Standard and non standard quantum limits for measurement errors

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1. Generalized Scheme of Indirect Measurements

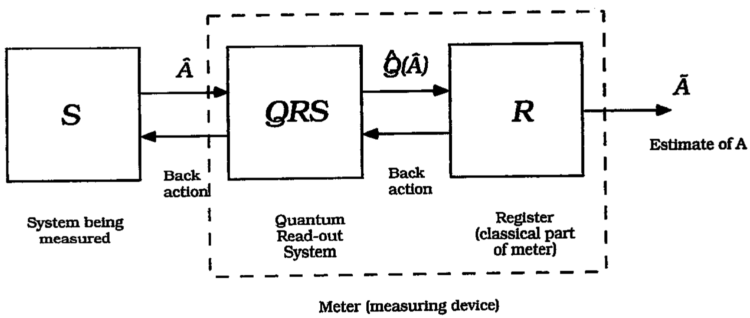

Figure 1.

The apparatus used in any measurement consists of a sequence of stages. Measurement theory asserts that, although the early stages of the apparatus may behave quantum mechanically, the later stages must be classical. There is no universally accepted definition of classical. We shall regard a stage as classical if the quantum mechanical uncertainties of subsequent stages have no significant influence on the overall accuracy of the measurement. First stages, which behave quantum mechanically, interact with a system under test reversibly. These stages were called Quantum Read-out System (QRS) or quantum transducer. The classical part of the measuring device may be called register. Interaction between the QRS and the register is irreversible.[1] In the register the so called "dequantization" of the signal takes place, that is a microscopic change in the QRS produces a macroscopic change in the register. Generalized scheme of measurement can be represented by the following figure 1 [1,2]. $\hat{A}$ is an observable being measured. $\hat{Q}(\hat{A})$ is QRS's observable changing by $\hat{A}$.

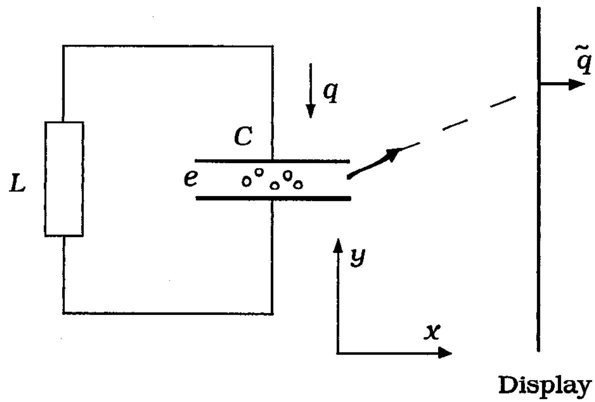

Figure 2.

Example. The scheme of a charge measurement by means of an electron beam. The LC-circuit $\to$ System. The electron beam $\to$ QRS. The display $\to$ Register. $\hat{q} \to \hat{A}$. The electron momentum $\hat{p}_y \to \hat{Q}(\hat{A})$. The QRS's interaction with the system and with the register occurs *subsequently* in time. It is of importance to emphasize that at least one stage of the measuring device interacts with the other impulsively. $\hat{\rho}_{1s}$ is the system's initial state perturbed through interaction of QRS with the system. This state is always mixed.

[1] If this is not the case it could be prepared the state of system in which the uncertainty relation is not valid.

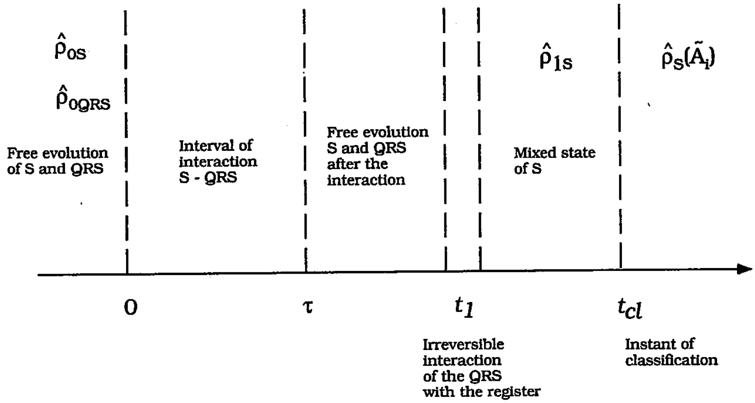

Figure 3. System state evolution during measurement.

$\hat{\rho}(\tilde{A}_j)$ is the system's states after the classification of system in accordance to the results of the measurements. In case of the exact measurement of A the state $\hat{\rho}_s(\tilde{A}_j)$ would be the eigenstate $|A_j\rangle$.

The classification (separation) of the system in accordance with the results of measurements requires a certain classical deeds. These deeds can be performed by an experimenter or automatically.

The interaction of the system with QRS can be described quantum-mechanically. Hamiltonian of the combined object can be represented as

$$\hat{H} = \hat{H}_s + \hat{H}_i + \hat{H}_Q, \tag{1}$$

where $\hat{H}_s$ depends only on the system observables and is called Hamiltonian of the system; $\hat{H}_Q$ is known as Hamiltonian of QRS. Hamiltonian of interaction $\hat{H}_i$ contains cross-terms.

The goal of measurement may be the value of an observable $\hat{A}$ that is related to: 1) the non-perturbed state of the system; 2) the state perturbed by measurement; and at the following moment:
a) $t = 0$,
b) $0 < t_1 < \tau$,
c) $t > \tau$.

Also the goal of measurement may be to prepare a new state of the system.

Each of the goals requires a specific initial state of QRS in order to obtain the least error of estimation.

The essential results which is necessary to emphasize in our discussion can be illustrated by the simplest example of measurement.

Example. Measurement of a free mass position.

In order for such a measurement to take place it is sufficient for QRS to interact with the mass in accordance to the Hamiltonian $\hat{H}_i = \alpha(t)xY$ for a certain time τ. (It can be $\hat{H}_i = \alpha(t)f(\hat{x})\hat{Y}$ as well). Here $\hat{x}$ is the position operator, $\hat{Y}$ is a certain operator of QRS, $\alpha(t)$ is the coupling function. We will consider $\alpha(t) = \alpha_0$ for $0 \leq t \leq \tau$, otherwise $\alpha = 0$.

The QRS can be represented by a free particle M. For the simplest solution we take

$$\hat{H} = \hat{p}^2/2m + \alpha(t)\hat{x}\hat{Y} + \hat{P}^2/2M$$

where $\hat{P}$ and $\hat{Y}$ are, respectively, the momentum operator and the position operator of the QRS.

This Hamiltonian is not practicable but permitted by the theory. In order to realize such an Hamiltonian it is necessary to compensate a positive rigidity due to the coupling between the S and QRS by means of additional negative rigidity.

Working in the Heisenberg picture we obtain

$$\begin{aligned} &a) \quad d\hat{x}/dt = \hat{p}/m \quad , \\ &b) \quad d\hat{p}/dt = -\alpha\hat{H} \quad , \\ &c) \quad d\hat{Y}/dt = \hat{P}/M \quad , \\ &d) \quad d\hat{P}/dt = -\alpha\hat{x}. \end{aligned} \tag{2}$$

For M sufficiently large $\hat{Y}$ can be considered constant during the interaction. We call it $\hat{Y}_0$. Sor for $0 < t < \tau$ we have from (2)

$$\begin{aligned} &a) \quad \hat{x}(t) = \hat{x}_0(t) - \alpha\hat{Y}_0 t^2/2m \quad , \\ &b) \quad \hat{p}(t) = \hat{p}_0(t) - \alpha\hat{Y}_0 t. \\ &c) \quad \hat{P}(\tau) = \hat{P}_0 - \alpha\int_0^\tau \hat{x}(t)dt \\ &\qquad\quad = \hat{P}_0 - \alpha\hat{x}_0(\tau/2)\tau - \alpha^2\hat{Y}_0\tau^3/6m \\ &\qquad\quad = \hat{P}_0 - \alpha\hat{x}(\tau/2)\tau - 7\alpha^2\hat{Y}_0\tau^3/24m. \end{aligned} \tag{3}$$

The equations (3c) can be rewritten as follows

$$\begin{aligned} &a) \quad \hat{x}_0(\tau/2) = \hat{P}_0 - \hat{P}(\tau))/\alpha\tau + \alpha\hat{Y}_0\tau^2/6m \quad , \\ &b) \quad \hat{x}(\tau/2) = \hat{P}_0 - \hat{P}(\tau))/\alpha\tau + 7\alpha\hat{Y}_0\tau^2/24m \quad , \end{aligned} \tag{4}$$

where

$$a)\hat{x}_0(\tau/2) = \hat{x}(0) + \hat{p}(0)\tau/2m$$

is the position operator in a non-perturbed state at $t = \tau/2$.

$$b)\hat{x}_0(\tau/2) = \hat{x}(0) + \hat{p}(0)\tau/2m - \alpha\hat{Y}_0\tau^2/8m \tag{5}$$

is that in a perturbed state.

Measuring $\hat{P}(\tau)$ with a r.m.s. errour $\Delta\tilde{P}$ we can estimate the value of $x_0(\tau/2)$ and $x(\tau/2)$.

If $\hat{P}_0$ is non correlated with $\hat{Y}_0$ we have the following r.m.s. error of the estimation of the position $x_0(\tau/2)$

$$(\Delta\tilde{x}_0)^2 = (1/\alpha\tau)^2[(\Delta\tilde{P}_0)^2] + (\alpha\tau^2/6m)^2(\Delta y_0)^2, \tag{6}$$

where $(\Delta P_0)^2$, $(\Delta y_0)^2$ are dispersions corresponding to the initial state of QRS. As soon as $(\Delta P_0)^2(\Delta y_0)^2 \geq \hbar^2/4$ minimizing the right-hand part of (6) with respect to $\alpha\tau$ we obtain

$$\Delta\tilde{x}_0 \geq (\hbar\tau/3m)^{1/2} \tag{7}$$

It is the so called standard quantum limit of the position's measurement error.

This limit can be beat by means of an appropriate correlation among $\hat{P}_0$ and $\hat{y}_0$. Let be

$$\hat{P}(0) = \hat{P}^0 - \alpha\tau^2\hat{y}_0/6m \tag{8}$$

where $\hat{P}^0$ is non-correlated with $\hat{y}_0$. This correlation can be produced by means of a field such as the field of an electron lens. In this case we have from (4)

$$(\Delta\tilde{x}_0)^2 = [(\Delta\tilde{P})^2 + (\Delta\tilde{P}^0)^2]/(\alpha\tau)^2 \to 0 \tag{9}$$

if $\alpha\tau \to \infty$.

Analogous picture takes place in case of the estimation of $x(\tau/2)$ if $\tilde{P}(0) = \hat{P}^0 - 7\alpha\tau^2\hat{y}_0/24m$.

Consequently the correlation helps us to remove the back action of the meter from the position's estimate. Nevertheless one cannot eliminate this action from the very position.

The equation (3a) gives us the variance of the position perturbance at time $\tau/2$

$$\Delta\tilde{x}(\tau/2) = \alpha\tau^2\Delta y_0/8m \tag{10}$$

As it follows from (2b) the momentum's variance is increased by

$$\Delta\tilde{p}(\tau) = \alpha\tau\Delta y_0 \tag{11}$$

The momentum perturbance originates a position perturbance also for the period following the measurement ($t > \tau$) and changes its variance for $t \gg \tau$ by

$$\Delta\tilde{x}(\tau) = \alpha\tau t\Delta y_0/m \tag{12}$$

Leaving out of account an uncertainty of the initial momentum $p(0)$ we obtain from (10), (12) that an overall dispersion of the position at time $t \gg \tau$ is equal to

$$(\Delta x(t))^2 = (\Delta\tilde{x}_0)^2 + (\Delta x(t))^2 \geq 2\Delta P^0\Delta y_0 t/m \geq \hbar t/m \tag{13}$$

Therefore a result of a subsequent measurement of a free particle's position cannot be predicted more precisely than [3] $(\hbar t/m)^{1/2}$ (V. Braginsky, Yu. Vorontsov, 1974).

The limiting error (13) is known as the standard quantum limit as well.

This limiting error of prediction was widely discussed by physicists in connection with the problem of gravity waves detection because this unprediction of position leads to a limit for sensitivity of force detection [4].

Quantum theory tells us that if an observable $\hat{A}_2$ does not commute with $\hat{A}_1$ then the measurement of $\hat{A}_1$ leads inevitably to unpredictable perturbance of the variable $\hat{A}_2$. In case of free evolution of a particle we have

$$[\hat{x}(t_1), \hat{x}(t_2)] = i\hbar(t_2 - t_1)/m. \tag{14}$$

Thus we are led to the conclusion: it is impossible to prepare a state of a free particle in which its position is entirely predictable at certain time t_2 by measuring of the position at time $t_1 < t_2$.

This desired state can be performed if after the first position's measurement an appropriate correlation between $\hat{x}(t)$ and $\hat{p}(t)$ is produced by means of a classical field like a focusing field.

2. Continuous Position's Measurements (Monitoring of Position)

Above we have discussed the measuring process in which only one particle serves as QRS. This measurement is not of practical interest. In real measuring process a flow of particles (or quasiparticles) is used such as electrons, photons, etc. In this case the force of the back action may be written as follows [2]

$$\hat{F}_{ba}(t) = \sum \hat{F}_j(t - t_j) \tag{15}$$

where $\hat{F}_j(t - t_j)$ is the force of the back action of one of the particles. The force $\hat{F}_j(t - t_j)$ acts in the interval $-\tau_j < t - t_j < 0$, where τ_j is a duration of the interaction of this particle with the system.

A result of an approximate measurement of an observable $\hat{A}(t)$ may be represented as a result of an exact measuring of sum $\hat{A}(t) + \hat{A}_a(t)$, where $\hat{A}_a(t)$ is a certain operator of the measuring device. In the above considered example $\hat{A}_a(t)$ was represented by $\hat{P}_0(t)/\alpha\tau$

In order to calculate a measuring error of position one can used the following equivalent scheme.

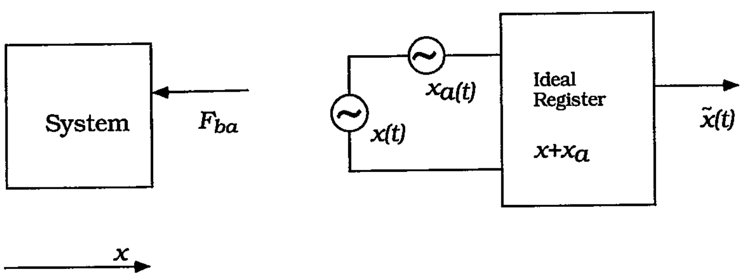

Figure 4. Equivalent scheme of position measuring.

It has been deduced [2] that in the case of stationary position measuring, the spectral density of the random functions $F_{ba}(t)$ and $x_a(t)$ obey

$$S_F(\omega)S_x(\omega) - |S_{Fx}(\omega)| \geq \hbar^2/4 + \hbar\omega|ImS_{Fx}(\omega)| \tag{16}$$

where $S_{Fx}(\omega)$ is a cross-correlation of $\hat{F}_{ba}(t)$ and $\hat{x}_a(t)$ and $ImS_{Fx}(\omega)$ is the imaginary part of it. (Yu. Vorontsov, F. Khalili, 1982).

The simplest formula $S_F S_x \geq \hbar^2/4$ was deduced in 1976 by R.P Giffard.

The output signal of the register is classical. Thus the classical theory of optimum estimation can be used to estimate a value of the variable of interest. In order to ascertain a quantum limit for the measuring error on a linear system it is sufficient to make a classical analysis of the equivalent scheme taking into account the equation (16). (See for details [2]).

It turns out than a r.m.s. error of estimation of a time average coordinate of a free particle obeys the inequality

$$\Delta\overline{x}^\tau \geq [(S_F S_x)^{1/2}\tau/m]^{1/2} \geq (\hbar\tau/2m)^{1/2} \tag{17}$$

(It was assumed that $S_{Fx} \equiv 0, S_F(\omega) = S_F, S_x(\omega) = S_x$).

An estimation error of a free particle's momentum under the same conditions obeys the inequality

$$\Delta\overline{p}^\tau \geq [(S_F S_x)^{1/2}m/\tau]^2 \geq (\hbar m/2\tau)^{1/2} \tag{18}$$

The amplitude a_0, the real $(\hat{X}_1)$ and imaginary $(\hat{X}_2)$ parts of the complex amplitude of a harmonic oscillator can be estimated in this way with the following imprecisions

$$\Delta\tilde{a}_0 \geq [(S_F S_x)^{1/2}m/\omega_0]^{1/2} \geq (\hbar/2m\omega_0)^{1/2} \tag{19}$$

$$\Delta X_1 = \Delta X_2 \geq (\hbar/2m\omega_0)^{1/2}. \tag{20}$$

3. Quantum Nondemolition (Nonperturbation) Measurements (QND) [1-3,5,6]

Quantum nondemolition measurement of an observable N is such that the back action of QRS on the system under measuring has no impact on the results of the first and subsequent measurements of this observable. Such measurements are called back-action-evading measurements as well.

QND observables are those which can be measured, in principle, by QND means. $\hat{N}$ is a QND observable, if and only if, when the system is evolving freely in the Heisenberg picture, $\hat{N}(t)$ commutes with itself at the different moments of time

$$[\hat{N}(t_j), \hat{N}(t_i)] = 0 \tag{1}$$

This condition is satisfied, in particular, by motion integrals. In case of a free particle the energy and momentum are conserved and thus are QND observables. For a harmonic oscillator the QND observables are the energy, the real ($\hat{X}_1$) and imaginary ($\hat{X}_2$) parts of the complex amplitude

a)
$$\hat{X}_1 = \hat{x}(t) \cos \omega_0 t - [\hat{p}(t)/m\omega_0] \sin \omega_0 t,$$

b)
$$\hat{X}_2 = \hat{x}(t) \sin \omega_0 t + [\hat{p}(t)/m\omega_0] \cos \omega_0 t. \tag{2}$$

Evolution of QND observables during QND Measurements [2,7,8]

There are two types of QND observables. Some of QND observables can be free from the influence of the back action even during the interaction of the system with the QRS. Others are perturbed unpredictably in the interval of the interaction but are restored to its original value when the interaction is turning off. In the latter case a certain observable $\hat{N}_1(t)$ is conserved during the interaction and is equal to the observable $\hat{N}(t)$ of the free system.

Such non-canonical QND observables as velocity, kinetic energy and others functions of generalized velocity are bound to be perturbed inevitably in a process of interaction between a system and QRS. If this would not be the case one could prepare such states of the system in which the uncertainty relation is violated. This conclusion may be explained thus. When the momentum is measured, the uncertainty of the coordinate should increase. A random variation of the coordinate can be caused by motion with an indeterminate velocity during a known time or motion with a determinate velocity during an indeterminate time.

However, if velocity could be monitored (tracked) its uncertainty at any moment should be equal to zero.

Example. Let us consider the following interaction

$$\hat{H} = \hat{p}^2/2m + a\hat{p}\hat{Y} + \hat{P}^2/2M. \tag{3}$$

Consequently, we have

$$d\hat{p}/dt = 0, \tag{4}$$

i.e. the generalized momentum $\hat{p}$ is conserved. On the other hand, the velocity is

$$d\hat{x}/dt = \hat{p}/m + a\hat{Y}(t), \tag{4a}$$

i.e. the velocity is perturbed during the interaction. Nevertheless the velocity is restored as the interaction is turning off ($a = 0$).

On the other hand, the Hamiltonian (3) can be rewritten as follows

$$\hat{H} = \hat{p}_0^2/2m - \hat{x}\hat{Y}(da/dt) + a^2\hat{x}^2/2M - a\hat{x}\hat{P}/M + \hat{P}^2/2M. \tag{5}$$

Consequently, we have

$$d\hat{x}/dt = \hat{p}_0/m \quad , \quad d\hat{p}_0/dt = \dot{a}\hat{Y} + a\hat{\dot{Y}}, \quad \text{i.e.} \quad (d/dt)(m\hat{\dot{x}} - a\hat{Y}) = 0. \tag{6}$$

In this case the generalized momentum $\hat{p}_0$ is equal to a kinetic momentum and thus is not conserved during the interaction. But the same value $(m\hat{\dot{x}} - a\hat{Y})$ that in the first case is conserved.

4. Conditions of realization of QND measurements [2]

Generalized condition of QND measurements can be formulated as follows. In order for the QND measurement of a certain QND observable $\hat{N}$ to take place, it is necessary and sufficient that the QRS carries after the interaction with the system an information about the $\hat{N}$ and no information about observables that do not commute with $\hat{N}$.

This condition is satisfied, in particular, by the following Hamiltonian of interaction

$$\hat{H} = \hat{H}_S + \alpha\hat{N}\hat{Y} + \hat{H}_a \tag{7}$$

It is generally believed that such an Hamiltonian is the necessary condition of the QND measurement. But strictly speaking such an Hamiltonian is a sufficient but not a necessary condition of the QND measurement.

Let as consider an example. In order to realize the QND measurement of $\hat{X}_2$ one can us an interaction which is characterized by the interaction Hamiltonian $\hat{H}_i = \alpha\hat{x}(t)\hat{Y}$ if the duration of the interaction is equal to one half period of oscillations. In this case the overall change of the momentum of the QRS will be equal to

$$\int_0^{\pi/\omega} \hat{x}(t)dt = \int (\hat{X}_1 \cos\omega t + \hat{X}_2 \sin\omega t)dt = \hat{X}_2 2/\omega, \qquad (8)$$

i.e. the QRS gets as a result an information only about the $\hat{X}_2$.

Nondemolition measurement of the energy of a harmonic oscillator [2,9]

Let us consider the scheme in fig. 5.

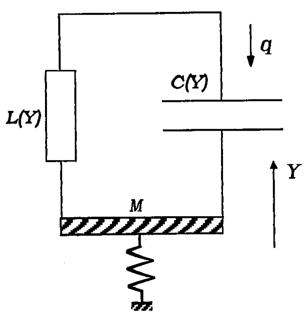

Figure 5. Harmonic oscillator.

Here the LC-circuit is the system under measuring. The QRS is represented by the mechanical oscillator (M, k) attached to movable parts of the inductor and of the capacitor. One can do so that it will be

$$1/L(\hat{Y}) = (1 + \alpha\hat{Y})/L_0 \quad , \quad 1/C(\hat{Y}) = (1 + \alpha\hat{Y})/C_0 \qquad (9)$$

where L_0, C_0 are the nonperturbed values of the circuit's parameters.

Consequently, we obtain

$$\hat{H} = (\hat{p}^2/2L_0 + \hat{q}^2/2C_0)[1+\alpha\hat{Y}] + \hat{H}_a = (\hat{n}+1/2)\hbar\omega_0[1+\alpha\hat{Y}] + \hat{H}_a, \qquad (10)$$

where $\hat{n}$ is the number operator,and $\omega_0 = (1/L_0C_0)^{1/2}$. In this case we have

$$d\hat{n}/dt = 0,$$

i.e. number of quanta remains fixed when the QRS is evolving. However, the frequency depends on the coordinate operator of the QRS and therefore is an operator itself:

$$\hat{\omega} = \omega_0(1 + \alpha\hat{Y}). \tag{11}$$

Accordingly, what does not change during the measurement process is not the current energy of the resonator, but the quantum number n.

Also we have

$$i\hbar d\hat{p}_y/dt = [\hat{p}_y, \hat{H}_a] + \hat{H}_0[\hat{p}_y, \alpha\hat{Y}], \tag{12}$$

where $\hat{H}_0 = (\hat{n} + 1/2)\hbar\omega_0$ is the operator of the oscillator's unperturbed energy.

Measuring $\hat{p}_y$ one can estimate values of $\hat{H}_0$ and $\hat{n}$. After the measurement the original value of the frequency can be restored, and, consequently, one can restore the initial energy.

Analyzing this scheme we have found, that the possibility of energy measurement of a conserved lumped circuit with an error

$$\Delta H_0 < \hbar/\tau$$

(τ is the duration of the measurement) does not contradict the principles of quantum mechanics. (In frames of validity of nonrelativistic theory).

An uncertainty of the frequency ω during the measuring will cause the increase of phase uncertainty by the amount

$$\tilde{\Delta}\dot{\varphi} = \Delta \int_0^\tau \hat{\omega}(t)dt \geq 1/2\Delta\tilde{n} \tag{13}$$

Also the following equation takes place

$$\Delta\tilde{H} \cdot \tilde{\Delta}H/H_0 \geq \hbar/2\tau \tag{14}$$

where $\Delta H = \Delta\tilde{n}\hbar\omega_0$ is the r.m.s. measuring error of the energy, $H_0 = (n+1/2)\hbar\omega_0$, $\tilde{\Delta}$ is the uncertainty of the energy perturbation during the measuring.

The equation (14) can be rewritten as follows

$$\Delta\tilde{H} \geq (\hbar/2\tau)(\omega_0/\tilde{\Delta}\overline{\omega}) \tag{15}$$

where $\tilde{\Delta}\overline{\omega} = \tilde{\Delta}\varphi/\tau$ is a perturbation uncertainty of the frequency averaged over τ.

Consequently, a necessary condition of the measuring of the oscillator energy with the error $\Delta\tilde{H} < \hbar/\tau$ is such an initial state of QRS that the relative uncertainty of the frequency will be above $0,5$.

During the energy measurement of a nonconservative oscillator a change of relaxation time of the oscillator takes place. As a result it is not possible to measure the energy with an error under $\hbar/\tau_0^*$ where τ_0^* is a relaxation time of the free oscillator [2].

Nondemolition Measurement of the Energy of Electromagnetic Waves

The energy of an electromagnetic wave in a volume V in a nondispersive substance is equal to

$$H = \int (1/8\pi)(\epsilon E^2 + \mu H^2)dV = \int (\epsilon\mu/8\pi)(E^2/\rho + \rho H^2)dV \qquad (1)$$

where $\rho = (\mu/\epsilon)^{1/2}$. The equation (1) tells us : in order for the nondemolition measurement of the energy to take place QRS must change the permeabilities ϵ and μ simultaneously and by such means that the value ρ remains fixed. In this case velocity ($v = 1/(\epsilon\mu)^{1/2}$) and frequency ($\omega = 2\pi v/\lambda$) of the wave will be changing but the number of quanta (n) and the wave length (λ) remain fixed. Up to now it does not spell out how such a measurement can be realized.

Before we proceed let us note an important result due to the theory of relativity. The variance of the change of the wave frequency at such a measurement is

$$\Delta\omega = 2\pi\Delta v/\lambda < 2\pi c_0/2\lambda = \omega_0/2$$

i.e. $\Delta\omega/\omega_0 < 1/2$.

Consequently, the error of the measuring of the energy of an electromagnetic wave cannot be under $\hbar/\tau$.

This resolution can be obtained by other mean. During the momentum measurement the perturbation of the conjugate coordinate must take place such that $\tilde{\Delta}x \geq h/2\Delta\tilde{p}$. Since in this case $\tilde{\Delta}x = \omega\tilde{\Delta}v < \tau c_0/2$ and the wave energy $H = pc_0$, we have

$$\Delta\tilde{H} \geq \hbar/\tau$$

This result contradicts that which was obtained from the analysis of the energy measurement on lumped circuits. The contradiction is resolved by the difference of the relations between momentum and energy in the first and in the second objects.

The method of energy measurement based on the optical Kerr's effect [10-12] is not the QND one. It is quasi-QND [13].

5. Detection of an external Action on the System

In experimental physics many phenomena are studied through so called test bodies (TB). Observing a change of a state of the TB, an experimenter obtains an information about fields which act on the TB. In this way, in particular, the problem of detection of force action on TB is solved. A process of detection consists of three stages. At the first stage one prepares a certain state of the TB. Subsequently, the system evolves. At the third stage, a suitable measurement is carried out and from its result using a certain criterion one decides whether the force has acted on the TB or not.

In order to ascertain an ultimate quantum limit for a probability of a wrong detection of the force we shall consider the case of a pure initial state of the TB. We shall designate : the initial state of the TB as $|\Psi(0)\rangle$, the Hamiltonian and the unitary transformation operator which correspond to the free evolution of the TB as $\hat{H}_0$ and as $\hat{U}_0(t)$, respectively, and those which correspond to the evolution in presence of the force action as $\hat{H}_1$ and as $\hat{U}_1(t)$.

The TB's state at time t can transform into

$$|\Psi_0(t)\rangle = \hat{U}_0(t)|\Psi(0)\rangle \quad , \qquad \text{or into} \quad |\Psi_1(t)\rangle = \hat{U}_1(t)|\Psi(0)\rangle.$$

It is known [1] that the quantum limit for the average probability of an error in discrimination of two states ($|\Psi_1\rangle$ and $|\Psi_0\rangle$) is described by the equation [14]

$$P_{w.d.} = [1 - (1 - 4\zeta_0\zeta_1|\gamma|^2)^{1/2}], \tag{1}$$

where ζ_0, ζ_1 are the a priori probabilities of the states $|\Psi_0\rangle, |\Psi_1\rangle$, respectively ; $|\gamma| = |\langle\Psi_0|\Psi_1\rangle|$.

This limit can be achieved by an optimal measurement that is predicted by the quantum theory of detection. In the case under consideration we have

$$|\gamma| = |\langle\Psi(0)|\hat{R}(t)|\Psi(0)\rangle| \tag{2}$$

The operator $\hat{R} = \hat{U}_0^+ \hat{U}_1$ obeys the equation

$$ihd\hat{R}/dt = \hat{H}_i^0 \hat{R}, \tag{5}$$

where

$$\hat{H}_i^0 = \hat{U}_0^+ (\hat{H}_1 - \hat{H}_0) \hat{U}_0. \tag{5}$$

The operator $\hat{R}(t)$ is similar to the scattering operator which is used in the interaction representation.

Representing $\hat{R}(t)$ in the form

$$\hat{R}(t) = e^{i\hat{\varphi}(t)}, \tag{6}$$

we obtain

$$|\gamma|^2 = |\int_{\infty}^{\infty} e^{i\varphi} d\Phi(\varphi)|^2 \tag{7}$$

where $\Phi(\varphi)$ is the probability's distribution function for the eigenvalues of the operator $\hat{\varphi}$.

Diverse functions $\Phi(\varphi)$ can give us $|\gamma| = 0$. In case of an optimal initial state $|\psi(O)\rangle_{opt}$ we have obtained [2]

$$|\gamma| = \cos^2 \Delta\varphi \quad \text{if} \quad \Delta\varphi < \pi/2 \quad , \quad 0 \quad \text{if} \quad \Delta\varphi > \pi/2 \tag{8}$$

The optimal state is that in which $|\gamma| = 0$ by a minimal value of $\Delta\varphi$. Such a state is characterized by the following distribution of the probability density of φ

$$W(\varphi) = [\delta(\Delta\varphi - \pi/2) + \delta(\Delta\varphi + \pi/2)]/2 \tag{9}$$

Let us consider the important particular case such that the operator $\hat{H}_i^0(t)$ obeys the following equation

$$[[\hat{H}_i^0(t_1), \hat{H}_i^0(t_2)], \hat{Q}] = 0 \tag{10}$$

where $\hat{Q}$ is any operator. This condition is fulfilled when a classical force acts on TB. We proved the formula

$$\hat{\varphi}(t) = -h^{-1} \int \hat{H}_i^0(t') dt' \tag{11}$$

In case of detection of the classical force $F(t)$ which acts for a time $\tilde{\tau}$ we have

$$\hat{H}_i^0(t) = F(t)\hat{x}^0(t) \tag{12}$$

and therefore

$$\hat{\varphi}(t) = \hbar^{-1} \int F(t)\hat{x}^0(t)dt \tag{13}$$

where

$$\hat{x}^0(t) = \hat{U}_0^+(t)\hat{x}(O)\hat{U}_0(t). \tag{14}$$

The formula (13) brings out a way to achieve a maximal value of the $\Delta\varphi$ if the function $F(t)$ is known.

Unfortunately, it is close to impossible to realize the optimal state $|\Psi(0)\rangle_{opt}$. Therefore we will consider a more practicable states which are characterized by the following function

$$W(\varphi) = \alpha \exp(-\varphi/2\sigma)[1 + f(\varphi)], \tag{15}$$

where $f(\varphi)$ is a finite-order polynomial, σ and α are parameters.

The distribution of the form (15) occurs, for example, when the classical force acts on an harmonic oscillator that has been prepared in one of the following states: in a coherent state ($f(\varphi) \equiv 0$), in a squeezed state ($f(\varphi) \equiv 0$), in a state with given energy ($(1 + f(\varphi))$ is the Hermite polynomial.

In order to ascertain a limit for the sensitivity of detectability we shall consider the case of $\Delta\varphi < 1$ and $\sigma < 1$. In this assumption it was obtained from (13) and (15)

$$|\gamma|^2 \approx 1 - (\Delta\varphi)^2 \tag{16}$$

Therefore, if $\zeta_0 = \zeta_1 = 1/2$ it follows from (1) and (16)

$$P_{w.d.} \approx (1 - \Delta\varphi)/2 \tag{17}$$

The equation (17) enables us to ascertain a quantum limit for a force detection sensitivity (FDS). We shall regard as the FDS such an amplitude of the force by which the P_{wd} amounts to a value established by the experimenter (the so called a threshold of detectability).

Considering some examples we will suppose that the threshold is equal to $P_{wd} \approx 0,25$. This P_{wd} takes place when the signal equals with a variance of the fluctuation.

Example 1.

$$F(t) = F_0, 0 < t < \tilde{\tau} \quad , \quad 0 \quad \text{otherwise} \quad . \tag{18}$$

It follows from (13), (17) that in this case the FDS is equal to

$$F_0\tilde{\tau} = h/2\Delta\overline{x} \tag{19}$$

where

$$\Delta\overline{x} = \Delta(1/\tilde{\tau})\int x^0(t)dt \tag{20}$$

a) If TB is represented by a free particle of mass m we have $\Delta\overline{x} = \Delta x(\tilde{\tau}/2)$. To illustrate the equation (19) we will answer the following question. Let us assume that we have a divergent beam of particles. At what time must the force act on this beam along axis x to get the minimal $P_{w.d.}$?

According to formula (19) the latter the action, the less the $P_{w.d.}$.

b) TB is represented by an harmonic oscillator (mass m, frequency ω_0).

Let be $\omega_0\tilde{\tau} \ll 1$. Then in case of a coherent initial state $|\psi(0)\rangle$ we obtain the following FDS

$$F_0\tilde{\tau} = \sqrt{\hbar m\omega_0/2} \tag{21}$$

If the initial state is an eigenstate $|n\rangle$ of the number operator and $n \gg 1$ we have

$$F_0\tilde{\tau} = \sqrt{\hbar m\omega_0/2n} \tag{22}$$

Prof. V.B. Braginsky was the first to obtain the equations (21) and (22) when he had been considering certain methods of force detection [15].

Example 2.

$$F(t) = F_0 \sin\Omega t \quad \text{if} \quad 0 \le t \le \tilde{\tau} \quad , \quad 0 \quad \text{otherwise} \quad .$$

a) This force acts on a free particle.

Let be $\Omega\tilde{\tau} = 2K\pi(k = 1, 2\ldots)$. In terms of the formula (13) we obtained

$$F_0\tilde{\tau} = \hbar m\Omega/2\Delta p \tag{23}$$

where Δp is the variance of particle's momentum. Subsequently, in case of action of this force on a divergent beam the FDS does not depend on the acting moment.

b) The force acts on an harmonic oscillator and $\omega_0 = \Omega$. In this case we obtain

$$F_0 \tilde{\tau} = \hbar / \Delta X_2 \tag{24}$$

where ΔX_2 is the variance of the imaginary part of the complex amplitude. In the coherent state $\Delta X_{2(c)} = (\hbar/2m\omega_0)^{1/2}$. In n-state $(|n\rangle)\Delta X_{2(n)} = \hbar n/2m\omega_0)^{1/2}$. Also it can be $\Delta X_2 \gg \Delta X_{2(c)}$ in a squeezed state.

It is of interest to emphasize that the minimum-uncertainty state $\Delta x \Delta p = \hbar/2$ has no advantage over other Gaussian's pure states. Nevertheless, the optimum measurement that is required to minimize the detection error depends on the initial state $|\psi(0)\rangle$ of TB and of the waveform of force. An optimal observable $\hat{A}$ which must be measured is given by the relation

$$\hat{A} = \hat{U}_1 |\psi(O)\rangle\langle\psi(O)| \hat{U}_1^+ - \hat{U}_0^+ |\psi(O)\rangle\langle\psi(O)| \hat{U}_0^+ .$$

6. Relation between Measurement Error and Uncertainty of Perturbation

It is known that in general case the uncertainties of any observables $\hat{A}$ and $\hat{B}$ are related by the following equation [16]

$$(\Delta A)^2 (\Delta B)^2 \geq \hbar^2 |\langle C \rangle|^2 / 4(1 - r^2) \tag{1}$$

where r is the coefficient of correlation among A and B at the given state of the system,

$$\hat{C} = [\hat{A}, \hat{B}]/i\hbar$$

The Heisenberg's relation is the particular case of the uncertainty relation (1) with the condition that $\hat{C} = 1$, $r = 0$.

The values ΔA and ΔB are not related to the measuring errors. To test the validity of the equation (1) it is necessary to perform a large number of exact measurements of : $A, B, C, (AB+BA)$ on different parts of an ensemble of systems in the given state. Nevertheless it is commonly believed that the measurement error $(\Delta\tilde{A})$ and the uncertainty of perturbation $(\Delta\tilde{B})$ are related by the same equation (1). To justify this opinion they give the example of the momentum perturbing the position measurement. They obtain

$$(\Delta\tilde{x})^2(\Delta\tilde{p})^2 \geq \hbar^2/4 \tag{2}$$

Analyzing this problem we have obtained that this opinion is valid only if C is not an operator and $r = 0$ [2].

The perturbing uncertainty $(\Delta\tilde{B})^2$ is determined by the equation

$$(\Delta\tilde{B})^2 = (\Delta B)_1^2 - (\delta B)_0^2 \tag{3}$$

where $(\Delta B)_0$, $(\Delta B)_1$ are the uncertainties of B respectively at the initial state (ρ_0) and at the mixed state (ρ_1) (which takes place after the interaction of QRS with the system), i.e. $(\Delta\tilde{B})^2$ is the change in the dispersion of B due to the interaction of the system with QRS. $(\Delta\tilde{B})^2$ can be negative. In case of $[[\hat{B},\hat{A}],\hat{A}] = 0$ we have obtained

$$(\Delta\hat{A})^2(\Delta\tilde{B})^2 \geq \hbar^2/4\langle|\hat{C}|^2\rangle \neq \hbar^2/4|\langle\hat{C}\rangle|^2. \tag{4}$$

For example, on measuring the free particle's energy it should be

$$(\Delta\tilde{H})^2(\Delta\tilde{x})^2 \geq (\hbar/2m)^2\langle\hat{p}^2\rangle. \tag{5}$$

If

$$[[\hat{B},\hat{A}]\hat{A}] \neq [[\hat{A},\hat{B}]\hat{B}],$$

we obtain

$$(\Delta\tilde{A})^2(\Delta\tilde{B})^2 \neq (\Delta\tilde{A})^2(\Delta\tilde{B})^2. \tag{6}$$

In the particular case we have

$$(\Delta\tilde{x})^2(\Delta\tilde{H})^2 \neq (\Delta\tilde{H})^2(\Delta\tilde{x})^2. \tag{7}$$

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