On the theory of the one-dimensional Fermi gaz with g/x^2 potential

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ABSTRACT. The one-dimensional Fermi gas with g/x^2 pair interaction potential is studied by the approximating hamiltonian method determining the reduced approximating hamiltonian which can be proved to be thermodynamically equivalent to original system one in thermodynamical limit and while the reduced parameter tends to zero.

RÉSUMÉ. On étudie un gaz de Fermi à une dimension avec un potentiel d'interaction des paires en g/x^2 par la méthode de l'hamiltonien approché. On détermine un hamiltonien réduit dont on montre l'équivalence thermodynamique au système initial à la limite thermodynamique quand le paramètre réduit tend vers zéro.

1 Introduction

In recent years different problems of statistical mechanics have been succesfully solved by the approximating hamiltonian method [1-3] which main idea is to choose by any way a thermodynamical equivalent approximating hamiltonian for some model. The thermodynamical equivalence of hamiltonians H_1 and H_2 is assumed as the following condition for associated free energy functions:

$$f[H_1] - f[H_2] = -\frac{\theta}{V} \cdot \ln \mathrm{Sp}\, e^{-\beta H_1} + \frac{\theta}{V} \cdot \ln \mathrm{Sp}\, e^{\beta H_2} = \mathrm{Cte} \qquad (1)$$

where V is the system volume, $\beta = \frac{1}{\theta}$ is the inverse absolute temperature.

This work demonstrates the usage of the approximating hamiltonian method to investigate the one-dimensional Fermi gas with g/x^2 pair potential, where x is the distance between particles, and *concretizes the results obtained in [4].*

1 - Statement of problem.

The spectrum of the fermi system hamiltonian (we assume hereafter $-\frac{1}{2} < g = \text{Cte}$ in order to prevent any collapse [6])

$$H_g = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + g \cdot \sum_{i<j} \frac{1}{(x_i - x_j)^2} \tag{2}$$

has been shown in [5] to be identical up to a constant to one of the hamiltonians having the following form in second quantization representation

$$H_\lambda = \sum_k (k^2 - \mu) a_k^+ a_k - \lambda \frac{\pi}{L} \cdot \sum_{k,k'} |k - k'| a_k^+ a_k a_{k'}^+ a_{k'} \tag{3}$$

where L is the length of the one-dimensional system; a_k, a_k^+ are creation and annihilation Fermi operators, $\{a_k, a_{k'}^+\} = a_k a_{k'}^+ + a_{k'}^+, a_k = \Delta(k-k')$; μ is the chemical potential, the momenta of particals $k = 2\pi m/L$ belong to the quasidiscrete set, $m \in Z$, Z is the set of integer number, $\lambda = (1/2)(1 - \sqrt{1+2g}) = \text{Cte} > 0$.

But that coincidence of spectra means the thermodynamical equivalence of H_g and H_λ in accordance with [1]. In this thesis we are going to investigate the more general system described by the hamiltonian

$$\Gamma = \sum_k (k^2 - \mu) a_k^+ a_k - \lambda \frac{\pi}{L} \cdot \sum_{k,k'} |k-k'| a_k^+ a_k a_{k'}^+ a_{k'} - \nu \cdot \sum_k (|k|+1) a_k^+ a_k \tag{4}$$

where the inserted "sources" $\nu \cdot \sum_k (|k|+1) a_k^+ a_k$ do not break the original system symmetry [3] contrary to the standard technics [2] of the approximating hamiltonian method.

We are going to use the approximation hamiltonian method to determine the quadratic approximating hamiltonian which will be proved to be thermodynamicaly equivalent to [4] in the limit $L \to \infty, L/N = \text{Cte}$, where N is the number of particles. All proofs will be provided for $\nu = \text{Cte} > 0$ and then ν will be put equal to zero after the thermodynamic limit transition.

2 - The thermodynamical equivalence of reduced hamiltonians.

Let us first consider the reduced problem with the following hamiltonian:

$$\Gamma^{\alpha} = \tau - \lambda \frac{\pi}{L} \sum_{k,k'} |k - k'| e^{-\alpha(|k|+|k'|)} \cdot n_k n_{k'} - \nu \sum_{k} (|k| + 1) e^{-\alpha |k|} n_k \quad (5)$$

L is the length of the one-dimensional system, $\alpha > 0$ is the reduced parameter, $\tau = \sum_k \varepsilon_k a_k^+ a_k$ is the free fermion hamiltonian, $n_k = a_k^+ a_k$ is the occupation number operator.

Let the approximating hamiltonian be

$$\begin{aligned} \Gamma_0^{\alpha}(C) = \tau - \lambda \frac{\pi}{L} \cdot \sum_{k,k'} |k - k'| e^{-\alpha(|k|+|k'|)} \cdot (n_k \cdot C_{k'} + n_{k'} \cdot C_k \\ - C_k \cdot C_{k'}) - \nu \sum_k (|k| + 1) e^{-\alpha|k|} n_k \cdot \end{aligned} \quad (6)$$

$\Gamma_0^{\alpha}(C)$ is depending on numeric number of parameters (C_k).

We denote the free energy function associated with the approximating hamiltonian as $f_0^{\alpha}(C)$ and write Bogolubov's inequality [2] with the following notations:

$$\begin{aligned} &< \cdots >_{\tilde{T}} = \frac{\mathrm{Sp}\,(\ldots e^{-\beta \tilde{T}})}{\mathrm{Sp}\, e^{-\beta \tilde{T}}} \\ &< \cdots >_{0,\alpha} = \frac{\mathrm{Sp}\,(\ldots e^{-\beta \Gamma_0^{\alpha}})}{\mathrm{Sp}\, e^{-\beta \Gamma_0^{\alpha}}}, \quad f_{\tilde{T}} = -\frac{\theta}{L} \cdot \ln \mathrm{Sp}\, e^{-\beta \tilde{T}} \\ &\tilde{T} = \tau - \nu \sum_k (|k| + 1) e^{-\alpha|k|} n_k \quad : \\ &\lambda \frac{\pi}{L^2} \sum_{k,k'} |k - k'| e^{-\alpha(|k|+|k'|)} \cdot (2 < n_k >_{\tilde{T}} \cdot C_{k'} - C_k C_{k'}) \leq \\ &\leq f_{\tilde{T}} - f_0^{\alpha}(C) \leq \\ &\leq \lambda \frac{\pi}{L^2} \cdot \sum_{k,k'} |k - k'| e^{-\alpha(|k|+|k'|)} \cdot (2 < n_k >_{0,\alpha} \cdot C_{k'} - C_k C_{k'}) \end{aligned} \quad (7)$$

It can be seen that $0 \leq f_0^{\alpha}(C) - f_{\tilde{T}}$ if $(C_k \,|\, \forall k \; C_k \leq 2)$. Taking in mind $f_0^{\alpha}(0) = f_{\tilde{T}}$ we have as result that if $(C_k \,|\, \forall k \; 0 < C_k < 2)$ the continious function $f_0^{\alpha}(C)$ has his absolute minimum at the "point" where

$$\forall k \frac{\delta f_0^{\alpha}(C)}{\delta C_k} = 0 \quad (8)$$

it's necessary to emphasize that [8] uses variational derivatives because parameters C_k are momentum (wave number) functions. Let's denote hereafter the condition [8] as

$$\frac{\delta f_0^{\alpha}(C)}{\delta C} = 0 \tag{9}$$

We need the following:

Lemma. If real numbers η_k realize

$$\forall f \qquad \sum_k |f-k|\eta_k = 0$$

where f and k belong to the quasidiscrete set $f = 2\pi l/L,\ k = 2\pi m/L$, l and m are integers, then for arbitrary $k,\ \eta_k = 0$.

Proof. Using the representation of f and k we denote

$$\eta_k = \eta(\frac{2\pi m}{L}) = \psi_m :$$

$$\sum_k |f-k|\eta_k = \frac{2\pi}{L}\sum_m |l-m|\psi_m = S(l) \quad , \quad \forall l \quad S(l) = 0$$

$$S(l+1) = \frac{2\pi}{L}\sum_m |l+1-m|\psi_m \quad ,$$

$$S(l) - S(l+1) = \frac{2\pi}{L}\sum_m (|l-m| - |l+1-m|)\cdot\psi_m$$

$$= \frac{2\pi}{L}\sum_m sgn(m-1-\frac{1}{2})\cdot\psi_m$$

where the sgn function

$$\mathrm{sgn}(x) = \begin{cases} 1 & \text{if } x>0 \\ 0 & \text{if } x=0 \\ -1 & \text{if } x<0 \end{cases}$$

By analogy

$$S(l-1) - S(l) = \frac{2\pi}{L}\sum_m (|l-1-m| - |l-m|)\cdot\psi_m$$

$$= \frac{2\pi}{L}\sum_m sgn(m-1+\frac{1}{2})\cdot\psi_m$$

Since $S(l-1) - S(l) - (S(l) - S(l+1)) = 0$,

$$\frac{2\pi}{L}\sum_m (sgn(m-l+\frac{1}{2}) - sgn(m-l-\frac{1}{2}) \cdot \psi_m = \frac{2\pi}{L} \cdot 2\psi_l = 0$$

It's true for arbitrary l, thus

$$\forall f \quad f = \frac{2\pi l}{L} \quad \psi_l = \eta_f = 0$$

The lemma is proved.

We are ready now to prove the following statement.

Theorem 1.

$$\frac{\delta f_0^\alpha(C)}{\delta C} = 0$$

if and only if

$$C = (C_k | \forall k C_k =< n_k >_{0,\alpha})$$

Proof. It is easy to see that if $< n_f >_{0,\alpha}= C_f$

$$\frac{\delta f_0^\alpha(C)}{\delta C_k} = -\frac{2\lambda\pi}{L^2}\sum_f e^{-\alpha(|k|+|f|)}|k-f|(< n_f >_{0,\alpha} - C_f) = 0 \qquad (10)$$

Let denote $\eta_f = \frac{2\lambda\pi}{L^2} \cdot e^{-\alpha|f|}(< n_f >_{0,\alpha} - C_f)$, than according to Lemma

$$\forall f \eta_f = 0, \quad i \cdot e \cdot < n_f >_{0,\alpha}= C_f$$

The theorem 1 is proved.

Let f^α be the free energy function of the model system (4). The Bogolubov inequality for f^α and $f_0^\alpha(C)$ is

$$\begin{aligned} &\lambda\frac{\pi}{L^2}\sum_{k,k'}|k-k'|e^{-\alpha(|k|+|k'|)} \cdot < (n_k - C_k)(n_{k'} - C_{k'}) >_{0,\alpha} \\ &\leq f_0^\alpha(C) - f^\alpha \qquad (11) \\ &\leq \lambda\frac{\pi}{L^2} \cdot \sum_{k,k'}|k-k'|e^{-\alpha(|k|+|k'|)} < (n_k - C_k)(n_{k'} - C_{k'}) >_\alpha \end{aligned}$$

where

$$< \dots >_\alpha = \frac{\mathrm{Sp}\,(\dots e^{-\beta\tau^\alpha}}{\mathrm{Sp}\, e^{-\beta\tau^\alpha}}$$

Since $f_0^\alpha(C)$ has the absolute minimum at the "point" (in accordance with Theorem 1)

$$\sigma^\alpha = [\sigma_k^\alpha | \forall k \sigma_k^\alpha =< n_k > \Gamma_0^\alpha(\sigma_k^\alpha) =< n_k >_{0,\alpha}] \tag{12}$$

the inequality [11] may be transformed as follows:

$$\begin{aligned} 0 &\le f_0^\alpha(\sigma^\alpha) - f^\alpha \le f_0^\alpha(C) - f^\alpha \\ &\le \lambda \frac{\pi}{L} \cdot \sum_{k,k'} |k-k'| e^{-\alpha(|k|+|k'|)} \cdot < (n_k - C_k)(n_{k'} - C_{k'}) > \alpha \end{aligned} \tag{13}$$

Let $C =< n_k >_\alpha$, then [13] may be rewritten as:

$$\begin{aligned} 0 &\le f_0^\alpha(\sigma^\alpha) - f^\alpha \le f_0^\alpha(< n_k >_\alpha) - f^\alpha \\ &\le \lambda \frac{\pi}{L^2} \cdot \sum_{k,k'} |k-k'| e^{-\alpha(|k|+|k'|)} < n_k n_{k'} >_\alpha - < n_k >_\alpha < n_{k'} >_\alpha \end{aligned} \tag{14}$$

We estimate the right side of (14) using the operator

$$U = \frac{1}{L} \sum_k (|k|+1) e^{-\alpha|k|} n_k$$

Then

$$\begin{aligned} &\lambda \frac{\pi}{L^2} \cdot \sum_{k,k'} |k-k'| e^{-\alpha(|k|+|k'|)} [< n_k n_{k'} >_\alpha - < n_k >_\alpha < n_{k'} >_\alpha] \\ &\le \lambda \frac{\pi}{L^2} \cdot \sum_{k,k'} (|k|+1)(|k'|+1) e^{-\alpha(|k|+|k'|)} \\ &\qquad [< n_k n_{k'} >_\alpha - < n_k >_\alpha < n_{k'} >_\alpha] \\ &= \lambda\pi(< U^2 >_\alpha - < U >_\alpha^2 = -\frac{\lambda\pi\theta}{L} \cdot \frac{\partial^2 f^\alpha}{\partial \nu^2} \end{aligned} \tag{15}$$

And now we are ready to prove the asymptotical equality of the free energy functions associated with the reduced model hamiltonian and approximating one if the free energy function of approximating system is taken at the "point" of absolute minimum.

Theorem 2.

If parameters (C_k) are the solution of the self-consistent equality (9) then the hamiltonians $\Gamma_0^\alpha(C)$ and Γ^α at $0 < \varphi \leq \nu$ are thermodynamically equivalent in the limit $L \to \infty$, $\frac{L}{N} = \mathrm{Cte}$ and

$$\lim_{L\to\infty} |f[\Gamma_0^\alpha(C) - f[\Gamma^\alpha]| = 0 \tag{16}$$

Proof. Let denote the solution of the self-consistent equality as (σ^α) and rewrite once more the inequality (14) using the estimation (15)

$$\begin{aligned} 0 &\leq f_0^\alpha(\sigma^\alpha) - f^\alpha \leq f_0^\alpha(< n_k >) - f^\alpha \\ &\leq \lambda\frac{\pi}{L^2} \cdot \sum_{k,k'} |k-k'|e^{-\alpha(|k|+|k'|)} < n_k n_{k'} > - < n_k >< n_{k'} > \\ &\leq -\frac{\lambda\pi\theta}{L} \cdot \frac{\partial^2 f^\alpha}{\partial\nu^2} \end{aligned} \tag{17}$$

Taking in mind

$$0 \leq -\frac{\partial f^\alpha}{\partial\nu} =< U >_\alpha = \frac{1}{L}\sum_k (|k|+1)e^{-\alpha|k|} < n_k >_\alpha \leq \Omega_1(\alpha) = \mathrm{Cte}$$

we integrate the product $\frac{\partial^2 f^\alpha}{\partial\nu^2} \cdot \nu$:

$$-\int_{\nu_0}^{\nu_1} \nu\frac{\partial^2 f^\alpha}{\partial\nu^2} \cdot d\nu = -\nu\frac{\partial f^\alpha}{\partial\nu}|_{\nu_0}^{\nu_1} + \int_{\nu_0}^{\nu_1} \frac{\partial f^\alpha}{\partial\nu} \cdot d\nu \leq (\nu_1 - \nu_0)\cdot\Omega_1(\alpha) \tag{18}$$

According to the mean-value theorem with $\nu_1 = \nu + 2l, \nu_0 = \nu + 1$

$$-\int_{\nu+1}^{\nu+2l} \nu\frac{\partial^2 f^\alpha}{\partial\nu^2} \cdot d\nu = -\frac{\partial^2 f^\alpha}{\partial\nu^2}(\xi) \cdot \frac{1}{2}[(\nu+2l)^2 - (\nu+1)^2] \tag{19}$$

where $\nu + 1 \leq \xi \leq \nu + 2l$. Joining (18) and (19) we have

$$0 \leq -\frac{\partial^2 f^\alpha}{\partial\nu^2}(\xi)\cdot \leq \frac{\Omega_1(\alpha)}{\nu + \frac{3}{2}l}$$

If l tends to zero

$$|\frac{\partial^2 f^\alpha}{\partial\nu^2}| \leq \Omega_2(\alpha) = \mathrm{Cte} \tag{20}$$

at arbitrary $\nu \geq \varphi > 0$ and this estimation does not depend on the system length L. Going back to (17) we obtain

$$0 \leq f[\Gamma_0^\alpha(\sigma^\alpha)] - f[\Gamma^\alpha] \leq \frac{\lambda\pi\theta}{L} \cdot \Omega_2(\alpha)$$

Hence Theorem 2 is proved.

3 - Transition to the limit of the reduce parameter.

The problem with the hamiltonians (4) and (5) becomes the original one when the reduce parameter α tends to zero. Thus let us to prove two additional theorems.

Theorem 3.

$$\lim_{\alpha\to 0}\lim_{L\to\infty} |f[\Gamma^\alpha] - f[\Gamma]| = 0 \tag{21}$$

Proof. The Bogolubov inequality for the free energy functions of the hamiltonians (4) and (5) is

$$\begin{aligned} &-\lambda\frac{\pi}{L}\cdot\sum_{k,k'}|k-k'|[1-e^{-\alpha(|k|+|k'|)}]\cdot <n_k n_{k'}> \\ &-\frac{1}{L}\sum_k(|k|+1) <n_k> \\ &\leq f[\Gamma]-f[\Gamma^\alpha] \\ &\leq -\lambda\frac{\pi}{L}\cdot\sum_{k,k'}|k-k'|[1-e^{-\alpha(|k|+|k'|)}] <n_k n_{k'}>_\alpha \\ &\quad -\frac{1}{L}\sum_k(|k|+1)e^{-\alpha|k|} <n_k>_\alpha \leq 0 \end{aligned} \tag{22}$$

where $<\cdots>_{\alpha=0}=<\cdots>$. Since limits $L\to\infty$ exist and

$$\lambda\frac{\pi}{L}\cdot\sum_{k_\infty,k'}|k-k'|[1-e^{-\alpha(|k|+|k'|)}]\cdot <n_k n_{k'}>$$

$$\xrightarrow[L\to\infty]{} \frac{\lambda}{4\pi}\int_{-\infty}|k-k'|[1-e^{-\alpha(|k|+|k'|)}]\cdot <n(k)n(k')> dkdk' \quad ,$$

$$\frac{1}{L}\sum_k(|k|+1)e^{-\alpha|k|} <n_k>$$

$$\xrightarrow[L\to\infty]{} \frac{1}{2\pi}\int_{-\infty}^{\infty}(|k|+1)e^{-\alpha|k|} <n(k)> dk$$

it is necessary to prove that

$$\lim_{\alpha\to 0}\frac{\lambda}{4\pi}\int_{-\infty}^{\infty}|k-k'|[1-e^{-\alpha(|k|+|k'|)}]<n(k)n(k')>\cdot dkdk'=0 \quad (23)$$

and

$$\lim_{\alpha\to 0}\frac{\nu}{2\pi}\int_{-\infty}^{\infty}(|k|+1)(1-e^{-\alpha|k|})<n(k)>\cdot dk=0 \quad (24)$$

The integral

$$\frac{\lambda}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|k-k'|[1-e^{-\alpha(|k|+|k'|)}]<n(k)n(k')>dkdk'$$

converges because of the convergence of

$$\frac{\lambda}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|k-k'|<n(k)n(k')>dkdk'$$

Thus

$$\exists R>0$$
$$\frac{\lambda}{4\pi}\int\int_{|k|,|k'|\geq R}|k-k'|[1-e^{-\alpha(|k|+|k'|)}]<n(k)n(k')>dk\,dk'<\frac{\varepsilon}{2} \quad (25)$$

The rest part of the integral is

$$\begin{aligned}&\frac{\lambda}{4\pi}\int_{-R}^{R}\int_{-R}^{R}|k-k'|[1-e^{-\alpha(|k|+|k'|)}]<n(k)n(k')>\cdot dkdk'\\&\leq\frac{\lambda}{4\pi}[1-e^{-2\alpha R}]\cdot\int_{-R}^{R}\int_{-R}^{R}|k-k'|<n(k)n(k')>\cdot dkdk'\\&\leq[1-e^{-2\alpha R}]\cdot Q_1\end{aligned}$$

where

$$\frac{\lambda}{4\pi}\cdot\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|k-k'|<n(k)n(k')>\cdot dkdk'\leq Q_1=\text{Cte}$$

We must have

$$[1-e^{-2\alpha R}]\cdot Q_1<\frac{\varepsilon}{2} \quad (26)$$

It is true for arbitrary $\alpha > 0$ if $\varepsilon \geq 2Q_1$ and for $1 - \varepsilon/(2Q_1) < e^{-2\alpha R}$ or $\alpha \leq -1/(2R) \cdot \ln[1 - \varepsilon/(2Q_1)]$ if $0 < \varepsilon < 2Q_1$. Joining (25) and (26) we have

$$\forall\varepsilon > 0 \quad \exists R > 0 \quad \forall\alpha \leq -\frac{1}{2R} \cdot \ln[1 - \frac{\varepsilon_1}{2Q_1}] \text{ where } \varepsilon_1 = \begin{cases} \varepsilon & \text{if} < 2Q_1 \\ Q_1 & \text{if } \varepsilon \geq 2Q_1 \end{cases}$$

$$\frac{\lambda}{4\pi} \cdot \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |k - k'| < n(k)n(k') > \cdot dkdk' \leq Q_1 = \text{Cte}$$
$$\frac{\lambda}{4\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |k - k'|[1 - e^{-\alpha(|k|+|k'|)}] < n(k)n(k') > \cdot dkdk' < \varepsilon$$

By analogy $\forall\nu\nu_0 \geq \nu \geq 0$

$$\frac{\nu}{2\pi} \int_{-\infty}^{\infty} (|k| + 1)[1 - e^{-\alpha|k|}] < n(k) > \cdot dk < \varepsilon$$

Going back to (22) and realizing the limit for $L \to \infty, L/N = \text{Cte}$ we have

$$\begin{aligned} 2\varepsilon > &\frac{\lambda}{4\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |k - k'|[1 - e^{-\alpha(|k|+|k'|)}] < n(k)n(k') > \cdot dkdk' \\ &+ \frac{\nu}{2\pi} \int_{-\infty}^{\infty} (|k| + 1)[1 - e^{-\alpha|k|}] < n(k) > \cdot dk \\ &\geq f_\infty[\Gamma^\alpha] - f_\infty[\Gamma] \geq 0 \end{aligned}$$

Theorem (3) is proved. And now we can prove the main theorem.

Theorem 4.

If (σ^α) is the "point" of absolute minimum of the free energy function $f[\Gamma_0^\alpha(C)] = -\frac{\theta}{L} \cdot \ln \operatorname{Sp} e^{-\beta\Gamma_0^\alpha(C)}$ then the hamiltonians

$$\Gamma = \tau - \lambda\frac{\pi}{L} \cdot \sum_{k,k'} |k - k'| \cdot n_k n_{k'} - \nu \sum_k (|k| + 1)n_k \tag{27}$$

and

$$\begin{aligned} \Gamma_0^\alpha(C) = \tau - \lambda\frac{\pi}{L} \cdot \sum_{k,k'} |k - k'| e^{-\alpha(|k|+|k'|)} \cdot (n_k \cdot \sigma_{k'}^\alpha + n_{k'} \cdot \sigma_k^\alpha) \\ + \lambda\frac{\pi}{L} \cdot \sum_{k,k'} |k - k'| e^{-\alpha(|k|+|k'|)} \sigma_k^\alpha \cdot \sigma_{k'}^\alpha - \nu \sum_k (|k| + 1)e^{-\alpha|k|} n_k \end{aligned} \tag{28}$$

are thermodynamicaly equivalent at the limit $L \to \infty, L/N = \text{Cte}$ while the reduce parameter tends to zero, i.e.

$$\lim_{\alpha\to 0}\lim_{L\to\infty}[f(\Gamma_0^\alpha(\sigma^\alpha)) - f(\Gamma)] = 0 \quad , \quad f(\Gamma) = -\frac{\theta}{L}\cdot \ln \mathrm{Sp} e^{-\beta\Gamma}$$

Proof:

$$\lim_{L\to\infty} f[\Gamma] = \lim_{L\to\infty}(f[\Gamma] - f[\Gamma_0^\alpha(\sigma^\alpha)] + f[\Gamma_0^\alpha(\sigma^\alpha)] - f[\Gamma^\alpha] + f[\Gamma])$$
$$= \lim_{L\to\infty}(f[\Gamma^\alpha] - f[\Gamma_0^\alpha(\sigma^\alpha)]) + \lim_{L\to\infty}(f[\Gamma] - f[\Gamma^\alpha]) + \lim_{L\to\infty} f[\Gamma_0^\alpha(\sigma^\alpha)] \tag{29}$$

Since the left side of [29] does not depend on α and all the limits exist we have

$$\lim_{L\to\infty} f[\Gamma] = \lim_{\alpha\to 0}\lim_{L\to\infty}(f[\Gamma^\alpha] - f[\Gamma_0^\alpha(\sigma^\alpha)])$$
$$+ \lim_{\alpha\to 0}\lim_{L\to\infty}(f[\Gamma] - f[\Gamma^\alpha]) + \lim_{\alpha\to 0}\lim_{L\to\infty} f[\Gamma_0^\alpha(\sigma^\alpha)]$$

According to Theorem 2 and Theorem 3

$$\lim_{\alpha\to 0}\lim_{L\to\infty}(f[\Gamma] - f[\Gamma^\alpha]) = 0$$
$$\lim_{\alpha\to 0}\lim_{L\to\infty}(f[\Gamma^\alpha] - f[\Gamma\alpha_0(\sigma^\alpha)]) = 0$$

hence $\lim_{L\to\infty} f[\Gamma] = \lim_{\alpha\to 0}\lim_{L\to\infty} f[\Gamma_0^\alpha(\sigma^\alpha)]$ and the Theorem 4 is proved.

Theorem 4 substantiates that we can determine the approximating hamiltonian (22) which is thermodynamically equivalent to the model one (3) at $\nu = 0$. It is necessary to emphasize the importance of the limit transitions order: the first is $L \to \infty, \alpha \to 0$ and the second one is $\nu \to 0$. It is in accordance with the idea of "quasi-averages" that have been suggested in [8].

Since the approximating hamiltonian is the quadratic form of field operators it allows in the thermodynamical limit and if the reduce parameter tend to zero the asymptotically exact evaluation of the free energy function (4) associated with (2) to investigate the system behaviour.

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