Another treatment of the relation between classical and quantum mechanics

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ABSTRACT. A model of field theory containing as its limits both the Schrödinger wave mechanics and the Newton classical mechanics is presented. All details are discussed explicitly on the example of the harmonic oscillator. A new fundamental constant connected with the distance of observation of physical phenomena is introduced. Its experimental value may be determined from the spectra of quantum mechanical systems.

RESUME. On présente un modèle de théorie des champs contenant comme limites, à la fois la mécanique ondulatoire de Schrödinger et la mécanique classique de Newton. Tous les détails sont discutés explicitement sur l'exemple de l'oscillateur harmonique. Une nouvelle constane fondamentale reliée à la distance d'observation d'un phénomène physique est introduite. Sa valeur expérimentale peut être déterminée à partir des spectres des systèmes quantiques.

According to a general belief [1], classical Newton's mechanics may be obtained as a "limit" of two different theories: the relativistic classical mechanics and the non-relativistic quantum mechanics. In the first case, the limit is reached when in all relativistic formulae the velocity of light is going to infinity. The limiting procedure changes the shapes of formulae but does not change either the number or the physical interpretation of the corresponding quantities. In this sense we may say that we understand satisfactorily the limiting procedure both from the mathematical and physical point of view [2].

Unfortunately, it is far to be true for the second case. All known ways of relating classical mechanics with "limits" of quantum mechanics suffer from the lack of precision [3,4]. In particular, it is not known, either from mathematical or physical points of view, what is the classical limit of the most fundamental object of quantum mechanics - the wave function - when the Planck constant is going to zero.

In the present paper we shall investigate the possibility of obtaining a well - defined classical limit of quantum mechanical wave function. We shall perform our investigation in a framework of a simple, mathematically consistent field theory, which as particular cases incorporates both the classical and the wave mechanics. The basic concepts of such a field theoretical approach, not only to Newton's and Schrödinger's equations, but to all fundamental equations of physics, have been reported in Ref. [5]. It has been shown there that the primary physical notions in each theory can be described in terms of four collections of basic fields. The first collection of fields $\psi_\alpha(x) = \psi_\alpha(x^0, x^1, x^2, x^3)(\alpha = 1, \ldots, N)$ contains all fields which in the case of mechanical theories describe the localization of physical objects. The second collection consists of fields $\phi_{\alpha\mu}(x)(\mu = 0, 1, 2, 3)$ which determine the space - time evolution of fields $\psi_\alpha(x)$. Altogether, these two collections describe the kinematical aspects of each theory. The dynamical aspects are described by the third collection of fields $\pi_\alpha(x)$ which realize the dynamical quantities of the theory and by the fourth collection of fields $\rho_\alpha(x)$ which describe all external influences acting on the considered physical system.

From their definition the fields introduced above satisfy the set of differential relations

$$\frac{\partial\ \psi_\alpha(x)}{\partial\ x^\mu} = \phi_{\mu\alpha}(x) \tag{1.a}$$

$$\frac{\partial\ \pi^\mu_\alpha(x)}{\partial\ x^\mu} = \rho_\alpha(x) \tag{1.b}$$

which are universal and do not contain any physical constants. In order to determine, in each case, the fields from these general relations we have to close them with the aid of particular constitutive relations. These relations distinguish various physical theories and introduce all necessary physical constants into general scheme.

In particular, the equations of Newton's classical mechanics of a material point moving in one dimension are obtained from (1a-b) by taking a single field $\psi(t)(N = 1)$ depending on time variable only, and by adopting the following constitutive relation

$$\pi^\mu(t) = m\ \phi_\mu(t) \tag{2}$$

where m is the mass of the material point. In this case we may interpret the field $\psi(t)$ as the trajectory of the point and the non - zero component of the field $\phi_\mu(t)$ as its velocity. The non - vanishing component of the field $\pi^\mu(t)$ is then equal to the momentum of the point and the field ρ is the force acting on it.

The Schrödinger wave equation for the wave function $\psi(x)$ of a scalar particle is obtained from (1a-b) by taking the following constitutive relations

$$\pi^0(x) \;=\; i\hbar\psi(x) \tag{3.a}$$

$$\pi^i(x) \;=\; -\,\frac{\hbar^2}{2m}\,\phi_i(x) \tag{3.b}$$

$$\rho(x) \;=\; -\,V(x)\,\psi(x) \tag{3.c}$$

where $\hbar$ is the Planck universal constant, m - the mass of the particle and $V(x)$ is the usual potential acting on it.

In order to investigate the relation between the Newtonian and Schrödinger mechanics let us consider the following more general constitutive relations

$$\pi^0(\vec{x},t) \;=\; m\,l_0^2\,\phi_0(\vec{x},t) \;+\; i\hbar\psi(\vec{x},t) \tag{4.a}$$

$$\pi^i(\vec{x},t) \;=\; -\,\frac{\hbar^2}{2m}\,\phi_i(\vec{x},t) \tag{4.b}$$

$$\rho(\vec{x},t) \;=\; -\,V(\vec{x},t)\psi(\vec{x},t) \;+\; l_0^2 F(\psi(\vec{x},t),\phi_\mu(\vec{x},t),\vec{x},t) \tag{4.c}$$

where l_0 is an arbitrary parameter with the dimension of length, and F describes all external influences not taken into account by the quantum mechanical potential $V(\vec{x},t)$. Substituting (4) into (1) we obtain the following equation for the field $\psi(\vec{x},t)$:

$$ml_0^2\,\frac{\partial^2\psi}{\partial t^2} \;+\; i\hbar\,\frac{\partial\psi}{\partial t} \;-\; \frac{\hbar^2}{2m}\,\Delta\psi \;+\; V\psi \;=\; l_0^2\,F \tag{5}$$

which is a generalization of the famous telegraphist's equation. It is now easy to see that in the limit $\hbar \;\to\; 0, V \;\to\; 0$ this equation takes the form of the Newton's equation

$$m\,\frac{d^2\psi}{dt^2} \;=\; F(\psi,\dot{\psi},t) \tag{6}$$

provided we are looking for the solutions which in this limit "loose" their x - dependence. In the limit $l_0 \rightarrow 0$ equation (5) goes to the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} - \frac{\hbar^2}{2m} \Delta \psi + V\psi = 0 \tag{7}$$

Therefore equation (5) provides a two parameter family of fields $\psi(\vec{x}, t; \hbar, l_0)$ which, as limiting points, contains both the quantum mechanical wave function $\psi_q(\vec{x}, t)$ and the classical trajectory $\psi_{cl}(t)$. We may therefore say that our approach is an alternative realization of the de Broglie idea of Double Solution [6]. In order to make the analogy between classical and quantum cases more exact, we must ensure that the force F in equation (6) and the potential V in equation (7) corresponds to the same physical interaction. We must therefore restrict ourselves to the field equation (5) in two dimensional space - time and put

$$F(\psi, \dot{\psi}, t)\Big|_{\psi(t)=x(t)} = - \frac{\partial V(x,t)}{\partial x}\Big|_{x = x(t)} \tag{8}$$

where the interpretation of the field $\psi(t)$ as the classical trajectory is explicitly taken into account. To maintain the possibility of taking in equation (5) the limit $V \rightarrow 0$ with $F \neq 0$ we must introduce into it one more dimensionless parameter, λ, which will multiply the potential V.

To simplify the analysis let us now consider the simplest example of the harmonic oscillator for which all the requirements can easily be fulfilled. In this case we have the basic equation of the form

$$ml_0^2 \frac{\partial \psi^2}{\partial t^2} + i\hbar \frac{\partial \psi}{\partial t} - \frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + \frac{m\omega_0^2}{2} x^2 \psi + l_0^2 k \psi = 0 \tag{9}$$

where we have put

$$\omega_0^2 = \lambda \frac{k}{m} \tag{10}$$

Note that the harmonic oscillator case, due to the relation (8), is the only case in which the basic equation (5) is linear. Therefore the full discussion of classical limits of quantum mechanics may be performed only in the framework of non - linear field equations which considerably complicates the problem.

In order to see some details of our limiting procedures, it is convenient to pass in equation (9) to dimensionless variables. The general form of such a change of variables is

$$t \;\rightarrow\; \tau \;=\; \omega_0 \, t \; A^{\alpha} \; B^{\beta} \tag{11.a}$$

$$x \;\rightarrow\; \xi \;=\; x \; l_0^{-1} \; A^{\gamma} \; B^{\delta} \tag{11.b}$$

where

$$A \;=\; \frac{\hbar}{m\omega_0 l_0^2} \tag{12.a}$$

and

$$B \;=\; \frac{k}{m\omega_0^2} \tag{12.b}$$

are two independent dimensionless constants constructed from our parameters. In the quantum limit the change of variables has to be independent of l_0 and therefore we must choose $\alpha \;=\; 0$ and $\gamma \;=\; -1/2$. Since in this case $\lambda \;=\; 1$, we have $B \;=\; 1$ and the powers β and δ are irrelevant. On the other hand, in the classical limit $\omega_0 \;\rightarrow\; 0$ and therefore $\beta \;=\; 1/2$. After the change of variables any solution of equation (9) depends on ξ, τ and in order to "loose" in it the x - dependence the remaining parameter δ must satisfy the inequality

$$\delta \leq \frac{1}{4} \tag{13}$$

This is the only way of passing from the general solution of equation (9) to a function which depends only on the time variable, as is required for solutions of the classical equations of motion. The traditional classical limit of quantum mechanics implemented by the limit $\hbar \;\rightarrow\; 0$ must therefore be performed after the limit $\omega_0 \;\rightarrow\; 0$ in order to keep meaning of the field $\psi(x,t)$. This fact is not seen from the equation (5), where both limits can be taken simultaneously.

More generally, the limit $\lambda \;\rightarrow\; 0$ has to be taken always before the limit $\hbar \;\rightarrow\; 0$. Therefore, in the classical limit all effects due to a "classical trace" of the quantum mechanical potentials disappear and in order to have a non - trivial classical limit of the quantum mechanical wave function we should in the usual approach start from the non - linear Schrödinger equation without any potential term. Otherwise, in the usual approach it is meaningless to speak about the classical limit of quantum mechanical wave equations and wave functions.

Since we are treating our basic equation (9) as a field equation and we adopt the usual quantum mechanical interpretation only for the limiting wave function $\psi_q(x,t)$ we do not impose the usual square - integrability condition on the field $\psi(x,t;\hbar,l_0)$.Instead of that we shall consider only those solutions of (9) which are bounded in the whole space - time. The general such solution may be written as

$$\psi(x,t;\hbar,l_0) \ = \ \sum_n \left(c_n e^{i\omega_n^+ t} \ + \ d_n e^{i\omega_n^- t}\right) \ H_n\left(x\sqrt{\frac{m\omega_0}{\hbar}}\right) \ e^{-\frac{m\omega_0^2}{2\hbar} \ x^2} \tag{14}$$

where $H_n(\xi)$ are the usual Hermite polynomials and

$$\omega_n^{\pm} == \frac{1}{2ml_0^2}\left[\hbar \pm \sqrt{\hbar^2 + 4\hbar\omega_0 m l_0^2 (n+\frac{1}{2}) + 4mkl_0^4}\right] \tag{15}$$

In the quantum limit $\omega_n^{(-)} \ \rightarrow \ \infty$ and hence $d_n = 0$, while

$$\omega_n^{(+)} \ \rightarrow \ \omega_0\left(n \ + \ \frac{1}{2}\right) \tag{16}$$

Thus, the solution (14) is going to the usual quantum mechanical oscillator wave function.

In the classical limit

$$\omega_n^{(\pm)} \ \rightarrow \ \pm\sqrt{\frac{k}{m}} \tag{17}$$

and taking the limit $\omega_0 \ \rightarrow \ 0$ before the limit $\hbar \ \rightarrow \ 0$ we get from (14) the classical solution

$$A \ \exp\left(\ i\sqrt{\frac{k}{m}} \ t \ \right) \ + \ B \ \exp\left(\ i\sqrt{\frac{k}{m}} \ t \ \right) \tag{18}$$

where

$$A \ = \sum_{n=0}^{\infty} c_n H_n(0) \tag{19.a}$$

and

$$B \ = \sum_{n=0}^{\infty} d_n H_n(0) \tag{19.b}$$

The formula (15) for small but non - zero l_0^2 gives a simple formula for the deviations of frequencies from usual quantum mechanical frequencies

$$\Delta\omega_n \approx \frac{2l_0^2}{\hbar}\left[k - m\omega_0^2\,(n + \frac{1}{2})\right] \tag{20}$$

If such deviations really are observed, (20) will determine the experimental value of l_0.

Up to now in our discussion we have notoriously manipulated the dimensional parameters. In order to make such manipulations more precise let us consider the Fourier transform of the field $\psi(x, t; \hbar, l_0)$ in the case when $\omega_0 = 0$. Then the field equation (9) takes the form

$$\left[l_0^2 m\,\left(\omega^2 - \omega_{cl}^2\right) + \hbar\left(\omega - \frac{\hbar\kappa^2}{2m}\right)\right]\psi(\omega, \kappa; \hbar, l_0^2) = 0 \tag{21}$$

For a classical particle

$$\omega \approx \omega_{cl} = \sqrt{\frac{k}{m}} \tag{22}$$

and ω differs considerably from $\frac{\hbar\kappa^2}{2m}$. Therefore the only way to satisfy the field equation is to put $\hbar = 0$. On the contrary, in the quantum case the frequency ω significantly differs from ω_{cl} while according to the basic de Broglie idea

$$\omega \approx \frac{\hbar\kappa^2}{2m} \tag{23}$$

Hence the only way to satisfy the field equation is to put $l_0 = 0$. More exactly, we are close to the classical case when

$$l_0^2 m\left|\frac{\omega^2 - \omega_{cl}^2}{\omega - \frac{\hbar}{2m}\kappa^2}\right| \ll \hbar \tag{24}$$

while we approach the quantum case when

$$l_0^2 \ll \left|\frac{\omega - \frac{\hbar}{2m}\kappa^2}{\omega^2 - \omega_{cl}^2}\right| \tag{25}$$

As we have seen, the parameter l_0 plays a crucial role in our approach and we need to give it a clear physical interpretation. To do this we would like to remind ,[8], that for any macroscopic theory it is necessary to define a finite length or resolution which is not explicitly present in the theory, but which determines the domain of applicability of the theory. Such a parameter should, however, appear explicitly in a theory which generalizes the given macroscopic theory. This is required in order to have the possibility to determine on the domain of applicability of the coarser initial theory. Our theory generalizes both the classical and the wave mechanics, and should therefore contain lengths which determine the domains of applicability of both these particular theories. For the classical mechanics we should forget about all effects connected with the Compton length proportional to $\hbar$. Therefore in the classical limit $\hbar \rightarrow 0$. The theory remains classical independently of the distance at which we are looking on the physical phenomena. In the quantum case we will be able to see the details of order of the Compton length only when we are looking at them from sufficiently small distances. Therefore we connect the length l_0 with the minimal distance of observation of phenomena and it is now clear that in order to see all quantum effects we must go with l_0 to zero just as our formalism requires.

The considerations presented above open a new way for a sufficiently precise discussion of the interrelations between the classical and the wave mechanics. As we have already mentioned, more insight into this problem can be obtained only when the basic field equation (5) becomes non-linear. This circumstance crucially complicates the problem however even if it cannot be solved it can at least be defined by our formalism.

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Footnotes and references

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is useful to read in order to understand the importance of the separation what is believed from what is established in physics. In particular it is useful for understanding the notion of a "limit" of a given theory.

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