Two-parameter quantum groups with noncanonical commutation relations

G.Brodimas[(o)(*)], A.Jannussis[(o)(x)(*)] and R.Mignani[(*)(∇)]

[(*)] Dipartimento di Fisica, Università "La Sapienza"
P.le A.Moro 2, 00185 Roma, Italy
and I.N.F.N. - Sezione di Roma, Italy
[(∇)] Dipartimento di Fisica "E. Amaldi", III Universitá di Roma

ABSTRACT We consider a deformed Heisenberg-Weyl algebra depending on two parameters, Q and q. The special case $q = 1$ corresponds to the Q-algebra introduced by Arik and Coon and Kuryshkin, while for $Q = 1/q$ one recovers the usual q-deformed algebra. The main properties of this two-parameter algebra are discussed by a non-standard approach, based on noncanonical commutation relations among the generators. We show that the two-parameter commutation rules of the creation and annihilation operators $\hat{A}, \hat{A}^+$ can be obtained from Q-analysis by means of a Bargmann realization. The explicit expressions of $\hat{A}, \hat{A}^+$ are derived by a bosonization procedure. The Fock representation is considered, leading to a non-canonical commutation relation between position and momentum operators. The quantum group $SU(2)_{Q,q}$, obtained by a noncanonical Jordan map of the generators, is briefly discussed. Finally, the time-evolution of the two-parameter deformed oscillator is found by exploiting the connection between deformed algebras and Lie-admissible algebras.

RÉSUMÉ On considère une algèbre déformée de Heisenberg-Weyl qui dépend de deux paramètres, Q et q, dotée de relations de commutation non canoniques entre les générateurs. Les principales propriétés de cette algèbre sont étudiées, en particulier ses réalisations de type Bargmann et de type boson, le groupe quantique associé $SU(2)_{Q,q}$, et l'évolution temporelle Lie-admissible de l'oscillateur déformé à deux paramètres.

[(o)] On leave from Dept. of Physics, Patras University - 26110 Patras, Greece.
[(x)] Also: IBR - Palm Harbor, Florida 34682-1577, U.S.A

Introduction

Quantum group theory started and developed in the last decade. The literature on this topic is presently more and more growing, so that it is very difficult to exhaustively account for all the papers which appear almost every day. An excellent synopsis as well as a historical review can be found in ref.[1], to we also refer the reader for the relevant literature on the subject.

Let us only recall that, after the pioneering (and largely ignored) papers by Arik and Coon [2] and Kuryshkin [3], quantum groups were recognized as the symmetries naturally arising in the study of integrable systems in statistical mechanics and quantum field theory (ruled by the Yang-Baxter equations) [4-6]. Then, their mathematical properties have been thoroughly investigated [7]. The main algebraic structure of quantum groups is the q-deformed algebra, that mathematically is a Hopf algebra [8]. q-deformed algebras allow the introduction of q-deformed oscillators and therefore, by a straightforward generalization of the Jordan-Wigner-Schwinger representation, the construction of deformations of the standard Lie groups [9]. q-deformed algebras are also involved in the study of parastatistics [10-12].

Since 1981, many aspects of the q-deformation of an oscillator algebra have been investigated by Jannussis and collaborators [10,11,13,14]. In particular, they realized (as early as 1981) [10] that the standard form of the commutation relation for a Q-harmonic oscillator [3]

$$\hat{A}\hat{A}^{+} - Q\hat{A}^{+}\hat{A} = \hat{I} \tag{1.1}$$

(where $Q\epsilon(-1,\infty), Q \neq 0$) corresponds to a (λ,μ) mutation algebra [15], i.e. a special case of a Lie-admissible algebra [16]. A Lie-admissible Q-algebra [14,17] is obtained when considering an operator $\hat{Q}$ (instead of a number Q) in the commutation relation, thus getting[1]

$$(\hat{A},\hat{A}^{+}) \equiv \hat{A}\hat{A}^{+} - \hat{A}^{+}\hat{Q}\hat{A} = \hat{I} \ . \tag{1.2}$$

Moreover, in ref.[10] a generalization of Q-algebra has been given in the form

$$\hat{A}\hat{A}^{+} - Q\hat{A}^{+}\hat{A} = f(\hat{n}) \tag{1.3}$$

[1] The connection between q-deformed algebras and Lie-admissible algebras is easily seen by noting that both of them involve a deformation (lifting) of the enveloping algebra, that, in turn, leads to a deformation of the corresponding Lie products.

where $\hat{n}$ is the usual number operator ($\hat{n} \mid n\rangle = n \mid n\rangle$) satisfying the standard commutation rules

$$[\hat{A}, \hat{n}] = \hat{A} \ ; [\hat{A}^+, \hat{n}] = -\hat{A}^+ \tag{1.4}$$

and the function $f(\hat{n})$ is a suitable (smooth) function. If

$$f(\hat{n}) = q^{\hat{n}} \tag{1.5}$$

we get a deformed algebra depending on two parameters Q and q [14]. We shall name it a (Q, q)-*algebra*, or a *two-parameter quantum group*[2].

In the last few years, multiparameter quantum groups have been considered independently by some authors [18-20] (besides the present ones) [14,21,22], in view of their applicability to a number of concrete physical models. In refs.[18-20] quantum algebras with many (in particular, two) deformation parameters are discussed in the framework of standard Heisenberg commutation relations with canonical form. However, we have recently stressed [23] the deep connection existing between q-deformed groups and noncanonical commutation rules [24-25]. The peculiar and remarkable feature of our "noncanonical" approach to quantum algebras is that the parameter of the q-deformation turns out, in general, to be a function of the physical constants characterizing the system considered [23,26]. As is well known, relativistic as well as quantum mechanics can be looked upon as "deformations" of classical mechanics, corresponding to the deformation parameters $\beta = v/c$ and $S/\hbar$. Therefore, in our opinion, the use of noncanonical commutation relations may reveal itself useful in the achievement of one of the ultimate goals in quantum group theory: understanding the very physical nature and origin of the deformation parameters and, possibly, connecting them with some new fundamental, dimensional physical constants. This point is also strictly related to the Lie-admissible nature of q-deformed algebras [cf. eq.(1.2), (1.3) above], on account of the possible capability of Lie-admissible theories to describing interactions structurally more general (e.g. of nonpotential, non-Hamiltonian type) than the Lie theories [27].

The aim of this paper is just to discuss some properties of two-parameter quantum groups in the noncanonical approach. In sect.2 we

---

[2] Clearly, for $Q = \frac{1}{q}$ one gets the usual q-deformed algebra, and for $q = 1$ the Q-algebra of Kuryshkin [3]. In the following, the harmonic oscillators corresponding to the above cases will be referred to, respectively, as a q-deformed oscillator and a Q-oscillator.

show that the (Q,q) algebra can be derived from Q-analysis [28] by a Bargmann differential realization of the operators $\hat{A},\hat{A}^+$. The bosonization of those operators is discussed in sect.3, and their Fock representation is obtained by explicitly imposing a noncanonical commutation relation between position and momentum. The eigenvalues of the corresponding (Q,q) harmonic oscillator are also found. Sect.4 provides an example of a two-parameter quantum group, $SU(2)_{Q,q}$, first introduced in ref.[14]. In sect.5 we derive the time-evolution of the operators $\hat{A},\hat{A}^+$ and $\hat{x},\hat{p}$ by exploiting the already quoted connection between deformed algebras and Lie-admissible algebras. Sect.6 concludes the paper.

Q-Analysis and Bargmann realization of a (Q,q) algebra

The relation between Q-analysis [28] and quantum groups has been investigated by some authors [13,29-31]. Essentially, the commutation relation of the Q-deformed algebra is realized in terms of the operators x, D_Q, where D_Q is the Q-derivative

$$D_Q = \frac{1}{x}\frac{Q^{xD}-1}{Q-1} \tag{2.1}$$

with $D=\frac{d}{dx}$ being the usual derivative operators.

We shall proceed in an analogous way. Firstly, let us define a *two-parameter derivative operator* $D_{Q,q}$ as follows. Consider a differentiable function $\varphi(x)$, and two points Qx and qx on the x-axis. We have

$$\frac{\varphi(qx)-\varphi(Qx)}{x(q-Q)} \equiv D_{Q,q}\varphi(x) = \frac{1}{x}\frac{q^{xD}-Q^{xD}}{q-Q}\varphi(x) \tag{2.2}$$

where

$$D_{Q,q} = \frac{1}{x}\frac{q^{xD}-Q^{xD}}{q-Q} = \frac{1}{x}\frac{Q^{xD}-q^{xD}}{Q-q} = D_{q,Q} \tag{2.3}$$

is just the (Q,q) derivative operator (symmetric under the exchange $Q \leftrightarrow q$). From the definition (2.3) we easily get the following commutation relations

$$D_{Q,q}x - Qx\,D_{Q,q} = q^{xD} \tag{2.4}$$

$$D_{q,Q}x - qx\,D_{q,Q} = Q^{xD} \tag{2.5}$$

which are changed into each other for $Q \leftrightarrow q$ and viceversa.

If we put $x = \frac{y}{q}$ or $x = \frac{y}{Q}(Q, q \neq 0)$ in eq.(2.2), the two-parameter derivative $D_{Q,q}$ reduces to the usual one-parameter Q-derivative:

$$\frac{\varphi(Qx) - \varphi(qx)}{x(Q - q)} = \frac{\varphi(\frac{Q}{q}\, y) - \varphi(y)}{y(\frac{Q}{q} - 1)} = \frac{\varphi(\lambda y) - \varphi(y)}{y(\lambda - 1)} \tag{2.6}$$

The above scale transformation can be given a physical meaning by noting that, under suitable transformations, the q-deformed harmonic oscillator is changed into the Q-harmonic oscillator [14].

The representation of the commutation relations (2.4) and (2.5) in Fock space can now be achieved by means of a Bargmann realization of the creation and annihilation operators as differential operators, i.e. setting

$$x = z; \qquad \hat{A}^{+} = z; \qquad \hat{A} = D_{Q,q}; \qquad \hat{n} = z\partial_z \ . \tag{2.7}$$

Then, we straightforwardly get the following relations

$$\hat{A}\hat{A}^{+} - Q\hat{A}^{+}\hat{A} = q^{\hat{n}} \ ; \tag{2.8}$$

$$\hat{A}\hat{A}^{+} - q\hat{A}^{+}\hat{A} = Q^{\hat{n}} \ ; \tag{2.9}$$

$$[\hat{A}, \hat{n}] = \hat{A} \ ; [\hat{A}^{+}, \hat{n}] = -\hat{A}^{+} \tag{2.10}$$

For $Q = 1/q$, the first two equations give exactly the well-known commutation relations of the q-deformed harmonic oscillator, i.e.

$$\hat{A}\hat{A}^{+} - \frac{1}{q}\hat{A}^{+}\hat{A} = q^{\hat{n}} \ ; \ \hat{A}\hat{A}^{+} - q\hat{A}^{+}\hat{A} = q^{-\hat{n}} \ . \tag{2.11}$$

which provide the connection between q-oscillators and quantum groups [32].

We therefore see that Q-analysis allows us to define in a natural way a (Q, q)-deformed algebra.

Bosonization of the operators $\hat{A}, \hat{A}^{+}$

We want now to find a boson realization of the annihilation and creation operator $\hat{A}, \hat{A}^{+}$ obeying the commutation rule (2.8).

Let us apply the bosonization method [10,11] and seek $\hat{A}, \hat{A}^+$ in the form

$$\hat{A} = F(\hat{n}+1)\hat{a} \ ; \ \hat{A}^+ = \hat{a}^+ F(\hat{n}+1) \ , \tag{3.1}$$

where $\hat{a}, \hat{a}^+$, $\hat{n} = \hat{a}^+\hat{a}$ are boson operators, satisfying the usual commutation relations.

Then, it is not difficult to show that

$$\hat{A} = \sqrt{\frac{[\hat{n}+1]_{Q,q}}{\hat{n}+1}}\,\hat{a} \ ; \ \hat{A}^+ = \hat{a}^+\sqrt{\frac{[\hat{n}+1]_{Q,q}}{\hat{n}+1}} \tag{3.2}$$

where

$$[\alpha]_{Q,q} = \frac{Q^\alpha - q^\alpha}{Q-q} = \frac{q^\alpha - Q^\alpha}{q-Q} \ . \tag{3.3}$$

Because the operators $\hat{A}^+\hat{A}$ and $\hat{a}^+\hat{a}$ commute, they have the same basis $|n\rangle$ (i.e. the usual Bose basis in Fock space).

From (3.2) it is easy to get the relations

$$\hat{A}\hat{A}^+ = [\hat{n}+1]_{Q,q} \ ; \ \hat{A}^+\hat{A} = [\hat{n}]_{Q,q} \ ; \tag{3.4}$$

$$[\hat{A}, \hat{A}^+] = [\hat{n}+1]_{Q,q} - [\hat{n}]_{Q,q} \ . \tag{3.5}$$

$$\{\hat{A}, \hat{A}^+\} = \hat{A}\hat{A}^+ + \hat{A}^+\hat{A} = [\hat{n}+1]_{Q,q} + [\hat{n}]_{Q,q} \ . \tag{3.6}$$

Moreover

$$\hat{A}|n\rangle = \sqrt{[n]_{Q,q}}\,|n-1\rangle\,; \ \hat{A}^+|n\rangle = \sqrt{[n+1]_{Q,q}}\,|n+1\rangle\,; \tag{3.7}$$

$$\hat{A}^+\hat{A}|n\rangle = [n]_{Q,q}|n\rangle \ . \tag{3.8}$$

Let us consider the Fock representation of $\hat{A}, \hat{A}^+$:

$$\hat{A} = \sqrt{\frac{1}{\lambda\hbar}}\Big(\sqrt{m\omega}\,\hat{x} + i\frac{\hat{p}}{\sqrt{m\omega}}\Big) \ ;$$

$$\hat{A}^+ = \sqrt{\frac{1}{\lambda\hbar}}\Big(\sqrt{m\omega}\,\hat{x} - i\frac{\hat{p}}{\sqrt{m\omega}}\Big) \tag{3.9}$$

where λ is a scale factor to be determined. Then, by eq.(3.5) the commutator of $\hat{x}, \hat{p}$ reads

$$[\hat{x}, \hat{p}] = i\,\frac{\lambda\hbar}{2}\Big([\hat{n}+1]_{Q,q} - [\hat{n}]_{Q,q}\Big) = = i\lambda\hbar\Big(\frac{Q^{\hat{n}+1} - q^{\hat{n}+1}}{Q-q} - \frac{Q^{\hat{n}} - q^{\hat{n}}}{Q-q}\Big) \tag{3.10}$$

The Hamiltonian of the two-parameter deformed oscillator is therefore given by

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m}{2}\omega^2\hat{x}^2 = \frac{\lambda\hbar\omega}{4}\Big\{\hat{A}, \hat{A}^+\Big\} = = \frac{\lambda\hbar\omega}{4}\Big([\hat{n}+1]_{Q,q} + [\hat{n}]_{Q,q}\Big)\ . \tag{3.11}$$

The corresponding energy eigenvalues are

$$E_n = \frac{\lambda\hbar\omega}{4}\Big([n+1]_{Q,q} + [n]_{Q,q}\Big)\ . \tag{3.12}$$

We want now to find the explicit expression of the scale factor λ. This will be accomplished by imposing that the commutator (3.10) takes the form of a non-canonical commutation relation, namely

$$[\hat{x}, \hat{p}] = i\hbar\Big(1 + \hat{H}\varphi(\hat{H})\Big) \tag{3.13}$$

where $\varphi(\hat{H})$ is an arbitrary hermitian operator function of the Hamiltonian [24,25,10][3]

To this aim, let us rewrite eq.(3.11) as

$$Q^{\hat{n}}(Q+1) - q^{\hat{n}}(q+1) = \frac{4\hat{H}(Q-q)}{\lambda\hbar\omega}\ . \tag{3.14}$$

A solution of the above equation for $\hat{n}$ can be found in the form of a power series:

$$\hat{n} = n_o + \frac{4\hat{H}(Q-q)}{\lambda\hbar\omega}n_1 + \Big[\frac{4\hat{H}(Q-q)}{\lambda\hbar\omega}\Big]^2 n_2 + \ldots + \Big[\frac{4\hat{H}(Q-q)}{\lambda\hbar\omega}\Big]^k n_k + \ldots \tag{3.15}$$

[3] In general, we can have [24] $[\hat{x}, \hat{p}] = i\hbar\, f(\hat{H})$ of which (3.13) is, of course, a special case.

After putting expansion (3.15) in eq.(3.14) we note that the constant term is exactly the relation from which we can determine n_o, i.e.:

$$Q^{n_o}(Q+1) = q^{n_o}(q+1) \ . \tag{3.16}$$

We get

$$n_o = \frac{ln\,\frac{q+1}{Q+1}}{ln\left(\frac{Q}{q}\right)} \ . \tag{3.17}$$

By replacing the expression (3.15) of $\hat{n}$ in the commutator (3.10), and re-arranging the different terms, we obtain finally

$$\begin{aligned}[\hat{x},\hat{p}] = \frac{i}{2}\lambda\hbar\Big\{&\frac{Q^{n_o}(Q-1)-q^{n_o}(q-1)}{Q-q} + \frac{4\hat{H}(Q-q)}{\lambda\hbar\omega}y_1 + \\ &\ldots + \Big[\frac{4\hat{H}(Q-q)}{\lambda\hbar\omega}\Big]^k y_k + \ldots\Big\}\end{aligned} \ . \tag{3.18}$$

The above commutator just takes the form (3.13) if

$$\frac{\lambda}{2}\Big[\frac{Q^{n_o}(Q-1)-q^{n_o}(q-1)}{Q-q}\Big] = 1 \ . \tag{3.19}$$

On account of eq.(3.16), (3.19) becomes

$$\frac{\lambda q^{n_o}}{Q+1} = 1 \ . \tag{3.20}$$

Due to the symmetry of (3.16) (and all equations (3.4)-(3.12)) under the exchange $Q \leftrightarrow q$, we have also

$$\lambda = (Q+1)q^{-n_o} = (q+1)Q^{-n_o} \ . \tag{3.21}$$

Then, from eq.(3.17), we get eventually

$$\lambda = (Q+1)q^{-\frac{ln\,\frac{q+1}{Q+1}}{ln(Q/q)}} \ . \tag{3.22}$$

It is now easy to check that the above relations reduce to the corresponding ones for the Q-oscillator ($q=1; \lambda = Q+1$) and for the q-deformed oscillator ($Q = 1/q; \lambda = q^{1/2} + q^{-1/2}$).

For further reference, let us explicitly write the final expressions of $\hat{A}, \hat{x}, \hat{p}$ and $\hat{H}$ for the two-parameter noncanonical oscillator.

We have

$$\hat{A} = \sqrt{\frac{1}{\hbar(Q+1)}}\, q^{(1/2)[ln(\frac{q+1}{Q+1})/ln(Q/q)]} \left(\sqrt{m\omega}\hat{x} - i\frac{\hat{p}}{\sqrt{m\omega}}\right) \;; \tag{3.23}$$

$$\hat{x} = \sqrt{\frac{\hbar(Q+1)}{2m\omega}}\, q^{-(1/2)[ln(\frac{q+1}{Q+1})/ln(Q/q)]} \,(\hat{A}^{+} + \hat{A}) \;; \tag{3.24}$$

$$\hat{p} = i\sqrt{\frac{m\hbar\omega(Q+1)}{2}}\, q^{-(1/2)[ln(\frac{q+1}{Q+1})/ln(Q/q)]} \,(\hat{A}^{+} - \hat{A}) \;; \tag{3.25}$$

$$\hat{H} = \frac{(Q+1)\hbar\omega}{4}\, q^{-[ln(\frac{q+1}{Q+1})/ln(Q/q)]} \,([\hat{n}+1]_{Q,q} + [\hat{n}]_{Q,q}) \;. \tag{3.26}$$

The noncanonical $SU_{Q,q}(2)$ quantum group

The two-parameter generalization of the $SU(2)$ group, based on noncanonical commutation relations among the generators, has been first introduced in ref.[14]. Let us briefly sketch its main features, in order to give an explicit example of a (Q, q) noncanonical quantum group.

Consider two independent oscillators with mutually commuting operators $\hat{A}_i, \hat{A}_i^{+} (i = 1, 2)$ and number operators $\hat{n}_i$. The Jordan-Wigner-Schwinger map for $\hat{J}_+, \hat{J}_-, \hat{J}_z$ gives

$$\hat{J}_+ = \hat{A}_1^{+}\hat{A}_2 \;;\quad \hat{J}_- = \hat{A}_2^{+}\hat{A}_1 \;;\quad \hat{J}_z = \frac{1}{2}\,(\hat{n}_1 - \hat{n}_2) \tag{4.1}$$

Moreover, we have

$$[\hat{J}_+, \hat{J}_-] = [2\hat{J}_z]_{Q,q} = \frac{q^{\hat{n}_1}Q^{\hat{n}_2} - Q^{\hat{n}_1}q^{\hat{n}_1}}{q-Q} \tag{4.2}$$

From the above relations we get

$$\begin{aligned} \hat{J}_z &= \frac{1}{2lnQ}\, arcsinh\, \frac{q-q^{-1}}{2(q-Q)}\left(q^{\hat{n}_1}Q^{\hat{n}_2} - q^{\hat{n}_2}Q^{\hat{n}_1}\right) = \\ &= \frac{1}{2lnq}\left[arcsinh\left(\frac{q-q^{-1}}{q-Q}\, sinh\,\frac{\hat{n}_1-\hat{n}_2}{2}\, ln\,\frac{q}{Q} e^{\frac{\hat{n}_1+\hat{n}_2}{2} lnQq}\right)\right] \end{aligned} \tag{4.3}$$

and the following non-canonical commutation relations for $\hat{J}_+, \hat{J}_z$:

$$[\hat{J}_z, \hat{J}_+] = \frac{1}{2lnQ}\left\{arcsinh\left[\frac{q-q^{-1}}{2(q-Q)}(q^{\hat{n}_1}Q^{\hat{n}_2} - q^{\hat{n}_2}Q^{\hat{n}_1})\right] - \right. \\ \left. - arcsinh\left[\frac{q-q^{-1}}{2(q-Q)}\left(q^{\hat{n}_1-1}Q^{\hat{n}_2+1} - q^{\hat{n}_2+1}Q^{\hat{n}_1-1}\right)\right]\right\} . \quad (4.4)$$

The above commutator can be given the form

$$[\hat{J}_z, \hat{J}_+] = [\hat{J}_z(\hat{n}_1, \hat{n}_2) - \hat{J}_z(\hat{n}_1 - 1, \hat{n}_2 + 1)]\hat{J}_+ \ . \quad (4.5)$$

Analogously, we have, for $\hat{J}_-, \hat{J}_z$:

$$[\hat{J}_z, \hat{J}_-] = -\hat{J}_-[\hat{J}_z(\hat{n}_1, \hat{n}_2) - \hat{J}_z(\hat{n}_1 - 1, \hat{n}_2 + 1)] \ . \quad (4.6)$$

In the basis $|j,m\rangle$, with $n_1 = j+m, n_2 = j-m$ we find

$$\hat{J}_z|j,m\rangle = \frac{1}{2lnQ}\, arcsinh\left(\frac{q-q^{-1}}{q-Q}\, sinh\, m\, ln\frac{q}{Q}e^{j\, lnQq}\right)|j,m\rangle \quad (4.7)$$

$$\hat{J}_+|j,m\rangle = \sqrt{[j+m+1]_{Q,q}\cdot[j-m]_{Q,q}}|j+m+1,\ j-m-1\rangle \quad (4.8)$$

$$\hat{J}_-|j,m\rangle = \sqrt{[j+m]_{Q,q}\cdot[j-m+1]_{Q,q}}|j+m-1,\ j-m+1\rangle \quad (4.9)$$

Of course, for $Q=1$ we recover the usual relations of the $SU(2)_Q$ group and for $Q=1/q$ those of the deformed $SU(2)_q$ group [9]. The reader is referred to refs.[14] for a more detailed discussion.

Lie-admissible time evolution for (Q,q) oscillators

We have already stressed in the introduction the connection between quantum groups and Lie-admissible algebras. In this section, we want to show that Lie-admissible theory allows to derive in a straightforward way the time-evolution of the operators of the (Q,q) deformed oscillator[21][4].

It is well known that the standard Heisenberg equations of motion are no longer valid for operators obeying non-canonical commutation

[4] The problem of the dynamical evolution of the Q-oscillator was solved in an analogous way in ref.[11]. Some preliminary considerations for q-deformed oscillators have been previously forwarded by the present authors [21].

relations. If, however, the commutation relations are in the form of the product of a Lie-admissible algebra, like eqs.(1.2) or (1.3), it has been shown by Santilli [27] that the following generalized Heisenberg equations hold:

$$i\hbar\frac{\partial\hat{A}}{\partial t} = (\hat{H},\hat{A}) \tag{5.1}$$

where (,) is the Lie-admissible bracket

$$(\hat{H},\hat{A}) = \hat{H}\hat{T}\hat{A} - \hat{A}\hat{R}\hat{H} \ . \tag{5.2}$$

In the above equations, $\hat{H}$ is the usual Hamiltonian (describing conservative forces) and $\hat{T},\hat{R}$ are suitable operators (supposed to represent in general, nonconservative interactions). The case $\hat{T} = \hat{R}$ and $\hat{H}\hat{T} \neq \hat{T}\hat{H}$ corresponds to the so-called Lie-isotopic case.

The integrated form of (5.1), when the operators $\hat{H},\hat{T},\hat{R}$ are time-independent, is given by [27]

$$\hat{A}(t) = exp\Big(\frac{it}{\hbar}\hat{R}\hat{H}\Big)\hat{A}(0)exp\Big(-\frac{it}{\hbar}\hat{H}\hat{T}\Big) \ . \tag{5.3}$$

Let us apply eqs.(5.1)-(5.3) to the creation and annihilation operators of the (Q,q)-deformed oscillator. A comparison of eq.(5.2) with (2.8) (see also eq.(1.2)) shows that, in this case

$$\hat{T} = \hat{I} \ ; \qquad \hat{R} = Q\hat{I} \ . \tag{5.4}$$

The Hamiltonian $\hat{H}$ is that of the two-parameter oscillator, eqs.(3.11), (3.26). We get, from (5.3), on account of (5.4):

$$\begin{aligned} \hat{A}(t) &= exp\Big(\frac{it}{\hbar}Q\hat{H}\Big)\hat{A}(0)exp\Big(-\frac{it}{\hbar}\hat{H}\Big) = \\ &= exp\Big\{it\omega\frac{(Q+1)}{4}\lambda Q\Big([\hat{n}+1]_{Q,q} + [\hat{n}]_{Q,q}\Big)\Big\}\hat{A}(0)\cdot \\ &\cdot exp\Big\{-it\omega\frac{(Q+1)}{4}\lambda\Big([\hat{n}+1]_{Q,q} + [\hat{n}]_{Q,q}\Big)\Big\} \ . \end{aligned} \tag{5.5}$$

By taking into account the explicit expression of $\hat{A}(0)$, eq.(3.2), we find, after some algebra

$$\begin{aligned} \hat{A}(t) = exp\Big\{it\omega\frac{(Q+1)}{4}\lambda\Big[Q([\hat{n}+1]_{Q,q} + [\hat{n}]_{Q,q}) - [\hat{n}+2]_{Q,q} - \\ - [\hat{n}+1]_{Q,q}\Big]\Big\}\hat{A}(0) = exp(-itK_{\hat{n}})\hat{A}(0) \end{aligned} \tag{5.6}$$

and

$$\hat{A}^{+}(t)=\hat{A}^{+}(0)exp(itK_{\hat{n}}) \ , \tag{5.7}$$

where we put

$$\begin{aligned} K_{\hat{n}} &= \omega\frac{(Q+1)(q+1)}{4}\,\lambda q^{\hat{n}} = \\ &= \omega\frac{(Q+1)^2(q+1)}{4}\,q^{[\hat{n}-\frac{ln(q+1)/(Q+1)}{ln(Q/q)}]} \ . \end{aligned} \tag{5.8}$$

From the above relations, using the Fock representation of the operators $\hat{x}(t),\hat{p}(t)$ (see eqs.(3.9), (3.24), (3.25)), we obtain

$$\hat{x}(t)=\frac{1}{2}\Big(e^{-itK_{\hat{n}}}+e^{it\frac{K_{\hat{n}}}{q}}\Big)\hat{x}(0)+\frac{i}{2m\omega}\Big(e^{-itK_{\hat{n}}}-e^{it\frac{K_{\hat{n}}}{q}}\Big)\hat{p}(0) \tag{5.9}$$

$$\hat{p}(t)=\frac{1}{2}\Big(e^{-itK_{\hat{n}}}+e^{it\frac{K_{\hat{n}}}{q}}\Big)\hat{p}(0)-\frac{i}{2}\,m\omega\Big(e^{-itK_{\hat{n}}}-e^{it\frac{K_{\hat{n}}}{q}}\Big)\hat{x}(0) \tag{5.10}$$

It is easy to check that, for $q=1$, eqs.(5.6)-(5.10) reduce to the analogous expressions for the Q-oscillator (already derived in ref.(11)). We have, in this case:

$$K_{\hat{n}}=\omega\frac{(Q+1)}{2} \ . \tag{5.11}$$

The time-evolution of the q-deformed oscillator are obtained from the above equations for $Q=1/q$ and

$$K_{\hat{n}}=\omega\frac{(q+1)^2}{4q}\,q^{\hat{n}+\frac{1}{2}} \ . \tag{5.12}$$

Of course, for $Q=q=1$ one recovers the well-known formulae for the harmonic oscillator.

Moreover, it is a lenghty but easy task to check that eqs.(5.6), (5.7) do satisfy the deformed commutation rule (2.8) at any time, namely

$$\hat{A}(t)\hat{A}^{+}(t)-Q\hat{A}^{+}(t)\hat{A}(t)=\hat{A}(0)\hat{A}^{+}(0)-Q\hat{A}^{+}(0)\hat{A}(0)=q^{\hat{n}} \ . \tag{5.13}$$

Conclusions

In the present paper we have introduced a deformed Heisenberg-Weyl algebra depending on two parameters. It contains, as special cases, both the Q-algebra by Arik and Coon [2] and Kuryshkin [3] and the q-deformed algebra. The commutation rules of the creation and annihilation operators $\hat{A}, \hat{A}^+$ can be derived from a straightforward generalization of Q-derivative by means of a differential Bargmann representation. The bosonization method has been applied to get the explicit expressions of $\hat{A}, \hat{A}^+$, and the corresponding Fock representation of the coordinate and momentum operators $\hat{x}, \hat{p}$, whose commutator turns out to be non-canonical.

We have also briefly sketched the main features of the two-parameter $SU(2)_Q, q$ group. In this case, too, the commutation relations of the $SU(2)_Q, q$ generators are non-canonical. The procedure we followed could be applied, in general, to define the quantum group generated by introducing N two-parameter deformed oscillators and using the Jordan-Wigner-Schwinger map. In this case, however, some problems are expected to arise in connection with the actual $U(N)_Q, q$ symmetry of the system, and with a consistent definition of the Casimir invariants [33]. We shall give a more careful discussion of this topics in a separate paper.

Needless to say, the two-parameter algebra can be used to define a noncanonical $SU(1,1)_{Q,q}$ group. Work on these lines is in progress.

Finally, we want to stress that the connection between deformation algebras and Lie-admissible algebras has allowed us to derive the time-evolution of the two-parameter operators. This has been accomplished by an almost straightforward application of the Lie-admissible Heisenberg equations introduced by Santilli [15]. This result seems to indicate that the dynamics underlying quantum groups is , in general, of the Lie-admissible type. Such a point can be physically seen by taking into account that systems of q-oscillators are interacting, and that their interactions are, in general, non-local [33].

Acknowledgements

Two of us (A.J. and G.B.) would like to thank the Dipartimento di Fisica, I Università di Roma "La Sapienza", for the kind hospitality.

References

[1] H.D.Doebner and J.D.Hennig, eds: *Quantum Groups* (Proc. 8^{th} Int. Workshop Math Phys., Clausthal, 1989)Springer-Verlag, Heidelberg, 1990). See in particular L.A.Takhtajan: *Introduction to Quantum Groups*, p.3; H.D.Doebner, J.D.Hennig and W.Lücke: *Mathematical Guide to Quantum Groups*, p.29.

[2] M.Arik and D.D.Coon: University of Pittsburgh preprint PITT-**116**, (November 1972).
[3] V.Kuryshkin:Dep. VINIT1. pp.3976-76 (1976); Annales de la Fondation Louis de Broglie **5**, 111 (1980).
[4] P.P.Kulish and N.Yu.Reshitikhin: Zap. Nauk. Semin. LOMI **101**, 101 (1981); J. Sov. Math.**23**, 2435 (1983).
[5] L.D.Faddeev: in Les Houches LecturesXXXIX, 1982, p.563 (North-Holland, Amsterdam, 1984).
[6] E.K.Sklyanin: Funk.Anal.Priloz.**16**, 27 (1982);**17**, 34 (1983).
[7] V.Drinfeld: Sov.Math.Dokl.**32**, 254 (1985); *Quantum Groups*,in Proc. Int. Congr. Math., Berkeley, CA, 1986 Vol. I, A.M. Gleason ed. (Providence, RI, American Math.Soc., p.798); M. Jimbo: Lett.Math.Phys.**10**, 63 (1985);**11**, 247 (1986); Comm. Math. Phys.**102**, 537 (1987); S. Woronovicz: Publ. RIMS (Kyoto Univ.)**23**, 117 (1987); Comm. Math. Phys.**93**, 35 (1988); Y.I.Manin: *Quantum Groups and Non-Commutative Geometry* (Montreal: Centre de Recherches Mathematiques, 1988); V. Pasquier: Nucl.Phys. B**295**, 491 (1988); Comm.Math.Phys.**118**, 355 (1988); G. Luststig:Adv. in Math.**70**, 237 (1988); L.L. Vaksmann and Ya.S. Soibelmann: Funk. Anal. Appl.**22**, 1 (1988); M. Rosso: Comm.Math.Phys.**124**, 307 (1989); L.D. Faddeev, N.Yu. Reshatiklin and L.A. Takhtajan: Algebra and Analysis**1**, 178 (1989); D. Bernard: Lett.Math.Phys.**17**, 239 (1989).
[8] For the notion of Hopf algebra, see ref.[1] and E.Abe: *Hopf Algebras*, Cambridge Tracts in Mathematics(Cambridge Univ. Press, 1980).
[9] A.J.Mac Farlane: J.Phys.A**22**, 4581 (1989);L.C.Biedenharn: J.Phys.A**22**, L873 (1989);C.P.Sen and H.C.Fu: J.Phys.A**22**, L983 (1989); M. Chaichian and P. Kulish: Phys.Lett.**234 B**, 72 (1990); M. Chaichian, P. Kulish and J. Lukierski: Phys.Lett.**237 B**, 401 (1990); D. Fairlie: J.Phys.A**23**, L183 (1990); P. Kulish and E.V. Damaskinsky: J.Phys.A**23**, L415 (1990); R. Floreanini, V.P. Spiridonov and L. Vinet: Phys.Lett.**242 B**, 383 (1990);S.K.Soni: J.Phys. A**23**, L951 (1990); T.Curtright and C.Zachos: Phys.Lett.B**243**, 237(1990); A.T. Filippov, D. Gangopadhyay and A.P.Isaev: J.Phys.A**24**, L63 (1991);C. Burdik and L. Hlavaty:J.Phys. A**24**, L165 (1991); L. Frappat, P. Sorba and A. Sciarrino: J.Phys. A**24**, L179 (1991);D. Bonatsos, E.N. Argyres and P.P. Raychev: J.Phys. A**24**, L455 (1991); F.J. Naragnes-Quijano: J.Phys. A**24**, 1699 (1991); E. Celeghini, T.D. Palev, M. Tarlini: Modern Phys. Lett. B**5**, 187 (1991).
[10] A. Jannussis, G. Brodimas, D. Sourlas, V. Zisis: Lett.N.Cim.**30**, 123 (1981); G. Brodimas, A. Jannussis, D. Sourlas, V.Zisis, P.Poulopoulos: Lett.N.Cim.**31**, 177 (1981), and references therein.
[11] A. Jannussis, G. Brodimas, D. Sourlas, L. Papaloucas, P. Poulopoulos, P. Siafarikas: Lett. N. Cim.**34**, 375 (1982); A. Jannussis, G. Brodimas, D. Sourlas, A. Streclas, P. Siafarikas, L. Papaloucas, N. Tsangas: Hadr.J.**5**, 1923 (1982), and references therein.
[12] R.N.Mohapatra: Phys. Lett. B **242**, 407 (1990); O.W.Greenberg: Phys. Rev. Lett.**64**, 705 (1990); "Q-mutators and violations of statistics", Proc. Spring Workshop on Quantum Groups (Argonne, May 1990) (World Sci-

entific, Singapore, in press).
[13] A. Jannussis, G. Brodimas, D. Sourlas, K. Vlachos, P. Siafarikas and L. Papaloukas: Hadr.J. **6**, 1653 (1983).
[14] A.Jannussis: "Quantum groups and the Lie-admissible Q-algebra": Hadr.J. (to appear); in Proc.Fifth Int.Conference on Hadronic Mechanics and Non-potential Interactions (Cedar Falls, Iowa, USA, 13-17 August 1990), H.C.Myung ed. (Nova Science, Commack, N.Y.), in press;G.Brodimas, S.Bascoutas, D.Vavougios and A.Jannussis: "Quantum group for linear combinations of Bose and Fermi operators", in Proc.Fifth Int. Conf. on Hadr. Mechanics and Non-potential Interactions (Cedar Falls, Iowa, USA, 13-17 August 1990), H.C.Myung ed. (Nova Science, Commack, N.Y), in press; A.Jannussis, C.Brodimas and S.Bascoutas: "Lie-admissible Q-algebra and the Pauli exclusion principle", Hadr.J. (to appear).
[15] R.M.Santilli: N.Cim.**51 A**, 570 (1967); Suppl.N.Cim.**6**, 1225 (1968).
[16] A.A.Albert: Proc. Mat. Acad. Sci. **35**, 317 (1949). See also H.C. Myung: *Malcev- Admissible Algebras* (Birkhauser, Boston, 1986), and references therein.
[17] D.L.Reiner: Discrete Mathematics**43**, 125 (1983).
[18] A.Sudbery: J. Phys. **A 23**, L697 (1990); M.Takeuchy:*Proc. Japan Acad.* **66 A**, 112 (1990).
[19] A.Schirrmacher, J. Wess and B. Zumino: Z. Phys.**C 49**, 317 (1991).
[20] R.Chakrabarti and R.Jagannathan: J. Phys.**A 24**, L711 (1991).
[21] A.Jannussis, G.Brodimas, R.Mignani: J. Phys. A**24**, L775 (1991).
[22] G.Brodimas, A.Jannussis and R. Mignani:INFN preprint n. 820 (1991).
[23] G.Brodimas, A.Jannussis and R. Mignani: J. Phys.**A 25**, L329 (1992).
[24] H.Ramphacher, H.Stumpf and F.Wagner: Fortschr. Phys.**13**, 385 (1965); I.Saavedra and C.Utreras: Phys. Lett.**98 B**, 74 (1981).
[25] A.Jannussis, A.Streclas, D.Sourlas and K.Vlachos: Lett. N. Cim.**19**, 163 (1977); Phys. Scripta**15**, 163 (1977).
[26] S.V.Shabanov: J. Phys. **A 25**, L1245 (1992).
[27] R.M.Santilli: Hadr. J.**1**, 754 (1978); *Lie-admissible Approach to Hadron Structure*, vol.I and II (Hadronic Press, Nonantum, Mass., U.S.A., 1978 and 1982).
[28] F.H.Jackson: Messenger Math.**38**, 62 (1909); Quart. J. Pure Appl. Math.**41**, 193 (1910);G.E.Andrews: Studies in Apll. Math.**50**, 345 (1971); G.C.Rota, D.Kalnamer and A.Odlysko: Jour. Math. Analysis and Applications **42**, 684 (1973); S.Roman and G.C.Rota: Advances in Math. **27**, 95 (1978); T.Cigler: Mh. Math. **88**, 87 (1979); **91**, 105 (1981); Archiv. der Math. **35**, 533 (1980); R.Askey and J.A.Wilson: Memoirs Amer. Math. Soc. **319** (1985); S.C.Milne: Advances in Math. **72**, 59 (1988); and references therein.
[29] H.Exton: *Q-Hypergeometric Functions and Applications*, (Ellis Horwood Publ. Clichestew, 1983).
[30] D.Fairlie: "Q-analysis and Quantum Groups", in *Symposium on Symmetries in Sciences*, V (Plenum, New York, in press).
[31] L.C.Biedernharn and M.A.Lohe: "Quantum Groups and Basic Hyper-

geometric Functions" (Proc. Workshop on q-groups, Argonne National Laboratory, April 1990), to appear.
[32] H.Yan: J. Phys. **A 23**, L1155 (1990).
[33] E.G.Floratos: J. Phys.**A 24**, 4739 (1991).

(Manuscrit reçu le 24 mai 1993)