Quaternion mechanics and electromagnetism

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ABSTRACT. We present here a new formulation of Maxwell's equations based on an extension of the Hamiltonian formalism for a single particle. We introduce a quaternion action, whose scalar part is the usual action S, and which contains a vector component $\mathbf{S}$. From this quaternion action the generalized momentum and generalized Hamiltonian are defined. After imposition of suitable constraints, S and $\mathbf{S}$ turn out to be the usual potential and potential vector of standard Electromagnetic theory. This approach allows to express the quantization of charge as a quantum condition of the Bohr-Sommerfeld type on the generalized momentum. In addition, gauge invariance and minimal coupling are unified into a single operation. Finally the fundamental difference between Newtonian gravitation and Electromagnetism is reviewed.

RESUME. Nous proposons une nouvelle formulation des équations de Maxwell, construite à partir d'une extension du formalisme Hamiltonien pour une particule simple. Nous introduisons une action quaternionienne dont la partie scalaire est l'action ordinaire S, et qui comprend une composante vectorielle $\mathbf{S}$. A partir de cette action quaternionienne nous définissons le moment et l'Hamiltonien généralisés. Après imposition de contraintes adéquates, S and $\mathbf{S}$ se révèlent être les usuels potentiels scalaire et vecteur de l'électromagnétisme standard. Cette approche permet l'expression de la quantification de la charge comme une condition quantique du type Bohr-Sommerfeld imposée au moment généralisé. De plus, l'invariance de jauge et le couplage minimal se trouvent réunis au sein d'une même opération. Enfin la différence fondamentale entre la gravitation Newtonienne et l'électromagnétisme est ré-examinée.

Introduction

The customary Hamiltonian formulation of Electromagnetism is based on a reformulation of Maxwell's equations in terms of canonically conjugated quantities **E** and **A** and Hamiltonian H equal to the electromagnetic energy. The advantage of this formulation is that the transition to QED is facilitated. However if the Hamiltonian formalism is retained, the connection to the mechanics of a point particle is lost, since the new canonical variables bear no relationship to position and ordinary momentum (mass $\times$ velocity).

We present here a different formulation, in which the Hamiltonian formalism is extended to the case where the Action is a quaternion in ordinary space, i.e. with a scalar and a vector part. The main reason for doing so is to be able to obtain a momentum which is not merely the gradient of a (scalar) action, but contains a rotational component as well. Although the original motivation was fluid mechanics, where the role of vorticity is crucial, this approach yields very interesting insights in electromagnetism, in particular gauge invariance and minimal coupling.

1. Generalized Hamiltonian and momentum and Maxwell's equations

In order to treat the case of rotational momentum we introduce a vector action **S** in addition to the scalar action S. We define the generalized momentum and Hamiltonian as follows :

$$-\frac{H}{c} + \mathbf{P} = (\frac{\partial}{c\partial t} + \nabla) * (S + \mathbf{S}) \tag{1}$$

where $*$ stands for quaternion multiplication [1]. Or

$$\mathbf{P} = \nabla S + \frac{\partial \mathbf{S}}{c\partial t} + \nabla \times \mathbf{S} \tag{2}$$

$$H = c\nabla \cdot S - \frac{\partial S}{\partial t} \tag{3}$$

When **S** equals zero we recover the usual definition of momentum and Hamiltonian. Note that the momentum **P** has nothing to do with the so-called electromagnetic momentum i.e. the Poynting vector.

Th expression for **P** suggests that we rewrite **P** in the form:

$$\mathbf{P} = -\mathbf{E} + \mathbf{B} \tag{4}$$

where **E** and **B** are electric and magnetic fields respectively. From this we see that two of Maxwell's equations are automatically satisfied, namely:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{c\partial t} \tag{5}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{6}$$

However the next assumptions are crucial to make **E** and **B** actual electric and magnetic fields, despite the fact that they already have the right expression in terms of S and $\mathbf{S}$. We therefore need to impose the following conditions:

$$\iint \mathbf{P} \cdot d\mathbf{A} = \frac{q}{\epsilon_0} \tag{7}$$

This defines the charge q. We interpret this condition on the momentum as a quantum condition of the Bohr-Sommerfeld type. The difference with the usual quantum condition on the ordinary momentum lies in the integration over area. In both cases however, quantification results in having the momentum scaling inversely with distance (q/r^2 for electromagnetism, h/λ for mechanics) instead of being independent of it. Physically, the quantum of action is connected to motion, whereas the quantum of charge is not. The reason is that the Gauss integral is defined for any area enclosing the source, so that in addition to possessing spherical symmetry, the generalized momentum **P** is defined for all values of r. This precludes its interpretation as a velocity, or even a velocity field. The quantification of charge is thus explained by the same type of mechanism as that of action, without the need of any extra hypothesis (such as Dirac monopoles [2]).

As fas as dimensions are concerned, the *rhs* of (7) has a dimension of mass $\times$ vol $\times$ time^{-1}. We choose q to have the dimension time $^{-1}$ and ϵ_0 the dimension $vol^{-1} \times mass^{-1}$ (see section 2 below).

$$\frac{\partial}{\partial t} = -\mathbf{v} \cdot \nabla \tag{8}$$

The fields are advected with a particle of velocity $\mathbf{v}$.

$$\frac{\partial}{\partial t}(\nabla S) = -\nabla(c\nabla \cdot \mathbf{S}) \tag{9}$$

This is the Lorentz condition which says that position and the S part of the generalized momentum (2) are canonically conjugated with respect to the **S** part of the generalized Hamiltonian (3).

These three conditions give the remaining Maxwell's equations as well as the inhomogeneous wave equations for the scalar and vector potential (see appendix).

2. Coupling with matter

It has been experimentally found that the electric force on a charge q is $\mathbf{F} = q\mathbf{E}$. Since $\mathbf{E}$ has the dimension of a momentum, q takes the dimension of $time^{-1}$.

The coupling of the electromagnetic field with matter results in the following substitutions in the ordinary Hamiltonian and momentum [3] :

$$\mathbf{p} = m\mathbf{v} \rightarrow \mathbf{p} = m\mathbf{v} + \frac{q}{c}\mathbf{A} \tag{10}$$

$$H = E \rightarrow H = E + q\phi \tag{11}$$

This should be compared to the expressions for the generalized Hamiltonian and the Electric part of the momentum (1), rewritten with the usual notation for vector and scalar potentials :

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{c\partial t} \tag{12}$$

$$H = c\nabla \cdot \mathbf{A} - \frac{\partial \phi}{\partial t} \tag{13}$$

In the process of coupling the EM field to a particle, only the explicit time dependent part of $\mathbf{E}$ and the generalized Hamiltonian are added to the usual momentum ($m\mathbf{v}$) and Hamiltonian H (Energy E), provided the following rule is used :

$$-\frac{\partial}{\partial t} \rightarrow q \quad \text{i.e.} \quad -\frac{\partial \phi}{\partial t} \rightarrow q\phi \quad , \quad -\frac{\partial \mathbf{A}}{c\partial t} \rightarrow q\mathbf{A} \tag{14}$$

Since the magnetic part of the generalized momentum does not contribute to the definition of the charge, it does not take part in the coupling as expected.

3. Gauge invariance

Maxwell's equations are invariant under the transformations [4] :

$$\mathbf{A} \rightarrow \mathbf{A} + c\nabla\chi \tag{15}$$

$$\phi \to \phi - \frac{\partial \chi}{\partial t} \tag{16}$$

Dimensional analysis yields that χ has the dimension of *Action* $\times$ *time*.

Multiplying (15) by q/c yields :

$$q\frac{\mathbf{A}}{c} \to q\frac{\mathbf{A}}{c} + \nabla(q\chi) \tag{17}$$

Now, according to the minimal coupling rule, $q\chi$ has the dimension of an action, hence

$$q\frac{\mathbf{A}}{c} \to q\frac{\mathbf{A}}{c} + \nabla(\text{action}) \tag{18}$$

$$q\frac{\mathbf{A}}{c} \to q\frac{\mathbf{A}}{c} + \mathbf{p} \tag{19}$$

Also (16) gives :

$$q\phi \to q\phi - \frac{\partial (q\chi)}{\partial t} \tag{20}$$

$$q\phi \to q\phi - \frac{\partial (\text{action})}{\partial t} \tag{21}$$

$$q\phi \to q\phi + H \tag{22}$$

where H and $\mathbf{p}$ are ordinary Hamiltonian (i.e. energy) and momentum (i.e. mass times velocity). We then obtain the gauge invariance as being the dual (or reverse) of the matter-field coupling i.e. :

$$\begin{array}{cc} \text{matter-field coupling} & \text{gauge invariance} \\ \mathbf{p} \to \mathbf{p} + q\mathbf{A}/c & q\mathbf{A}/c \to q\mathbf{A}/c + \mathbf{p} \\ H \to H + q\varphi & q\varphi \to q\varphi + H \end{array} \tag{23}$$

Hence gauge invariance and minimal coupling are unified into a single operation. In other words the transformation that leaves Hamilton's equation invariant in presence of an electromagnetic field, and the one that leaves Maxwell's equations invariant are parts of a single operation.

Conclusion.

This formulation of electromagnetism yields the following results.

The charge plays the role of the quantum of flux of momentum, in the same way the Planck constant is the quantum of circulation of momentum (or action).

The identification of the dimension of charge as $time^{-1}$ allows minimal coupling and gauge invariance to be unified into a single operation.

The invariance of charge takes on a different formulation, as does the Lorentz transformation : Instead of deriving Maxwell's equations by going over from the static case to the dynamic (uniform velocity) one, the complete expression for $\mathbf{E}$ and $\mathbf{B}$ is first obtained, and then constraints are imposed. This means that the covariance of Maxwell's equations can be explained (or viewed) in a very different fashion. Namely it is the imposition of the set of constraints which results in having $\mathbf{S}$ proportional to the velocity of the charge. Only at this point can we separate the velocity dependent and independent effects in EM and derive the Lorentz transformation. This suggests that the principle of covariance (in the special relativistic sense) is a composite principle. Extending this formalism to treat gravitation might result in decomposing the principle of general covariance as well. This will require richer objects than ordinary quaternions.

Finally the profound difference between (Newtonian) gravitation and Electromagnetism is revealed in the expressions for mass and charge :

$$m = \iint \mathbf{g} \cdot d\mathbf{A}$$

$$q = \iint \mathbf{P} \cdot d\mathbf{A}$$

where $\mathbf{P}$ is a momentum whereas $\mathbf{g}$ is the gradient of a potential. This results in having the force being the time derivative of the ordinary momentum ($m\mathbf{v}$), while being proportional to the electric momentum. As a consequence the charge is quantized whereas mass is not. This illustrates well the limits of the present formalism and the need to extend it.

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Appendix

The derivation follows that of Rosser [5], but in different order. Also the constant μ_0 is introduced with a different relationship to ϵ_0. It is reproduced here for clarity.

From

$$\nabla \cdot \mathbf{E} = \frac{q}{\epsilon_0} \tag{1}$$

using Lorentz condition, we obtain

$$\Box \phi = -\frac{q}{\epsilon_0} \tag{2}$$

hence

$$\phi = \frac{q}{4\pi\epsilon_0 s} \tag{3}$$

where

$$s = r - \frac{\mathbf{r} \cdot \mathbf{v}}{c} \tag{4}$$

and $\mathbf{v}$ is the velocity of the particle.

Using Lorentz condition again yields

$$\nabla \cdot \mathbf{A} = -\frac{1}{c}\frac{\partial \phi}{\partial t} = \mathbf{v} \cdot \nabla \phi = \frac{q}{4\pi c \epsilon_0} \mathbf{v} \cdot \nabla (\frac{1}{s}) \tag{5}$$

since $\mathbf{v}$ is constant we obtain

$$\mathbf{A} = \phi \frac{\mathbf{v}}{c} \tag{6}$$

and therefore

$$\Box \mathbf{A} = -\frac{p\mathbf{v}}{c\epsilon_0} = -\mu_0 \mathbf{J} \tag{7}$$

where

$$\mu_0 \equiv \frac{1}{\epsilon_0 c} \tag{8}$$

Adding equation (7) and the Lorentz condition gives

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{\partial \mathbf{E}}{c \partial t} \tag{9}$$

In vacuum, waves propagate with velocity c. This identifies c with the velocity of light. It is straightforward to check that $\epsilon_0 \mathbf{E}^2$ and $\mathbf{B}^2/c\mu_0$ have dimension of energy density.

Références

[1] A. Kyrala, *Theoretical physics* (W.B. Saunders & co, 1967) p. 262.
[2] P. Horvathy, *Introduction to monopoles* (Napoli, Bibliopolis, 1988) p. 21.
[3] J. Aharoni, *The special theory of relativity* (Oxford University Press, 1965) p. 175.
[4] W. Panofsky, *Classical Electricity and magnetism* (Addison-Wesley, 1955) p. 241.
[5] W.G. Rosser, *Classical Electromagnetism via relativity* (Plenum press, 1968) Chap. 5.

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