Generalized parametric oscillator in phase space

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ABSTRACT. We study the time evolution of the generalized parametric oscillator in phase space. The corresponding time-dependent Wigner equation is solved by an operator ordering method of the Wei-Norman type, based on the $SU(1,1)$ structure of the Wigner operator. The expression of the propagator is explicitly derived.

RÉSUMÉ. Nous étudions, dans l'espace de phase, l'évolution au cours du temps de l'oscillateur paramétrique généralisé. L'équation de Wigner correspondante, dépendant du temps, est résolue à l'aide d'une méthode de type Wei-Norman, en ordonnant les opérateurs, et basée sur la structure $SU(1,1)$ *de l'opérateur de Wigner. On en déduit l'expression explicite du propagateur.*

The generalized, parametric harmonic oscillator is a time-dependent, one-dimensional quantum system driven by the Hamiltonian

$$H(t) = \frac{1}{2}[\frac{Z(t)}{m}p^2 + \gamma Y(t)(pq+qp) + m\omega^2 X(t)q^2] \qquad (1)$$

(where $X(t), Y(t), Z(t)$ are non-singular, real functions of time) that has received in the last years a great deal of attention(1−8). Indeed, the parametric oscillator is a system that exhibits a geometrical Berry phase. Moreover, for a suitable choice of the time-dependent functions, it provides a phenomenological model of open quantum systems.

In particular, in the previous paper [8] we have found the exact solution of the time evolution for Hamiltonian (1), both in the Schrödinger and in the Heisenberg representation, by means of a time-ordering method of the Wei-Norman (WN) type(9), based on the SU(1,1) structure of H (10).

There has been recently renewed interest [11-13] in the Wigner distribution function[14] and the related phase-space formalism, due to their applications to a number of different physical fields[11-13]. In this connection, it seems worth to study Hamiltonian (1) in phase space. As we shall see, the solution of the corresponding Wigner equation can be found by exploiting the Wei-Norman expression of the evolution operator[10](*).

As is well known the Wigner distribution function is defined as

$$f(q,p,t) = \frac{1}{2\pi\hbar}\int dx\psi^*(q-\frac{x}{2},t)\psi(q+\frac{x}{2},t)e^{\frac{i}{\hbar}px}. \qquad (2)$$

The time-dependent Wigner equation reads

$$i\hbar\frac{\partial f}{\partial t} = Wf(q,p,t) \qquad (3)$$

where the Wigner operator W can be written in the form[17-18]

$$W(q,p,\frac{\partial}{\partial q},\frac{\partial}{\partial p}) = H(P,Q) - H(P^*,Q^*) \qquad (4)$$

with the operators P,Q given by

$$P = p - \frac{i\hbar}{2}\frac{\partial}{\partial q} \;\; ; Q = q + i\frac{\hbar}{2}\frac{\partial}{\partial p} \qquad (5)$$

Then, for the parametric oscillator the Wigner operator is

$$W = -i\hbar\frac{Z(t)}{m}p\frac{\partial}{\partial q} + i\hbar\gamma Y(t)(p\frac{\partial}{\partial p} - q\frac{\partial}{\partial q}) + i\hbar m\omega^2 X(t)q\frac{\partial}{\partial p}. \qquad (6)$$

Therefore eq.(3) becomes

$$\frac{\partial f}{\partial t} = [-\frac{Z(t)}{m}p\frac{\partial}{\partial q} + \gamma Y(t)(p\frac{\partial}{\partial p} - q\frac{\partial}{\partial q}) + m\omega^2 X(t)q\frac{\partial}{\partial p}]f(q,p,t). \quad (7)$$

The above equation is exactly the Liouville equation for the classical parametric oscillator, as expected, since - as is well known - the Wigner

* For a discussion, by different approaches, of two special cases of eq.(1) in phase space, i.e. the Caldirola-Kanai Hamiltonian and the damped oscillator, see refs[15,16].

equation for quadratic Hamiltonians coincides with the classical Liouville equation(17).

Eq.(7) can also be written in the form

$$\frac{\partial f}{\partial t} + [\frac{Z(t)}{m}p + \gamma Y(t)q]\frac{\partial f}{\partial q} - [m\omega^2 X(t)q + \gamma Y(t)p]\frac{\partial f}{\partial p} = 0. \quad (8)$$

The characteristic curves of such a partial-differential equation are solutions of the system

$$dt = \frac{dq}{\frac{Z(t)}{m}p + \gamma Y(t)q} = -\frac{dp}{m\omega^2 X(t)q + \gamma Y(t)p} \quad (9)$$

or

$$\frac{dq}{dt} = \frac{Z(t)}{m}p + \gamma Y(t)q; \ \frac{dp}{dt} = -m\omega^2 X(t)q - \gamma Y(t)p \quad (10)$$

which are just the Hamilton equations of motion (or the Heisenberg equations, if q and p are the corresponding operators).

In order to elucidate the group structure of eq.(7), let us consider the operators

$$K_+ = -p\frac{\partial}{\partial q}, \quad K_- = q\frac{\partial}{\partial p}; \quad K_o = \frac{1}{2}(p\frac{\partial}{\partial p} - q\frac{\partial}{\partial q}). \quad (11)$$

It is easy to check that they satisfy the SU(1,1) commutation rules

$$[K_+, K_-] = -2K_o; \quad [K_o, K_\pm] = \pm K_\pm. \quad (12)$$

Therefore the Wigner equation for the parametric oscillator takes the form

$$\frac{\partial f}{\partial t} = [\frac{Z(t)}{m}K_+ + 2\gamma Y(t)K_o + m\omega^2 X(t)K_-]f. \quad (13)$$

We can now apply the WN method and write the solution of eq.(13) in the form

$$\begin{aligned} f(t) &= e^{-A(t)K_+}e^{B(t)K_-}e^{2C(t)K_o}f(0) \\ &= e^{A(t)p\frac{\partial}{\partial q}}e^{B(t)q\frac{\partial}{\partial p}}e^{C(t)(p\frac{\partial}{\partial p} - q\frac{\partial}{\partial q})}f(0). \end{aligned} \quad (14)$$

Replacing (14) in eq.(7), it is easily seen, after some algebra, that the WN characteristic functions $A(t), B(t), C(t)$ satisfy the system of differential equations

$$\begin{cases} \dot{A} - 2\gamma Y A + m\omega^2 X A^2 = -\frac{Z}{m}; \\ \dot{B} + 2B\dot{C} = m\omega^2 X; \\ \dot{C} = \gamma Y - m\omega^2 X A, \end{cases} \tag{15}$$

(where the dot denotes time derivative) with initial conditions $A(0) = B(0) = C(0) = 0$. Putting

$$A = \frac{1}{m\omega^2 X} \frac{\dot{\sigma}}{\sigma} \tag{16}$$

in the first eq.(15) leads to the following second-order differential equation for $\sigma(t)$:

$$\ddot{\sigma} - (\frac{\dot{X}}{X} + 2\gamma Y)\sigma + \omega^2 X Z \sigma = 0 \tag{17}$$

which has two independent solutions $\sigma_1(t)$ and $\sigma_2(t)$. We get therefore for $A(t)$ the solution

$$A(t) = \frac{1}{m\omega^2 X(t)} \frac{\dot{\sigma}_1(t)\dot{\sigma}_2(o) - \dot{\sigma}_1(o)\dot{\sigma}_2(t)}{\sigma_1(t)\dot{\sigma}_2(o) - \dot{\sigma}_1(o)\sigma_2(t)} \tag{18}$$

and similarly

$$C(t) = \gamma \int_o^t Y(t')dt' - ln \frac{\dot{\sigma}_1(t)\dot{\sigma}_2(o) - \dot{\sigma}_1(o)\dot{\sigma}_2(t)}{\sigma_1(o)\dot{\sigma}_2(o) - \dot{\sigma}_1(o)\sigma_2(o)}; \tag{19}$$

$$B(t) = m\omega^2 e^{-2C(t)} \int_o^t e^{2C(t')} X(t')dt'. \tag{20}$$

The above equations fully specify the time evolution of the parametric oscillator in phase space. Clearly, the choice of the initial Wigner function $f(o)$ must be done according to the specific problem considered.

Finally, it is an easy task to derive the expression of the propagator of the parametric oscillator in phase space. Indeed, setting

$$f(o) = \delta(q - q')\delta(p - p') = \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} e^{ik(q-q')} dk \int_{-\infty}^{+\infty} e^{is(p-p')} ds \tag{21}$$

in eq.(14), we get:

$$G(q,p,q',p',t) = \frac{1}{4\pi^2} e^{A(t)p\frac{\partial}{\partial q}} e^{B(t)q\frac{\partial}{\partial p}} \int_{-\infty}^{+\infty} e^{ik(e^{C(t)}q-q')} dk \int_{-\infty}^{+\infty} e^{is(e^{-C(t)}p-p')} ds$$
$$= \frac{1}{4\pi^2} e^{A(t)p\frac{\partial}{\partial q}} \int_{-\infty}^{+\infty} e^{ik(e^{C(t)}q-q')} dk \int_{-\infty}^{+\infty} e^{is\{e^{-C(t)}[p+B(t)q]-q'\}} ds$$
$$= \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} e^{ik\{e^{C(t)}[q+A(t)p]-q'\}} dk \int_{-\infty}^{+\infty} e^{is\{e^{-C(t)}[p+B(t)(q+A(t)p)-q']\}} ds \tag{22}$$

or

$$G(q,p,q',p',t) = \delta\{q' - e^{C(t)}[q + A(t)p]\} \\ \delta\{p' - e^{-C(t)}[p(1 + A(t)B(t) + B(t)q]\}. \tag{23}$$

In conclusion, we have shown that the Wigner equation for the generalized parametric oscillator in phase space does possess an algebraic structure (namely, the Wigner operator is a linear combination of the SU(1,1) generators), and, therefore, its solution can be obtained by expressing the corresponding time-evolution operator in a Wei-Norman form. Physical applications of this formalism will be considered elsewhere.

Note added in proof. After the completion of this work, we became aware of some papers(19), in which a special case of Wigner equation (1.4) (derived in the study of charged beam transport through linear magnetic systems) is solved by the WN method. We are very grateful to G.Dattoli for communicating us the preprints of his papers.

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