The ideal fluid as the classical limit of free quantum fields*

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ABSTRACT. We establish the relation among Quantum Field Theory variables, like amplitudes and phases, with macroscopic fluid variables: proper energies and four-velocities. This approach puts into evidence the low energy relation between relativistic fluid dynamics and Quantum Field Theory. We perform the WKB expansion of the Lagrangians and the energy-momentum tensors corresponding to free spin 0 and spin 1/2 (Dirac and Majorana) massive fields. We keep classical terms and prove how non interacting Lagrangian fields conduce, for both spin 0 and spin 1/2 cases, to the Lagrangians of ideal fluids without pressure.

RESUME. Nous signalons la relation entre les variables de la théorie quantique des champs (par exemple amplitudes et phases), avec les variables macroscopiques d'un fluide : énergies propres et 4-vitesses. Cette approche met en évidence la relation que, à faibles énergies, il y a entre la dynamique d'un fluide relativiste et la théorie quantique des champs. Nous faisons l'expansion WKB des lagrangiens et des tenseurs d'énergie-moment correspondant aux champs massifs libres de spin 0 *et spin* 1/2 *(Dirac et Majorana). Nous gardons les termes classiques et prouvons, de quelle façon les lagrangiens des champs sans interaction, ramènent, aussi bien pour les cas de spin* 0 *comme de spin* 1/2*, aux lagrangiens de fluides parfaits sans pression.*

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1. Introduction

Quantum Field Theory (QFT) is successful in describing microscopical behavior of matter. Here we are interested in studing its macroscopical limit. We show how the Lagrangian of free spin 0 and spin 1/2 massive fields conduce, up to the lowest order in the Planck constant $\hbar$, to the dynamics of ideal fluids.

In II we perform the WKB expansion of the scalar field and find the main contributions of the Laurent series in $\hbar$ in both the Lagrangian and the energy-momentum tensor. We are able to interpret them as the classical Lagrangian and the classical energy-momentum tensor. We find the proper energy density expression in terms of the highest order amplitudes and the mass. We use this magnitude and the four-velocity definition in order to demonstrate that we are dealing with an ideal fluid without pressure. In III we apply essencially the same procedure for the spin 1/2 Dirac and Majorana fields and a straight-foward computation shows that even spinorial matter conduces, in the classical limit, to an ideal fluid. In IV we discuss our results.

Before describing our approach, we discuss some general aspects of the WKB expansion. In the first quantification scheme, the complex wave function of a particle can be expressed as a path integral:

$$\psi(x_0, x_f) = N \int e^{iS_{cl}[x]/\hbar} \mathcal{D}x \tag{1.1}$$

where $S_{cl}[x]$ is the classical action, a functional of the trajectory $x(t)$ which depends on the initial and final fixed data (x_0, x_f). N is a normalizing factor. If S is the action of a non-relativistic particle then ψ is the wave function which satisfies the Schroedinger equation [1]. On the other hand, for a relativistic action $S_{cl}[x]$, ψ must be considered a field which satisfies the relativistic field equation corresponding to the respective representation of the Lorentz group. This can be seen by explicit computation over the expression (1.1), taking into account that the x dependence in the path integral is through the initial and final points of $S_{cl}[x]$, and

$$\partial_\alpha S_{cl} = P_\alpha \tag{1.2}$$

is the four-momentum; $\alpha, \beta \ldots = 0, 1, 2, 3$. The field equations may also be obtained extreming the action S with respect to the field

$$\frac{\delta S[\psi]}{\delta \psi} = 0 \tag{1.3}$$

The expansion of (1.1) in powers of $\hbar$:

$$\psi = N e^{i/\hbar S_{cl}[x_{cl}]} \int e^{\int \frac{1}{2} \frac{\delta S_{cl}}{\delta x_1 \delta x_2}[x] \mathcal{D}x_1 \mathcal{D}x_2 + \dots} \mathcal{D}x = \sum_{n=0}^{\infty} (-i\hbar)^n \psi_n e^{iS/\hbar} \tag{1.4}$$

with x_{cl} the classical trajectory, is known as the WKB expansion of the field. Replacing this expansion in S for any group representation, we are able to identify the S_{cl} term in

$$S[\psi] = S_{cl}[x] + O(\hbar) \tag{1.5}$$

(In this sense the $\hbar = 0$ limit of $S[\psi]$ in a first quantification scheme conduces to $S_{cl}[x]$ in the same way as the $\hbar = 0$ limit of the effective action conduces, in a second quantification scheme, to $S[\psi]$).

2. Spin 0 field

We start from the action functional S which is constructed from a Lagrangian density $\mathcal{L}$, a function of the fields ψ and their first derivatives :

$$S[\psi] = \int \mathcal{L}(\psi, \partial\psi) d^4x \tag{2.1}$$

The Euler-Lagrange equations (1.3) read:

$$\partial_\alpha \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \psi)} - \frac{\partial \mathcal{L}}{\partial \psi} = 0 \tag{2.2}$$

where $\alpha, \beta, \ldots = 0, 1, 2, 3$. We take $c = 1$ and $\eta_{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1)$.

Translation invariance of (2.1) conduces "on shell", i.e. using (2.2), to the conservation of the energy-momentum tensor via the Noether theorem

$$\partial_\alpha T^{\alpha\beta} = 0 \tag{2.3}$$

where

$$T^{\alpha\beta} = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \psi)} \partial^\beta \psi - \eta^{\alpha\beta} \mathcal{L} \tag{2.4}$$

First we study the massive spin 0 field ϕ. The corresponding Lagrangian density reads

$$\mathcal{L} = \partial_\alpha \phi \partial^\alpha \phi^* + \frac{m^2}{\hbar^2} \phi^* \phi \tag{2.5}$$

where m is the mass. In this case, equations of motion (2.2) are

$$(\Box - \frac{m^2}{\hbar^2})\phi = 0 \quad , \quad (\Box - \frac{m^2}{\hbar^2})\phi^* = 0 \tag{2.6a,b}$$

$\Box$ means $\eta^{\alpha\beta}\partial_\alpha\partial_\beta$. The energy-momentum tensor (2.4) for this field is

$$T^{\alpha\beta} = \partial^\alpha\phi\partial^\beta\phi^* - \frac{1}{2}\eta^{\alpha\beta}\partial_\gamma\phi\partial^\gamma\phi^* - \frac{1}{2}\eta^{\alpha\beta}\frac{m^2}{\hbar^2}\phi\phi^* \tag{2.7}$$

In order to study the classical behavior of the scalar field, we perform its expansion in powers of $(-i\hbar)$, i.e., the WKB expansion:

$$\phi = \sum_{n=0}^{\infty}(-i\hbar)^n\phi_n(x)\exp[iS_\phi(x)/\hbar] \tag{2.8}$$

Field (2.8) will represent a classical particle if the phase S_ϕ is the Hamiltonian principal function of a particle whose canonical momentum is

$$P^\alpha = \partial^\alpha S = mu^\alpha \tag{2.9}$$

Replacing (2.8) into (2.5) we obtain a Laurent series in powers of $\hbar$. The highest (classical) contributions of this series are the $\hbar^{-2}$ terms

$$\overset{(\hbar^{-2})}{\mathcal{L}_\phi} = \frac{1}{\hbar^2}\phi_0\phi_0^*\partial_\alpha S\partial^\alpha S + \frac{m^2}{\hbar^2}\phi_0\phi_0^* \tag{2.10}$$

On the other hand, the main contribution of (2.8) to the energy-momentum tensor (2.7) is:

$$\overset{(\hbar^{-2})}{T^{\alpha\beta}} = \frac{1}{\hbar^2}[\partial^\alpha S\partial^\beta S - \frac{1}{2}\eta^{\alpha\beta}\partial_\gamma S\partial^\gamma S - \frac{1}{2}\eta^{\alpha\beta}m^2]\phi_0\phi_0^* \tag{2.11}$$

Using the fact that $U_\alpha U^\alpha = -1$, expression (2.11) becomes

$$\overset{(\hbar^{-2})}{T^{\alpha\beta}} = \frac{m^2}{\hbar^2}\phi_0\phi_0^* U^\alpha U^\beta \tag{2.12}$$

Now we are able to find the proper energy density ρ_ϕ of the scalar field which is nothing but the 00 component of (2.12) in the reference system where the particle is at rest:

$$\rho_\phi = \frac{m^2}{\hbar^2}\phi_0\phi_0^* = T^{00}_{rest} \tag{2.13}$$

This last result shows the relation among field variables (amplitudes) and a classical fluid variable (proper energy density). So, $\overset{(\hbar^{-2})}{T^{\alpha\beta}}$ may be written as

$$\overset{(\hbar^{-2})}{T^{\alpha\beta}} = \rho_\phi U^\alpha U^\beta \tag{2.14}$$

In (2.14) we recognize the expression of the energy-momentum tensor of an ideal fluid without pressure.

We now return to the Lagrangian density (2.10) and write it in terms of the macroscopical variables ρ_ϕ and U^α:

$$\overset{(\hbar^{-2})}{\mathcal{L}_\phi} = \rho_\phi \eta^{\alpha\beta} U_\alpha U_\beta - \rho_\phi \tag{2.15}$$

Integrating expression (2.15), we obtain the final form of the Lagrangian $\overset{(\hbar^{-2})}{L}$ and its corresponding classical action

$$\overset{(\hbar^{-2})}{L_\phi} = \int \rho_\phi \eta^{\alpha\beta} U_\alpha U_\beta d^3x \tag{2.16}$$

$$S_\phi[x] = \int \overset{(\hbar^{-2})}{L_\phi} \, dt \tag{2.17}$$

where $x(t)$ is the trajectory of the particle. In (2.17) we have omitted the constant

$$M_\phi = \int \rho_\phi d^3x$$

Finally, from $\delta S/\delta x = 0$ we obtain

$$U^\beta \partial_\beta U^\alpha = 0 \tag{2.18}$$

i.e. straight line trajectories, as it is expected for free particles.

3. Spin 1/2 field

The Lagrangian density for the spin 1/2 massive Dirac field λ reads:

$$\mathcal{L} = \frac{1}{2}\bar{\lambda}\partial\!\!\!/\lambda + \frac{1}{2}\frac{m}{\hbar}\bar{\lambda}\lambda \tag{3.1}$$

where $\partial\!\!\!/ = \gamma^\alpha \partial_\alpha$, γ^α are the Dirac matrices and $\bar{\lambda}$ is the Dirac adjoint of λ. Then, the equation of motion is

$$\partial\!\!\!/\lambda + \frac{m}{\hbar}\lambda = 0 \tag{3.2}$$

We expand the spin 1/2 field in powers of $(-i\hbar)$

$$\lambda = \sum_{n=0}^{\infty} (-i\hbar)^n \lambda_n(x) \exp[iS_\lambda(x)/\hbar] \tag{3.3}$$

The corresponding canonical momentum of the spin 1/2 particle is defined as in (2.9). Replacing (3.3) into (3.1) we see that the highest contributions to the Laurent series are now of $\hbar^{-1}$ order

$$\overset{(\hbar^{-1})}{\mathcal{L}_\lambda} = \frac{1}{2\hbar}\bar{\lambda}_0 \gamma^\alpha \lambda_0 m U_\alpha + \frac{1}{2}\frac{m}{\hbar}\bar{\lambda}_0\lambda_0 \tag{3.4}$$

The main contribution of (3.3) to the energy-momentum tensor (2.4) is:

$$\overset{(\hbar^{-1})}{T^{\alpha\beta}} = \frac{\bar{\lambda}_0 \gamma^\alpha \lambda_0 m U^\beta}{\hbar} \tag{3.5}$$

Let us find the proper energy density ρ_λ of the spin 1/2 Dirac field, i.e. the 00 component of (3.5), in the reference system where the particle is at rest.

$$\overset{(\hbar^{-1})}{T^{00}_{\text{rest}}} = \rho_\lambda = \frac{1}{\hbar}\bar{\lambda}_{(0)}\gamma^O U_0 \lambda_{(0)} m = \frac{1}{\hbar}\bar{\lambda}_{(0)}\lambda_{(0)} m \tag{3.6}$$

We have made use of the fact that the lowest order corresponding to the field equation (3.2) written in that reference system implies:

$$\gamma^0 U_0 \lambda_{(0)} = \lambda_{(0)}$$

Replacing (3.6) into (3.5), $\overset{(\hbar^{-1})}{T^{\alpha\beta}}$ reads:

$$\overset{(\hbar^{-1})}{T^{\alpha\beta}} = \rho_\lambda U^\alpha U^\beta \tag{3.7}$$

as in the scalar field case.

Now we return to the Lagrangian density (3.4). Making use of the relation

$$\gamma^\alpha P_\alpha \lambda_0 = \lambda_0$$

arising from the field equation, we obtain

$$\overset{(\hbar^{-1})}{\mathcal{L}_\lambda} = \frac{1}{2\hbar}\bar{\lambda}_0 \gamma_\alpha U^\alpha m \gamma_\beta U^\beta \lambda_0 + \frac{1}{2}\frac{m}{\hbar}\bar{\lambda}_0 \lambda_0 \tag{3.8}$$

Using γ matrices properties $\{\gamma^\alpha, \gamma^\beta\} = 2\eta^{\alpha\beta}$ and definition of ρ_λ given by (3.6), eq. (3.8) becomes

$$\overset{(\hbar^{-1})}{\mathcal{L}_\lambda} = \frac{1}{2}\rho_\lambda U_\alpha U^\alpha + \frac{1}{2}\rho_\lambda \tag{3.9}$$

So the macroscopical Lagrangian and the macroscopical action read:

$$\overset{(\hbar^{-1})}{L_\lambda} = \int \frac{1}{2}\rho_\lambda \eta^{\alpha\beta} U_\alpha U_\beta d^3x \tag{3.10}$$

$$S_\lambda = \int \overset{(\hbar^{-1})}{L_\lambda}\, dt \tag{3.11}$$

which correspond to a free classical fluid moving along straight line trajectories.

Up to now we have been dealing with the Dirac representation of the spin 1/2 field for which the operation of parity is well defined. But if we need equivalence between spin 1/2 particles and anti-particles, that is, self-conjugation under charge conjugation operation, we must use the Majorana representation [2]. Another relevant feature of this representation is that it is usually used in supersymmetric theories which admit a transformation of bosons into fermions and viceversa, keeping the invariance of the whole Lagrangian. Majorana representation describes objects with have as many degrees of freedom as Dirac spinors. For a Majorana field with mass m, the Lagrangian density reads:

$$\mathcal{L}_\lambda = -\frac{1}{2}[\bar{\lambda}\partial\!\!\!/\lambda + (i\frac{m}{\hbar})\bar{\lambda}\lambda] \tag{3.12}$$

where

$$\lambda = \begin{pmatrix} \lambda_L \\ \lambda_R \end{pmatrix} = \begin{pmatrix} \lambda_L \\ -\sigma_2 \lambda_L^* \end{pmatrix} \quad , \quad \bar{\lambda} = \lambda^+ \gamma^0$$

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Spinors λ_L and λ_R are the left and right handed components of the field, respectively. We can perform the WKB expansion of the Weyl spinor λ_L:

$$\lambda_L = \sum_{n=0}^{\infty} \lambda_L^n (-i\hbar)^n \exp[iS/\hbar] \tag{3.13}$$

Replacing (3.13) into (3.12) and taking into account that the Lagrangian density (3.12) is of first order, classical contributions arise from its $\hbar^{-1}$ terms ($\mathcal{L}^{(\hbar^{-1})}$). After a straightforward computation, we are able to obtain the highest contributions which are of $\hbar^{-1}$ order for the Lagrangian density, the energy-momentum tensor and the rest energy. In the four-component formalism, these results read:

$$\overset{(\hbar^{-1})}{\mathcal{L}_\lambda} = \frac{1}{2}[i\dot{\bar{\lambda}}\gamma^\alpha\gamma_5\dot{\lambda}P_\alpha/\hbar + i\frac{m}{\hbar}\dot{\bar{\lambda}}\dot{\lambda}] \tag{3.14}$$

$$\overset{(\hbar^{-1})}{T^{\alpha\beta}} = i\dot{\bar{\lambda}}\gamma^\alpha\gamma_5\dot{\lambda}P^\beta/\hbar \tag{3.15}$$

$$\overset{(\hbar^{-1})}{T^{00}_{\text{rest}}} = i\dot{\bar{\lambda}}\gamma^0\gamma_5\dot{\lambda}mU^0/\hbar \tag{3.16}$$

where

$$\dot{\lambda} = \lambda_L^0 \exp[iS_\lambda(x)/\hbar]$$

$$P^\alpha = mU^\alpha = \partial S/\partial x^\alpha$$

$$\overset{(\hbar^{-1})}{\partial_\alpha \lambda} = i\gamma_5\dot{\lambda}P_\alpha/\hbar$$

$$\gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix} \quad , \quad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad i = 1, 2, 3$$

On the other hand, the $\hbar^{-1}$ order of the field equation:

$$\partial\!\!\!/\lambda = -i\frac{m}{\hbar}\lambda \tag{3.17}$$

may be written as

$$\dot{\lambda} = -\gamma^\alpha\gamma_5\dot{\lambda}U_\alpha \tag{3.18}$$

In the proper system, eq. (3.18) reads:

$$\dot{\tilde{\lambda}} = -\gamma^0 \gamma_5 \dot{\tilde{\lambda}} U_0 \tag{3.19}$$

Replacing (3.19) into (3.16), we obtain

$$\overset{(\hbar^{-1})}{T^{00}_{\text{rest}}} = \rho_\lambda = \frac{-i \dot{\tilde{\bar{\lambda}}} \dot{\tilde{\lambda}} m}{\hbar} \tag{3.20}$$

If we now replace eq. (3.18) into (3.14), we have

$$\overset{(\hbar^{-1})}{\mathcal{L}} = \frac{1}{2\hbar} (i \dot{\tilde{\bar{\lambda}}} \gamma^\alpha \gamma_5 \gamma^\beta \gamma_5 \dot{\tilde{\lambda}} U_\beta P_\alpha + i m \dot{\tilde{\bar{\lambda}}} \dot{\tilde{\lambda}}) \tag{3.21}$$

Using the properties of the γ matrices

$$\{\gamma^5, \gamma^\alpha\} = 0 \quad , \quad \{\gamma^\alpha, \gamma^\beta\} = 2\eta^{\alpha\beta} \quad , \quad (\gamma^5)^2 = 1$$

in eq. (3.21), we obtain

$$\overset{(\hbar^{-1})}{\mathcal{L}} = \frac{i}{2\hbar} \dot{\tilde{\bar{\lambda}}} \eta^{\alpha\beta} \dot{\tilde{\lambda}} U_\beta m U_\alpha + \frac{i}{2\hbar} m \dot{\tilde{\bar{\lambda}}} \dot{\tilde{\lambda}} \tag{3.22}$$

Substituting (3.20) into (3.22), the Lagrangian density reads

$$\overset{(\hbar^{-1})}{\mathcal{L}} = -\frac{1}{2} \rho_\lambda U^\alpha U^\beta \eta_{\alpha\beta} - \frac{1}{2} \rho_\lambda \tag{3.23}$$

Integrating eq. (3.23), we obtain the final expression for the Majorana spin 1/2 classical Lagrangian $L_\lambda^{(\hbar^{-1})}$:

$$\overset{(\hbar^{-1})}{L_\lambda} = -\frac{1}{2} \int \rho_\lambda U^\alpha U^\beta \eta_{\alpha\beta} d^3x \tag{3.24}$$

where we have again omitted a (mass) constant. Finally, the classical contribution to the energy-momentum tensor reads:

$$\overset{(\hbar^{-1})}{T^{\alpha\beta}} = \rho_\lambda U^\alpha U^\beta \tag{3.25}$$

This corresponds again to the energy-momentum tensor of an ideal fluid without pressure.

4. Discussion

We have found the macroscopical variables ρ and U^α which describe the classical behavior of bosonic and fermionic fields. We have shown that these variables can be built up as a combination of the first terms of WKB expansions and its derivatives. This gives a fluid like interpretation of the WKB variables and renders evident the relation between micro and macro variables.

Our treatment acquires interest in curved backgrounds where some results, obtained via the WKB expansion of the field equations (see for example ref. 3 and 4 for the Dirac case), may now be understood as a covariantization of the special relativistic expressions obtained here. The formalism applied to free fields may be easily extended to interacting cases, giving interesting tools for describing classical consequences of QFT. In fact, fields appearing in interacting terms may also be replaced by classical fluid variables: we have shown how supersymmetric matter fields interacting with gravity in Supergravity, are expressed in terms of fluid variables [5],[6]. Moreover, this “supersymmetric fluid”, when used as the source of a Robertson-Walker cosmology, reproduces the correct inflationary phase [7].

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