega_\mu \eta$$

Note that it follows from the factorization (1), as well as choosing the proper normalization, that

$$g^{\mu\nu} = -\frac{1}{2}(q^\mu \tilde{q}^\nu + q^\nu \tilde{q}^\mu)$$

The local (flat spacetime) limit of the metric tensor is the diagonal form, $(1-1-1-1)\delta_{\mu\nu}$, corresponding in this limit to the limiting quaternions, $q^\mu \to \sigma^\mu$, where σ^0 is the unit two-dimensional matrix and σ^k are the three Pauli matrices (with $k = 1, 2, 3$). The latter four matrices are the basis elements of the quaternion variable q^μ. The conjugated elements of the quaternion above are its time- (or space-) reflected fields, with $\tilde{\sigma}^\mu = (-\sigma^0, \sigma^k)$.

The explicit form of the Riemann curvature tensor, in terms of the quaternion metrical field, $R^\mu_{\rho\lambda\kappa}$, follows from $q^\mu_{;\rho;\lambda} - q^\mu_{;\lambda;\rho} = 0$, taking account of the behavior of q^μ as a vector as well as a second-rank spinor (GRM, Sec. 6.12). From this we determine the Ricci tensor, $R^\mu_{\rho\lambda\mu} = R_{\rho\lambda}$, and its contraction with $g^{\rho\lambda}$ yields the Riemann scalar curvature, having the form :

$$R = \frac{1}{2}[K_{\rho\lambda} q^\lambda \tilde{q}^\rho - q^\rho \tilde{q}^\lambda K_{\rho\lambda} + q^\lambda K^\dagger_{\rho\lambda} \tilde{q}^\rho - q^\rho K^\dagger_{\rho\lambda} \tilde{q}^\lambda] \tag{3}$$

where $K_{\rho\lambda}$ is the spin-curvature tensor, defined as follows :

$$\eta_{;\rho;\lambda} - \eta_{;\lambda;\rho} = [(\partial_\lambda \Omega_\rho + \Omega_\lambda \Omega_\rho) - (\partial_\rho \Omega_\lambda + \Omega_\rho \Omega_\lambda)]\eta \equiv K_{\lambda\rho}\eta \tag{4}$$

where η is the two-component spinor variable.

The factorized field equations for the metrical field are determined with the variational principle, where $\mathcal{L} = \mathcal{L}_q + \mathcal{L}_m$ is the total Lagrangian density, $\mathcal{L}_q$ is the metrical part, given by the trace of R (eq. (3)) and $\mathcal{L}_m$ is the matter part of the Lagrangian density that yields the source of the metrical field in the field equations. Taking the variation of $\mathcal{L}$ with respect to the quaternion variables, we find :

$$-\frac{\delta \mathcal{L}_q}{\delta \tilde{q}^\rho} = \frac{\delta \mathcal{L}_m}{\delta \tilde{q}^\rho} \to \frac{1}{4}(K_{\rho\lambda} q^\lambda + q^\lambda K^\dagger_{\rho\lambda}) + \frac{1}{8} R q_\rho = k\mathcal{T}_\rho \tag{5a}$$

and its conjugate (i.e. reflected) equation:

$$-\frac{\delta \mathcal{L}_q}{\delta q^\rho} = \frac{\delta \mathcal{L}_m}{\delta q^\rho} \to -\frac{1}{4}(K^\dagger_{\rho\lambda} \tilde{q}^\lambda + \tilde{q}^\lambda K_{\rho\lambda}) + \frac{1}{8} R \tilde{q}_\rho = k\tilde{\mathcal{T}}_\rho \tag{5b}$$

The fundamental metrical field equations, that are a factorization of Einstein's second-rank tensor field equation, is then given in eq. (5a), or, equivalently, by its time- (or space-) reflected equation which is its conjugate equation (5b).

The field equations (5a) are 16 independent relations at each spacetime point. To see that this formalism contains Einstein's tensor formalism (the 10 relations that predict the gravitational effects) and Maxwell's formalism (the 6 relations that predict electromagnetic effects), we may proceed as follows: Multiplying eq. (5a) on the right with the conjugated quaternion solution $\tilde{q}_\gamma$, and multiplying eq. (5b) on the left with the quaternion solution q_γ, and then adding and subtracting the two second-rank tensor equations so- constructed - a procedure that is unique only up to a constant k or k' on the right –the following equations are obtained :

$$\frac{1}{2}(K_{\rho\lambda}q^\lambda\tilde{q}_\gamma \mp q_\gamma\tilde{q}^\lambda K_{\rho\lambda} + q^\lambda K^\dagger_{\rho\lambda}\tilde{q}_\gamma \mp q_\gamma K^\dagger_{\rho\lambda}\tilde{q}^\lambda) + \frac{1}{4}(q_\rho\tilde{q}_\gamma \pm q_\gamma\tilde{q}_\rho)R = 2\begin{pmatrix} k \\ k' \end{pmatrix}(\mathcal{T}_\rho\tilde{q}_\gamma \pm q_\gamma\tilde{\mathcal{T}}_\rho) \quad (6\pm)$$

The field equation (6+) is in one-to-one correspondence with the structure of Einstein's equation :

$$R_{\rho\gamma} - \frac{1}{2}g_{\rho\gamma}R = kT_{\rho\gamma} \quad (7+)$$

This follows from the dependence of the corresponding tensors of the Riemannian spacetime on the quaternion variables, as discussed above, [GRM][1].

3. The Maxwell field implicit in the quaternion formalism.

The field equation (6-) may be put into the form of Maxwell's field equations for electromagnetism by taking the covariant divergence of both sides of this equation, yielding

$$F^{;\rho}_{\rho\gamma} = (4\pi/c)j_\gamma \quad (7\text{-})$$

where the antisymmetric second-rank tensor on the left is :

$$F_{\rho\gamma} = Q[\frac{1}{4}(K_{\rho\lambda}q^\lambda\tilde{q}_\gamma + q_\gamma\tilde{q}^\lambda K_{\rho\lambda} + q^\lambda K^\dagger_{\rho\lambda}\tilde{q}_\gamma + q_\gamma K^\dagger_{\rho\lambda}\tilde{q}^\lambda) + \frac{1}{8}(q_\rho\tilde{q}_\gamma - q_\gamma\tilde{q}_\rho)R] \quad (8)$$

and the four-current source term is

$$j_\gamma = (cQk'/4\pi)(\mathcal{T}_\rho^{;\rho}\tilde{q}_\gamma - q_\gamma \tilde{\mathcal{T}}_\rho^{;\rho}) \tag{9}$$

The constant Q in eqs. (8) and (9) is a constant electric charge inserted on both sides of the field equation (6-) in order to give the solutions the proper dimensions of electromagnetic field intensity and current density in the equation (7-). In deriving the form (9), use was made of the fact that the covariant derivatives of the quanternion variables q^μ are identically zero.

What has been done so far is to take a formalism given by the field equations (5a) –16 relations at each spacetime point that are neither even nor odd with respect to reflections in space or time– and re-express them in a form that is a sum of an even part under reflections (10 relations) and an odd part (6 relations).

The fact that the four-current density source term for the Maxwell field is odd under reflections is then compatible with the identification of the iterated equation (6-) and the Maxwell formalism. But it is to be noted that the (arbitrary) charge Q is not derived here. It is only the mathematical form of the laws of electrical charges that is derived from general relativity, when the reflections are dropped from the underlying covariance group. This is the requirement of a unified field theory.

4. Probability density of wave mechanics in a local limit.

Because $F_{\rho\gamma}$ is an antisymmetric second-rank tensor, it follows automatically that the covariant divergence $F_{\rho\gamma}^{;\rho;\gamma}$ must vanish. Thus equation (7-) leads to the vanishing of the covariant divergence of the four-current density, $j_\gamma^{;\gamma} = 0$. Integration of the local limit of this equation, together with Gauss'theorem and the boundary condition that j_k vanishes on the bounding surface of the volume that contains the charge Q, then gives the time-conservation of charge :

$$\sigma^0 Q = \int j_0 d^3x = \quad \text{constant} \tag{10}$$

The insertion of the unit matrix σ^0 on the left takes account of the matrix structure (9) of the four-currrent in this quaternion formalism.

The local limit of j_0 in eq. (9) is the following :

$$j_0 = -(cQk'/4\pi)\partial^\rho(\mathcal{T}_\rho^{(1)} + \tilde{\mathcal{T}}_\rho^{(1)})$$

where $\mathcal{T}_\rho^{(1)}$ is the local limit of the matter source of the metric equations, $\delta\mathcal{L}_m/\delta\tilde{q}^\rho$. Taking the determinant of both sides of eq. (10) then yields the value of the constant k' as follows :

$$k' = -(4\pi/c)/|\int \partial^\rho(\mathcal{T}_\rho^{(1)} + \tilde{\mathcal{T}}_\rho^{(1)})d^3x|$$

Thus the four-current density j_γ of this expression of the Maxwell theory, has the general form :

$$j_\gamma = Q(\mathcal{T}_\rho^{;\rho}\tilde{q}_\gamma - q_\gamma\tilde{\mathcal{T}}_\rho^{;\rho})/|\int \partial^\rho(\mathcal{T}_\rho^{(1)} + \tilde{\mathcal{T}}_\rho^{(1)})d^3x| \qquad (11)$$

The matter density, interpreted in quantum mechanics as a 'probability density', is then

$$\sigma^0\rho = j_0/Q = (q_0\tilde{\mathcal{T}}_\rho^{;\rho} - \mathcal{T}_\rho^{;\rho}\tilde{q}_0)/|\int \partial^\rho(\mathcal{T}_\rho^{(1)} + \tilde{\mathcal{T}}_\rho^{(1)})d^3x| \qquad (12)$$

In the local limit, $q_0 \to \sigma_0$, $\tilde{q}_0 \to \tilde{\sigma}_0 = -\sigma_0$ and $\mathcal{T}_\rho^{;\rho} \to \partial^\rho\mathcal{T}_\rho^{(1)}$, so that

$$\mathrm{loc}\lim|\int \rho d^3x| = 1 \qquad (13)$$

This derived normalization condition was originally imposed by Born, in his interpretation of nonrelativistic quantum mechanics as a probability calculus. As we see above, however, this normalization is not a general relation, in the full, generally covariant expression of this field theory [9].

We also see from the general form (11) of the current density that the three-current part of this four-vector,

$$j_k/Q = (q_k\tilde{\mathcal{T}}_\rho^{;\rho} - \mathcal{T}_\rho^{;\rho}\tilde{q}_k)/|\int \partial^\rho(\mathcal{T}_\rho^{(1)} + \tilde{\mathcal{T}}_\rho^{(1)})d^3x|$$

predicts a coupling of a 'gravitational field', q_k, to the matter field components $\mathcal{T}_\rho = \delta\mathcal{L}_m/\delta\tilde{q}^\rho$, to define a gravitational current term, not foreseen in the conventional theories that neglect the gravitational coupling to matter fields. It is anticipated that this current must entail physical effects in the domain where gravitation and electromagnetism are on an equal footing regarding their relative magnitudes of coupling.

In the nonrelativistic quantum mechanical formalism, the conserved density is $\rho_\psi = \psi^* \psi$, where ψ is a scalar, complex number-valued field, interpreted as a 'probability amplitude'. In the relativistic extension, the conserved density is $\psi^\dagger \psi$, where ψ is a four-component bispinor whose law is covariant with respect to space and time reflections. It is made up of the two-component spinor fields, η and χ as follows :

$$\psi \sim \begin{pmatrix} \eta + \chi \\ \eta - \chi \end{pmatrix}$$

where the field equations satisfied by η and χ are the most general regarding the symmetry of relativity theory –in the sense that they are not covariant with respect to reflections. In the bispinor form, $\psi^\dagger$ is the hermitian conjugate, defined as the complex conjugate of the transposed bispinor. In the two-component spinor formalism, χ is the time- (or space-) reflection of η. In the latter formalism in general relativity, the four-current that satisfies the equation of continuity is $J_\gamma = (\eta^\dagger q_\gamma \eta - \chi^\dagger \tilde{q}_\gamma \chi)$. Note that the second part of J_γ is the time reversal of the first part, so that this current is odd with respect to reflections in time, as well as space, as it should. In the local limit, the time-component of this four-current is proportional to

$$J_0 \sim (\eta^\dagger \eta + \chi^\dagger \chi)$$

The conserved density that corresponds to the quantum mechanical probability current, in the local limit, is then the addition of the gravitational current discussed above, j_γ, shown in eq. (11), and J_γ. That is, it is the four-current $(J_\gamma + j_\gamma)$ that obeys the law of continuity, and the law of conservation that follows.

References.

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