Algebraic inconsistencies of a class of equations for the description of open systems and their resolution via Lie-admissible formulation

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ABSTRACT. In the present paper we study some algebraic and physical inconsistencies inherent in the Hamilton and Liouville-von Newmann equations with external terms, widely used for the dynamics of open Classical and Quantal systems. The treatment of these systems in the context of the Lie-admissible theory removes these inconsistencies and allows the formulation of a consistent theory for the open systems.

RESUME. Dans cet article nous étudions quelques incohérences algébriques et physiques inhérentes aux équations de Hamilton et de Liouville-von Neumann, avec termes externes, et qui sont largement employées dans la dynamique des systèmes ouverts classiques et quantiques. Le traitement de ces systèmes dans le cadre d'une théorie Lie-admissible supprime ces incohérences et permet la formulation d'une théorie cohérente des systèmes ouverts.

1. Introduction

It is well known that the dynamics of a closed isolated classical system is described by the Hamilton equations while that of the corresponding Quantum system is represented by a monoparametric group of unitary transformations in Hilbert space. This formalism is however insufficient for the description of irreversible phenomena characterizing a direction in time, such as dissipation in the Classical case and the quantum measurement, exponential decay in the Quantum case etc [1].

Various models have been proposed to include irreversibility and dissipativity, i.e. for the treatment of the open (non-Hamiltonian) Classical and Quantum systems.

In the classical case, we have the well known Hamilton equations with external terms representing the non-Hamiltonian forces [2].

In the Quantum regime, the dynamical description is achieved by the generalized master equations [3], or stochastic models, or the so-called Brussels School Theory [4].

The simplest dynamics for an open quantum system is represented by a semigroup of transformations. The most general form of the generators of these semigroups was identified by Lindblad [5], and applied in various physical phenomena [6,7], such as the damping of collective modes in deep inelastic collisions [8] and the unified dynamics of the microscopic and macroscopic systems [9].

Lindblad's approach leads to a linear equation of motion for the density operator containing, in general, some external terms.

The introduction of external terms in the equations of motion, in the Classical, as well as in the Quantum case, gives rise to some algebraic inconsistencies, which are resolved in the context of a new theory for the description of such systems, the so-called Lie-admissible theory.

Indeed, this theory provides the most general framework for the description of the dynamics of open systems, essentially based on the mathematical concept of Lie-admissible algebras.

The study of the algebraic and physical inconsistencies of equations of motion with external terms in the Classical and Quantum level and their resolutions via Lie-admissible formulation is the aim of this paper.

The paper is organized as follows: In section 2, the Hamilton equations with external terms are studied, i.e. the classical case. In section 3, we consider the Liouville-von Neumann equations with external terms, i.e. the Quantum case. Finally, the section 4 is devoted to concluding remarks.

2. Lie-admissible structure of Hamilton equations with external terms

At present, there are several well-established and interrelated formulations of Classical mechanics for the description of Newtonian systems with forces derivable from a potential, i.e.

$$m\ddot{q} - f(t, q, \dot{q}) = 0 \tag{2.1}$$

$$f = -\frac{\partial U}{\partial q} + \frac{d}{dt}\frac{\partial U}{\partial \dot{q}} \tag{2.2}$$

All these formulations are centrally dependent on the fundamental analytical equations of the theory, which are the conventional Lagrange equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0 \tag{2.3}$$

$$L = T(\dot{q}) - U(t, q, \dot{q}) \tag{2.4}$$

and the Hamilton equations:

$$\dot{q} = \frac{\partial H}{\partial p} \quad , \quad \dot{p} = -\frac{\partial H}{\partial q} \tag{2.5}$$

$$H = T(p) + U(t, q, p) \tag{2.6}$$

with the interconnecting Legendre transformation

$$p = \frac{\partial L}{\partial \dot{q}} \quad , \quad H = p\dot{q} - L \tag{2.7}$$

The time evolution law for a quantity $A(q, p)$, in phase space, is governed by the equation

$$\dot{A}(q, p) = [A, H] \tag{2.8}$$

where

$$[A, B] = \frac{\partial A}{\partial q}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial q} \tag{2.9}$$

is the Poisson bracket.

The algebraic structure underlying all formulations of classical mechanics is a Lie-algebra. This is due to the fact that the Poisson brackets satisfy the defining identities of a Lie algebra.

$$[A, B] + [B, A] = 0 \tag{2.10}$$

$$[[A, B], C] + [[B, A], C] + [[C, A], B] = 0 \tag{2.11}$$

In the case of existence of forces nonderivable from a potential, which are of the form:

$$F(t, q, \dot{q}) \neq -\frac{\partial U}{\partial q} + \frac{d}{dt}\frac{\partial U}{\partial \dot{q}}, \tag{2.12}$$

one of the methodological approaches is that which uses the following equations, originally proposed by Lagrange and Hamilton [2]

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = F(t, q, \dot{q}) \tag{2.13}$$

$$\dot{q} = \frac{\partial H}{\partial p} \quad , \quad \dot{p} = -\frac{\partial H}{\partial q} + F(t,q,p) \tag{2.14}$$

$$p = \frac{\partial L}{\partial \dot{q}} \tag{2.15}$$

$$H = p\dot{q} - L \tag{2.16}$$

In these equations the Hamiltonian can characterize the total energy, i.e., the sum of the kinetic and potential energies of all forces admitting a potential function, while all forces that do not admit a potential function are represented by the external terms. It must be noted here that we consider the total energy of a nonconservative (open) system which by definition is not conserved.

The time evolution of the quantity $A(q,p)$ is now given by the equation:

$$\dot{A}(q,p) = (A,H) = A \times H = \frac{\partial A}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial H}{\partial q} + \frac{\partial A}{\partial p}F(t,q,p) \tag{2.17}$$

It is easily proved that the brackets $(,)$ violate the conditions caracterizing any algebra [10].

Indeed, these brackets violate the right scalar and right distributive laws, i.e.,

$$\alpha \times (A \times B) = A \times (\alpha \times B) = (\alpha \times A) \times B \tag{2.18a}$$

$$(A \times B) \times \alpha \neq A \times (B \times \alpha) \neq (A \times \alpha) \times B \tag{2.18b}$$

and

$$(A + B) \times C = A \times C + B \times C \tag{2.19a}$$

$$A \times (B + C) \neq A \times B + A \times C \tag{2.19b}$$

Therefore, the transition from the contemporary Hamilton equations to their original form with external terms, implies not only the loss of Lie algebras but the loss of all algebras.

The above loss of algebras has various mathematical and physical implications in the Classical as well as in the Quantum theory. This will be extensively discussed in section 3 of this paper.

However, a reformulation of Hamilton equations is possible in an analytically identical way (in order to avoid alterations of the equations of motion), which is admitting of a consistent algebraic structure.

This algebraic structure must have two main characteristics. First, it must permit the representation of the time rate of variation of energy:

$$\dot{H} = (H, H) = \frac{\partial H}{\partial p} F \neq 0 \tag{2.20}$$

Then, the bracket $(,)$ cannot be symmetric.

Second, one can recover the Lie algebras, as a partial case, when we have not forces nonderivable from a potential, i.e.,

$$(A, B) = [A, B] \quad , \quad F = 0 \tag{2.21}$$

It has been pointed out that the above conditions identify the so-called Lie-admissible algebras which introduced in Mathematics by Albert [11].

In anticipation of section 3, where more information is given, we give now a "preliminary" definition of Lie-admissible algebras.

An algebra U, with elements (abstract) $a, b, c, \cdots$ and (generally nonassociative) product ab over a field F, is called a Lie-admissible algebra, when the attached algebra U^-, with the same vector space as U, but equipped with the product:

$$U^- : [a, b]_U = ab - ba \tag{2.22}$$

is Lie.

The most general possible algebras of the type considered are called general Lie-admissible algebras U when they verify no condition other than the Lie-admissibility law which can be written:

$$(a, b, c) + (b, c, a) + (c, a, b) = (c, b, a) + (b, a, c) + (a, c, b) \tag{2.23}$$

where

$$(a, b, c) = a(bc) - (ab)c \tag{2.24}$$

is called associator.

Further discussion of this topic demands a more effective formulation of the Hamilton equations [2].

Let us consider the 2n-component contravariant vector

$$\begin{aligned} a^\mu &= q^\mu \quad , \quad \mu = 1, 2, \cdots, n \\ a^\mu &= p_{\mu - n} \quad , \quad \mu = n+1, n+2, \cdots, 2n \end{aligned} \tag{2.25}$$

which spans a phase space, by assumption.

The Hamiltonian H can be written

$$H(t,q,p) = H(t,a) = H(t,a^\mu) \tag{2.26}$$

and by use of the contravariant form:

$$(\omega^{\mu\nu}) = \begin{pmatrix} 0_{n\times n} & 1_{n\times n} \\ -1_{n\times n} & 0_{n\times n} \end{pmatrix} \tag{2.27}$$

we can write:

$$\begin{pmatrix} \frac{\partial H}{\partial p} \\ -\frac{\partial H}{\partial q} \end{pmatrix} = (\omega^{\mu\nu}\frac{\partial H}{\partial a^\nu}) \tag{2.28}$$

The Hamilton equations (2.5), take now the form:

$$\dot{a}^\mu - \omega^{\mu\nu}\frac{\partial H}{\partial a^\nu} = 0 \quad , \quad \mu = 1,2,\cdots,2n \tag{2.29}$$

The Hamilton equations (2.14), with external terms, can be written in the form:

$$\dot{a}^\mu = S^{\mu\nu}\frac{\partial H}{\partial a^\nu} = \omega^{\mu\nu}\frac{\partial H}{\partial a^\nu} + T^{\mu\nu}\frac{\partial H}{\partial a^\nu} \tag{2.30a}$$

$$T^{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & -s \end{pmatrix} \quad , \quad s = \mathrm{diag}(0, F/(p/m)) \quad , \quad F = -s\frac{\partial H}{\partial p} \tag{2.30b}$$

Formulas (2.30) are used instead of (2.14) for the Hamilton equations with external terms because the equations (2.30) admit a consistent algebraic structure. Indeed, they admit the time evolution bracket $(,)$ of the equation

$$\dot{A}(a) \overset{\mathrm{def}}{=} (A,H) = \frac{\partial A}{\partial a^\mu}S^{\mu\nu}\frac{\partial A}{\partial a^\nu} = \frac{\partial A}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial H}{\partial q} + \frac{\partial A}{\partial p_i}S_{ij}\frac{\partial H}{\partial p_j} \tag{2.31}$$

which satisfies the right and left distributive and scalar laws. The brackets (A,H) characterizes an algebra and this is the Lie-admissible generalization of the Lie algebra.

In fact, the brackets (A,B) when written in the form:

$$(A,B) = \frac{\partial A}{\partial a^\mu}S^{\mu\nu}(t,a)\frac{\partial B}{\partial a^\nu} \tag{2.32}$$

with the S-tensor given by the symmetric form

$$S^{\mu\nu} = \omega^{\mu\nu} + T^{\mu\nu}(t,a) \tag{2.33}$$

verify firstly both right and left scalar and distributive laws and secondly they characterize a Lie-admissible algebra, because the attached brackets are Lie.

$$(A,B) - (B,A) = 2[A,B] \quad , \quad S^{\mu\nu} - S^{\nu\mu} = 2\omega^{\mu\nu} \tag{2.34}$$

The equations (2.30a) can be written in the form:

$$\dot{a}^\mu = S^{\mu\nu}\frac{\partial H}{\partial a^\nu} = (a^\mu, H) \tag{2.35}$$

and we call them Hamilton-admissible equations.

From the above it seems that the introduction of Lie-admissible algebras permits the regaining of a consistent mathematical structure for the description of the open Classical systems.

3. Lie-admissible structure of the Liouville-von Neumann equations with external terms

The Lie-admissible structure of Statistical mechanics has been initially investigated by Fronteau et al [12]. In the context of this theory we can define the Lie-admissible Liouville-von Neumann equation for the density matrix ρ, i.e. :

$$i\dot{\rho} = (H,\rho) = HS\rho - \rho S^{+}H \quad , \quad \hbar = 1 \tag{3.1}$$

where H the usual Hamilton operator of the system and $S \neq S^{+}$, is the operator describing, in general, nonconservative interactions.

The study and the applications of eq.(3.1) will be the subject of a forthcoming paper. The aim of this section is to identify the main mathematical and physical reasons supporting our assumption for the consistence of this equation.

First, let us review the case of the density matrix evolution without collisions, i.e.

$$-i\dot{\rho} = [\rho, H] = \rho H - H\rho \quad , \quad H = H^{+} \quad , \quad \hbar = 1 \tag{3.2}$$

As well known, this case is fully consistent at all mathematical as well as corresponding physical levels. The fundamental mathematical structure is the enveloping associative algebras ξ of operators $A, B, \cdots$ and trivial associative product "AB" with fundamental unit I characterized by

$$\xi : IA = AI = A \quad , \quad \forall A \in \xi \quad , \quad I = \mathrm{diag}(1, 1, 1, \cdots, 1) \tag{3.3}$$

The existence of a consistent enveloping associative algebra has numerous mathematical and physical implications. On mathematical grounds, it implies the existence of a consistent Lie algebra as the algebra ξ^- attached to ξ with brackets

$$\xi^- : [A, B] = AB - BA = LIE \tag{3.4}$$

The envelope ξ allow a consistent exponentiation to the group structure

$$U = e^{i\theta A} \quad , \quad U^+ = U^{-1} \quad , \quad A^+ = A \tag{3.5}$$

which, as well known, characterizes a unitary transformation. Finally, the existence of a consistent envelope ξ allow the construction of the representation theory, as well as numerous additional methodological procedures such as symplectic or naive quantization, etc.

The implications from a physical profile are far reaching. First, the existence of a consistent envelope with a consistent unit for *all* operators (i.e., for all physical quantities) allows the very formulation of fundamental physical quantities, such as that of spin, or of parity, or of "elementary particle" at large (which is exactly a representation of the enveloping associative algebra of a given space-time Lie algebra).

Last but not least, all mathematical algorithms of Eq.(3.2) have a clear physical meaning in the sense that the operator "$\vec{p}$" is the linear momentum, "H" is the total energy, "$\vec{M}$"="$\vec{r} \times \vec{p}$" is the angular momentum, etc. Finally, formulation (3.2) is form-invariant under unitary transformations, that is, physical laws can be formulated in an invariant way valid throughout the universe. All these aspects are well treated in any sound textbook on the foundations of quantum mechanics or of quantum statistical mechanics.

We pass now to the examination of the modification of the case (3.2) caused by collision or other external terms

$$-i\dot{\rho} = [\rho, H] + \Gamma \quad , \quad \hbar = 1 \tag{3.6}$$

with specific reference to the *Lindblad equations* [5], i.e.,

$$-i\dot{\rho} = [\rho, H] + \frac{1}{2} i \sum_j (\{V_j^* V_j, \rho\} - 2V_j \rho V_j) \quad , \quad \hbar = 1 \tag{3.7}$$

The following series of breakdowns of the mathematical and physical foundations of the case (3.2) were identified by Santilli (see ref. 13) already in his original proposal of Eqs.(3.1), and then elaborated in more details in the subsequent literature [14-20]. When applied to the Lindblad equations, these problematic aspects can be summarized as follows.

1 *Lindblad equations (3.7) imply the breakdown of the universal enveloping associative algebra ξ of Eq.(3.2).* This is evidently due to the fact that the trivial associative products as used, say, in the terms $H\rho$ or $V\rho$ or V^*V are indeed defined in the theory, but the algebra looses its "universal" and independently, its "enveloping" character. In particular, this implies the loss of the existence of the ordered infinite-dimensional basis of the consistent envelope ξ, the loss of the Poincaré-Birkhoff-Witt theorem, etc.

2 *Lindblad equations (3.7) imply the loss of the unit as an element of the center of the associative envelope.* In fact, the trivial unit element I can be introduced as in Eqs.(3.7) but, since there is no consistent associative envelope (Problematic Aspect 1), it loses its meaning as an element of the center of the algebra. In turn, this has profound physical implications (see below).

3 *The brackets of Lindblad equations (3.7) do not characterize a consistent algebra.* In the pure case (3.2), the brackets are given by the familiar form $A \times B \stackrel{\text{def}}{=} [A, B] = AB - BA$ which, first of all, verifies all conditions to characterize an algebra (scalar law, left and right distributive law, etc.), and, second, that algebra turns out to be Lie. To study the corresponding situation for Lindblad equations, one can introduce the "product"

$$A \times B \stackrel{\text{def}}{=} [A, B] + \frac{1}{2} i \sum_j (\{V_j^* V_j, A\} - 2V_j A V_j) \tag{3.8}$$

which is defined for fixed V_j and V_j^*. It is easy to see that product (3.2) violates the condition to characterize any algebra.

The physical implications of the above mathematical problematic aspects are very deep indeed, and they were also identified by Santilli beginning with the original proposal [13] of his Lie-admissible equations (3.1). In fact, by specializing again the latter analysis to Eqs.(3.7), we have the following consequences:

4 *There exist serious problematic aspects for the consistent definition of the measurement theory for Lindblad equations (3.7).* This is, clearly, a direct physical consequence of Problematic Aspect 2 (loss of the unit of the theory). In fact, predictions can be correctly formulated for the case (3.2) and the experimental measurements consistently associated to such theoretical predictions, because Eq.(3.2) possess a consistent unit I as per Eqs.(3.3). The expectation value of an observable A is obtained from the formula: $< A >= Tr(\rho A I)$, while the probability of finding the eigenvalue n, where $A|n >= n|n >$, from the formula $p_n = Tr(\Lambda_n \rho I)$, where Λ_n the projector in the one dimensional subspace spanned by the eigenfunction $|n >$ and I the fundamental unit of equation (3.3). In Lindblad equations (3.7), there is the loss of the notion of unit from various counts (loss of the enveloping algebra, loss of the center of such algebra, etc). As a result, the measurement theory cannot be consistently *defined* for Eqs.(3.7), let alone treated.

5 *Lindblad equations (3.7) imply the loss of a form-invariant formulation of the Laws of Nature.* As also well known, the case (3.2) allows a form invariant formulation of the Laws of Nature because the brackets $[A, B]$ are form invariant under unitary transformations, i.e.

$$U[A, B]U^+ = [A', B'] \quad , \quad U^+ = U^{-1} \tag{3.9}$$

In particular, the brackets are invariant under the time evolution characterized by themselves (which is precisely unitary). In the case of Lindblad equations (3.7), this additional fundamental property of quantum mechanics is lost. In fact, the equations characterize, this time, a *nonunitary* time evolution. It is then a matter of simple calculations to see that the transformed product (3.2) under nonunitary transformations is not algebraically equivalent to the original product, i.e.

$$U(A \times B)U^+ \neq A' \times B' \quad , \quad U^+ \neq U^{-1} \tag{3.10}$$

Thus, Lindblad equations (3.7) are not form invariant under the time evolution characterized by themselves and, consequently, there is no known possibility of formulating Laws of Nature in a form invariant way.

6 *Lindblad equations (3.7) imply the loss of the contemporary notion of particle.* In fact, by "particle" we mean a certain representation of a given space-time symmetry which, mathematically, means a representation of the universal enveloping *associative* algebra of the given space-time Lie algebra. In Lindblad equations (3.7) we have, not only the loss of any consistent algebra in the product of the time evolution (Problematic Aspect 3), but actually the more fundamental loss of the underlying associative envelope. The contemporary notion of "particle" is then irreconcilably lost. The statistics considered must be referred to an ensemble of unidentifyable or otherwise unknown objects.

7 *Lindblad equations (3.7) imply the loss of the notion of spin and any other physical characteristics centrally dependent on the existence of a consistent algebra in the brackets of the time evolution.* We are here trying to express the fact that the loss of the notion of particle at the representation level (Problematic Aspect 6) is only a first layer of problems. The second layer exists, specifically, at the level of individual quantities, such as the spin. In fact, for the case (3.2) we can consistently define spin because of the consistent formulation of the $SU(2)$-spin algebra. More specifically, the latter algebra has mathematical and physical meaning for equations (3.2) because the brackets of the spin algebra coincide with the fundamental brackets of the time evolution law. All this is irreconcilably gone for Lindblad equations (3.7). In fact, one could aprioristically introduce the words "$SU(2)$-spin" and talk about, say, an ensemble of "particles with spin 1/2", but this statement would have not hopes for mathematical or physical consistency, evidently because the brackets $[A, B]$ of the $SU(2)$ algebra have no meaning for Lindblad equations (3.7), mathematically and physically.

The above seven problematic aspects of Eqs.(3.7) are only a part of the problematic aspects identified by Santilli in his study of algebraically inconsistent modification of the Lie brackets $[A, B]$, such as: Eqs.(3.7) imply the breakdown of quantization (e.g., prequantization in symplectic geometry) which is notoriously dependent on the existence of consistent algebraic structures both before and after the mapping (classical and operator levels); the notion of quantum of energy has problematic aspects in its very introduction; there exist numerous other operator-type problematic aspects for which we refer the interested reader to Santilli's original studies [13-20].

We now pass to a review of the central aspects of this paper, namely, the resolution of the above problematic aspects. In fact, *Santilli proposed his fundamental Lie-admissible equations (3.1) because they are capable of resolving all problematics aspects* 1 *through* 7 *above.* To see this occurence, one must first understand the mathematical structure of the Lie-admissible equations and then study the physical profiles.

Eqs.(3.1) are based on the existence of a consistent, bimodular, left and right generalization of the envelope of quantum mechanics. Select one direction of time, say the forward. Then the equations underlying (3.1) are given in their Schrödinger-type form by [18,19]

$$i\frac{\partial}{\partial t}\Psi = H \triangleright \Psi \stackrel{\text{def}}{=} HS\Psi \tag{3.11}$$

with corresponding, inequivalent form for the backward motion,

$$-i\Phi\frac{\overleftarrow{\partial}}{\partial t} = \Phi \triangleleft H \stackrel{\text{def}}{=} \Phi RH \quad , \quad S^{+} = R \neq S, \tag{3.12}$$

This is done to ensure the *nonconservative* nature of the physical system considered via his intrinsic *irreversible* structure, as typical of all *nonunitary* time evolutions. Conservation/reversibility is a trivial particular case, when desired, when $S = S^{+} = R$.

Each equation (3.11) and (3.12) is *modular-isotopic* in the sense that the conventional modular action of quantum mechanics is lifted "isotopically" that is in a linearity preserving and associativity preserving way.

$$H\Psi \to H \triangleright \Psi \quad , \quad \Phi H \to \Phi \triangleleft H \tag{3.13}$$

This implies the generalization of the conventional envelope ξ, Eq.(3.3), into two different forms, one per each direction in time

$$\xi^{\triangleright} : A \triangleright B = ASB \quad , \quad I^{\triangleright} A = A \triangleright I^{\triangleright} \quad , \quad I^{\triangleright} = S^{-1} \tag{3.14a}$$

$$^{\triangleleft}\xi : A \triangleleft B = ARB \quad , \quad {}^{\triangleleft}I \triangleleft A = A \triangleleft^{\triangleleft} I \quad , \quad {}^{\triangleleft}I = R^{-1} \tag{3.14b}$$

The above ultimate structure of Santilli's Lie-admissible equations resolves Problematic Aspects 1 and 2 of Lindblad equations (3.7) by its very conception (existence of generalized but consistent envelope and related units). In particular, specific mathematical studies have proved that the envelopes $\xi^{\triangleright}$ and ${}^{\triangleleft}\xi$ preserve all the properties of the old envelopes (Poincaré-Birkhoff-Witt theorem, ordered infinite-dimensional basis, etc.). In particular, they allow a consistent exponentiation, of course, one per each direction of time, according to the rules

$$\xi^{\triangleright} : U^{\triangleright} = e^{iA\theta}|_{\xi^{\triangleright}} = I^{\triangleright} e^{iA\triangleright\theta}|_{\xi} = I^{\triangleright} e^{iAS\theta} \tag{3.15a}$$

$$^{\triangleleft}\xi :^{\triangleleft} U = e^{i\theta A}|_{^{\triangleleft}\xi} =^{\triangleleft} I e^{i\theta\triangleleft A}|_{\xi} = e^{i\theta RA\triangleleft} I \tag{3.15b}$$

This confirms the existence of a consistent representation theory, of course, of generalized nature (the so-called bi-representations on bimodular vector spaces).

The reader should note the transition in Eqs.(3.15) from the exponentiation in the new envelopes $\xi^{\triangleright}$ and ${}^{\triangleleft}\xi$ to their reformulation in terms of the old envelope ξ for facility of explicit expressions, with the understanding that the mathematically correct exponentians are those in the new envelopes. For more details on these aspects, the reader may consult ref. 18, 19, where the underlying Hilbert space structure is identified too.

The reader should be aware that the transition for envelope ξ to one of its generalizations is nontrivial. In fact, squares (and higher powers) of operators such as $p^2 = pp$ are inconsistent in the generalized theory (they break the linearity condition [18]). The consistent square is instead given by $p \triangleright p$ or $p \triangleleft p$ depending on the time arrow.

We now pass to Heisenberg-type formulations. A generalization of the standard transition from Schrödinger to Heisenberg equations leads to Santilli's fundamental time evolution law associated to Eqs.(3.11) and (3.12) [13].

$$i\dot{A} = (A, H) \stackrel{\text{def}}{=} A \triangleleft H - H \triangleright A = ARH - HSA \tag{3.16}$$

with integrated form

$$A(t) = e^{iHSt} A(0) e^{-itRH} = I^{\triangleright} e^{iH\triangleright t} \triangleright A(0) \triangleleft e^{-it\triangleleft H\triangleleft} I \tag{3.17}$$

which is evidently *nonunitary* by conception.

The fundamental product of Santilli's formulation is now given by

$$U : A \times B \stackrel{\text{def}}{=} (A, B) \tag{3.18}$$

It is important for the reader to know that, unlike Lindblad product (3.8), Santilli's product (A, B), first and above all verifies all the conditions to characterize an algebra (2.18), (2.19) i.e., the distributive and scalar laws

$$\begin{aligned} (A+B)\times C &= A\times C + B\times C \\ A\times(B+C) &= A\times B + A\times C \\ \alpha(B\times C) &= (\alpha B)\times C \\ (A\times B)\alpha &= A\times(B\alpha) \end{aligned} \tag{3.19}$$

Secondly, the algebra turns out to be Lie-admissible in the sense that the attached algebra U^-, which is the same vector space as U, but equipped with the attached antisymmetric product

$$U^- = [A, B]_U = (A, B) - (B, A) = ATB - BTA \quad , \quad T = R + S \tag{3.20}$$

is Lie. In fact, it is instructive for the interested reader to check that, despite its generalized structure, product (3.20) verifies all Lie algebra axioms. In actually, product (3.20) characterizes an important generalization of Lie's theory called by Santilli *Lie-isotopic theory* [2,14], and which is evidently an intermediary formulation between the simplest possible product $AB - BA$, and the most general possible Lie-admissible form $ARB - BSA$.

The above findings resolve, by conception, Problematic Aspect 3 (loss of a consistent algebra in the brackets of the time evolution).

To summarize these mathematical aspects, Santilli's fundamental Lie-admissible equations are based on a mathematically consistent generalization of the enveloping associative algebra of quantum mechanics which, from the original simple form ξ with product AB valid for both directions of time, is generalized into forms (3.14) which first of all, are still associative, and, still universal and enveloping, although different for forward and backward in time (to achieve the desired physical property of characterizing nonconservative irreversible processes). Second, the simplest conceivable Lie brackets $AB - BA$ are generalized into the form $ARB - BSA$ which, first of all, characterize a consistent algebra and, then, that algebra turn out to be of the Lie-admissible type, i.e., possessing the well defined Lie content (3.20). Third, unlike Lindblad equations (3.7), exponentiation in Santilli's theory can be consistently defined, thus leading to the formulation of a consistent exponential form, Eqs.(3.17) for the characterization of the time evolution in a finite form.

Equivalently, we can say, that Eqs.(3.1) are those emerging from the finite form (3.17) when considered in the neighbourhood of the generalized identities.

The above consistent mathematical structure underlying Eqs.(3.1) readily allows the resolution of all remaining problematic aspects of Lindblad equations.

To begin, *the measurement theory can now be consistently defined for Eqs.(3.1)*, because of the existence, first, of consistent envelopes, and, more specifically, of consistent centers with well defined units. Of course, the measurement theory is now divided into two, one for measurements forward in time, and the other for measurements backward in time, with the understanding that such measurements of the *same* physical quantities must be *different* under irreversibility for physical consistency of the theory. This resolves Problematic Aspect 4, with the understanding that the measurement theory can be consistently *formulated* for Eqs.(3.1), but the studies are just at the beginning.

Next, *Santilli's fundamental equations (3.16) are form-invariant under the most general possible, generally nonunitary transformations* as known since their original proposal [13]. In fact, the Lie-admissible character persists under transformations (3.17)

$$U^{\triangleright} \triangleright (A, B) \,{}^{\triangleleft}\!\triangleleft\, U = (A', B') \tag{3.21}$$

In particular, such Lie-admissible character is preserved by the time evolution characterized by the brackets themselves. Fundamental Equations (3.1), not only allow the form invariant expression of the Laws of Nature, but also the form invariance under the most general possible transformations (recall that in case (3.2) the form invariance exists only under *unitary* transformations).

Next, *Santilli's fundamental equations (3.16) allow the consistent definition of the notion of "particle", of course, as a suitable generalization of the conventional notion for the pure case (3.2).* This is due to the existence of consistent envelopes which permit the consistent definition of the representation theory, and Problematic Aspect 5 of Lindblad equations (3.7) is resolved. We should stress here that we have a consistent *formulation* of particles in Santilli's theory, but its actual, explicit computation is under way. In actuality, we are referring here to one of the most complex notions in contemporary mathematics. *Santilli's notion of particle is a bi-representation of a Lie-admissible algebra of operators on* *bimodular vector spaces* [15]. Predictably, such a notion implies the generalization of all conventional characteristics under sole electromagnetic interactions without collisions as in the case (3.2), including a generalization of the intrinsic characteristics. This point is so fundamental that it was identified already in the original proposal of Lie-admissible equations (3.1), see in Ref.(13) Sect. 4.11 on the theory of "mutation" of particles. This should not be surprising to the attentive reader. In fact, the conventional notion of particle holds under local, potential, action-at-a-distance, unitary interactions, while the particle under consideration here is under the most general dynamical conditions that are mathematically conceivable today, and represents extreme physical conditions, such as a proton in the core of a star undergoing gravitational collapse.

Furthermore, *the existence of a consistent algebra in the brackets of the time evolution (3.16) allows the consistent formulation of physical characteristics, such as spin, while under extreme nonconservative/irreversible conditions* [17]. For instance, the $SU(2)$-spin for the case (3.2) is replaced by the $SU(2)$-admissible quantity for Eqs.(3.16). We are again referring to the fact that the problem of spin and other physical characteristics can be consistently *formulated* for Eqs.(3.1), because the brackets characterize a consistent algebra, and that algebra turns out to have a well defined Lie algebra content, but the explicit study of the emerging new context is under way.

Finally, we should mention the property that *Santilli's fundamental Lie-admissible equations in their infinitesimal form (3.16) or finite form (3.17) are directly universal, in the sense that they represent all possible nonunitary time evolutions (universality), directly in the frame of the experimenter without any need of local transformations (direct universality).* In fact, under certain continuity restrictions, any non-Hermitean Hamiltonian H can be always decomposed into the form

$$H = H_0 S \quad , \quad H^+ = R H_0 \quad , \quad H_0^+ = H_0 \quad , \quad S^+ = R \tag{3.22}$$

thus leading to Lie-admissible form (3.16).

4. Conclusion

In this paper we identify some mathematical and physical inconsistencies of a class of equations used for the dynamical description of open systems. We then point out that these inconsistencies are removed in the context of the new more general theory of Lie-admissible theory.

At the level of Classical mechanics the Hamilton equations with external terms (2.14) are reformulated in the form of (2.35) permitting the recovering of a consistent algebraic structure.

At the Quantum mechanical level, any given nonconservative, nonunitary equation, including Lindblad equations (3.7), can be always rewritten in Santilli's fundamental Lie-admissible form (3.16). This reformulation is not purely formal, because it allows the resolution of truly fundamental, mathematical and physical, problematic aspects which are inherent to algebraically inconsistent time evolutions. In turn, the rigorous formulation of physical problems prevents predictable misconceptions, such as the rather general beliefs that the spin holds for Eqs.(3.7) without any change as compared to the case (3.2). Finally, the identification of the ultimate mathematical structure underlying nonconservative, irreversible and nonunitary processes permits the systematic study of the entire class, rather than one individual element. To put it differently, Lindblad equations (3.7) can indeed be written in Santilli's Lie-admissible form (3.16). The point is that the latter also represent infinite varieties of equations structurally more general than the formers.

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