On a generalized Kaluza constraint for the dimensional reduction of multidimensional theories with torsion *

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ABSTRACT. We show the possibility of defining in a simple way a generalized Kaluza constraint for $(4+N)$ -dimensional spacetimes with torsion resembling the original Kaluza constraint on the multidimensional metric. The somewhat intriguing conclusion is reached that the application of this generalized constraint leads directly to the dimensional reduction of a first order action in a very general multidimensional model with non-vanishing torsion. Consequences of this result are investigated.

RÉSUMÉ. On montre la possibilité de définir simplement une liaison généralisée de Kaluza pour espace-temps à $4+N$ dimensions avec torsion ressemblant la liaison originelle de Kaluza sur la métrique multidimensionelle. On arrive à la conclusion que l'application de cette liaison généralisée mène directement à la réduction dimensionelle d'une action de premier ordre dans un modèle multidimensionel tres généralisé avec torsion. On évalue les consequences de ce résultat.

1. Introduction.

It is well known that the so-called Kaluza-Klein theory was worked out in the 20s [1,2] in order to unify gravity and electromagnetism in a common 5- dimensional geometrical scheme. Since the first years, however, Klein [2] and de Broglie [3] saw a natural link between the 5-dimensional theory and the quantum behaviour of the microscopic matter as prescribed by the new wave mechanics. Klein's article is historically analysed in an interesting paper by Bergia *et al.* [4]; De Broglie,

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on the other hand, asserted the validity of an equivalence principle in five dimensions, the role of which was recently discussed in refs.[5,6], and studied a 5- dimensional covariant wave equation leading to a 4- dimensional projection in form of a Klein-Gordon equation with (minimal) electromagnetic coupling.

The original Kaluza-Klein theory was further developed by Einstein, Bergmann, Jordan and Thiry [7-9] in a 5-dimensional context. More recently multidimensional extensions were elaborated by "adding" a non-Abelian Lie group to the usual 4-dimensional spacetime (see, for instance, the early proposals in refs.[10-13] and definite mathematical formulations in refs.[14,15]). This approach leads to interesting consequences for the physics of the fundamental interactions. Recently, moreover, following an Einstein- Bergmann approach, Bergia *et al.*[16] discussed the possibility that metrical fluctuations in the fifth dimension can supply a physical subquantum background for quantum mechanics. A modification of the quantum evolution equation determined by multidimensionality has been also envisaged [17]. In particular, these papers bear witness to the relevance of studies about Kaluza- Klein theories even for the foundations of quantum mechanics.

Even if the modern approach to Kaluza-Klein theories is that of the so- called spontaneous compactification [18-20], we want to stress that an euristic "Ansatz" can still turn out to be helpful in a formal preliminar treatment of new extensions.

In the 80s, several theorists started working on multidimensional models of the Kaluza-Klein type in spacetimes with non-vanishing torsion [21-36]. Most of these articles deal with very specific models characterized by precise requirements on the form of torsion.

The only papers dealing with a general framework for multidimensional theories with torsion are those listed in refs.[22,33]. In this note, taking up and somewhat extending their treatment by a general and more explicit formalism, it is our aim to show the possibility of defining in a simple way a generalized Kaluza constraint for extended spacetimes with torsion extending in a very natural way the original Kaluza constraint on the $(4 + N)$-dimensional metric γ. The somewhat intriguing conclusion is reached that the simple application of this generalized constraint leads to the dimensional reduction of a first order action in a very general multidimensional model with non-vanishing torsion. At this stage, the role of the generalized Kaluza constraint is especially formal and euristic. Nevertheless it enables us to obtain a general metrical model even with non-vanishing torsion in the usual four dimensions and it provides a general background in which new compactification mechanisms could be analysed.

This paper is structured as follows: in section 2 we introduce some geometrical notations on a principal fibre bundle representing our extended spacetime. In section 3 we "translate" the original Kaluza constraint on a 5-dimensional metric in our general formalism for a $(4+N)$-dimensional extended spacetime and perform the usual dimensional reduction of standard multidimensional unified theories. In section 4 the generalized Kaluza constraint is defined and its application to the dimensional reduction of a first order action with torsion is accurately described. Finally, in the last section, several comments are made on the consequences of the result previously presented.

2. Some geometrical notations.

In this section we introduce some elements of the general formalism that we adopt in order to deal with multidimensional theories and theories with torsion. For a complete treatment of this geometrical subject one can see refs.[37,15,5,38].

The two fundamental elements of our geometrical treatment are a principal fibre bundle with a rule of horizontality (bundle connection), representing our extended spacetime, and a non-vanishing torsion on the usual spacetime as well as on the internal dimensions. In our notations about indices of tensorial quantities defined on a D-dimensional extended spacetime, capital italic indices refer to the whole manifold (that is to say $M, N, \ldots = 1, \ldots, D$), Greek ones refer, as usual, to the four dimensions of ordinary spacetime ($\mu, \nu, \ldots = 1, \ldots, 4$) and small italic ones refer to the remaining $N = D - 4$ dimensions ($i, j, ... = 5, ..., D$).

On a generical D-dimensional Riemann-Cartan manifold U_D, in a basis $\{\mathbf{e_M}\}_{\mathbf{M=1^D}}$, we can define torsion as

$$\begin{aligned}\mathbf{S}(\mathbf{e_M}, \mathbf{e_N}) &= \frac{1}{2}\left(\bigtriangledown_{\mathbf{e_M}}\mathbf{e_N} - \bigtriangledown_{\mathbf{e_N}}\mathbf{e_M} + [\mathbf{e_M}, \mathbf{e_N}]\right) \\ &= \frac{1}{2}\left(\Gamma_{MN}{}^{P} - \Gamma_{NM}{}^{P} + c_{MN}{}^{P}\right)\mathbf{e_P} = \mathbf{S_{MN}{}^{P}e_P}.\end{aligned} \tag{1}$$

As one can see by eq.(1), in a more usual holonomic basis torsion is simply the antisymmetric part of connection. It is not difficult to prove that the most general form of the connection coefficients in a Riemann-Cartan manifold is

$$\Gamma_{MN}{}^{P} = \overset{\circ}{\Gamma}{}_{MN}{}^{P} - K_{MN}{}^{P} \tag{2}$$

where $\overset{\circ}{\Gamma}$ is the well-known Levi-Civita connection and

$$K_{MN}{}^{P} = -S_{MN}{}^{P} - S^{P}{}_{MN} + S_{N}{}^{P}{}_{M} \quad (= -K_{M}{}^{P}{}_{N}) \tag{3}$$

is the so-called contortion tensor.

Our extended spacetime is a principal fibre bundle (M_D, π, M_4, G), where M_D and M_4 are two differentiable manifolds,π is a map (the so-called "projection")

$$\pi : M_D \longrightarrow M_4 \tag{4}$$

with rank 4 at each point, and G is a compact and semi- simple Lie group.

We can define a bundle connection such that, $\forall p \in M_D$, the tangent space in p is decomposable in two subspaces, a horizontal subspace H_p and a vertical subspace V_p:

$$T_p(M_D) = H_p \oplus V_p \tag{5}$$

Through this property we are able to decompose every vector field in its horizontal and vertical parts; in short, the bundle connection supplies a rule of horizontality. In particular, if we choose 4 fields $\mathbf{V}_\mu$:

$$\mathbf{V}_\mu = \partial_\mu \text{locally} \tag{6}$$

as a basis of M_4, and N left-invariant fields $\mathbf{V}_i$ as a basis of the Lie algebra $\mathcal{L}(G)$ of the non-Abelian Lie group G, then a basis of vector fields on M_D is

$$\left\{ \widehat{\mathbf{V}}_\mu, \, \mathbf{V}_i^* \right\} \, , \mu = 1, \ldots 4, \quad i = 5, \ldots D, \tag{7}$$

where

$$\widehat{\mathbf{V}}_\mu = (V_\mu^1,\ V_\mu^2,\ V_\mu^3,\ V_\mu^4,\ \underbrace{0,\ldots,0}_{N}\ ) \tag{8}$$

are the horizontal liftings of the fields $\mathbf{V}_\mu$ and $\mathbf{V}_i^*$ are the N fundamental fields of the Lie algebra $\chi(M_D)$, induced by the $\mathbf{V}_i$ of $\mathcal{L}\,(G)$. (We remark that $\forall p \in M_D$ the fields $\mathbf{V}_i^*$ are tangent to the fibre of $x = \pi(p)$ and therefore they are vertical). The commutation rules relating to the basis (7) are

$$\left[\widehat{\mathbf{V}}_\mu, \widehat{\mathbf{V}}_\nu\right] = c_{\mu\nu}{}^k\, \mathbf{V}_k^* = -F_{\mu\nu}{}^k \mathbf{V}_k^*, \tag{9}$$

$$\left[\widehat{\mathbf{V}}_\mu, \mathbf{V}_i^*\right] = 0, \tag{10}$$

$$\left[\mathbf{V}_i^*, \mathbf{V}_j^*\right] = c_{ij}{}^k\, \mathbf{V}_k^* = -f_{ij}{}^k\, \mathbf{V}_k^*, \tag{11}$$

As far as the metric on our D-dimensional principal fibre bundle is concerned, we choose a non- degenerate metric γ with signature $D-2$ (that is the further N dimensions are space-like). The application of the Kaluza constraint to γ determines its final form.

3. The dimensional reduction of a generalized Riemannian gravitational action.

In order to implement physics on a geometrical background like that just outlined, in a theory without matter fields it is sufficient to write an action for the multidimensional spacetime continuum. A usual choice is represented by ($z \in M_D$)

$$\overset{\circ}{A}_g = const. \int \overset{\circ}{R}\,(\det \gamma_{MN})^{\frac{1}{2}}\, d^D z \tag{12}$$

(the Lagrangian is that of general relativity in $4+N$ dimensions). It is interesting to see how constraints on the geometry allow the dimensional reduction of this action.

In the original 5-dimensional Kaluza theory, he assumed that one can find local coordinates on the Riemannian manifold V_5 such that

$$\partial_5 \gamma_{MN} = 0. \tag{13}$$

This constraint (the so-called Kaluza constraint) leads to restrict the metric γ to the form (x belongs to the usual Riemannian spacetime V_4):

$$\gamma_{MN} = \left(\begin{array}{c|c} g_{\mu\nu} + a_\mu a_\nu & a_\mu \\ \hline a_\nu & \phi(x) \end{array}\right) \tag{14}$$

The further application of the Klein constraint

$$\partial_M \gamma_{55} = 0 \tag{15}$$

leads to the well-known Kaluza-Klein metric, with a constant γ_{55} component. Here we apply the Kaluza constraint only, so dealing with metrics of the Jordan- Thiry type such as (14). In the lifted system $\widehat{\partial}_M = (\widehat{\mathbf{V}}_\mu, \widehat{\partial}_5) = (\partial_\mu - a_\mu \partial_5, \partial_5)$, γ reads as

$$\widehat{\gamma}_{MN} = \left(\begin{array}{c|c} g_{\mu\nu}(x) & 0 \\ \hline 0 & \phi(x) \end{array}\right) \tag{16}$$

Here and afterwards the symbol $\widehat{}$ denotes quantities referring to lifted systems. We stress that the Kaluza constraint on the multidimensional metric γ uniquely determines the 4-dimensional metric: $g_{\mu\nu} = g_{\mu\nu}(x)$.

In the more intrinsic language of the general formalism outlined in the previous section, the Kaluza constraint on a $(4+N)$-dimensional metric γ reads as

$$\mathcal{L}_{V_i^*}\gamma = 0 \qquad (i = 5, \ldots, D = 4 + N) \tag{17}$$

where $\mathcal{L}$ denotes the Lie derivative. The further constraint of the γ -orthogonality condition

$$\widehat{\gamma}_{\mu i} = \gamma(\widehat{\mathbf{V}}_\mu, \mathbf{V}_i^*) = 0 \tag{18}$$

and the restriction to a Killing-Cartan bi-invariant vertical metric (for which f_{ijk} is totally antisymmetric), lead to work with

$$\widehat{\gamma}_{MN} = \left(\begin{array}{c|c} g_{\mu\nu}(x) & 0 \\ \hline 0 & \phi(x)\, \delta_{ij} \end{array}\right) \tag{19}$$

in the lifted system $(\widehat{\mathbf{V}}_\mu, \mathbf{V}^*_i)$. At this point one can make the geometry of the Riemannian principal fibre bundle (V_D, π, V_4, G) completely explicit, deriving the connection coefficients $\widehat{\overset{\circ}{\Gamma}}{}^P{}_{MN}$, the curvature tensor $\widehat{\overset{\circ}{R}}{}^Q{}_{MNP}$, the Ricci tensor $\widehat{\overset{\circ}{R}}{}_{MN}$ and the curvature scalar $\widehat{\overset{\circ}{R}}$ in the lifted system (we use the symbol $\circ$ over quantities built in terms of the Levi-Civita connection and with vanishing torsion). We are interested in the form of $\widehat{\overset{\circ}{R}}$ $(= \overset{\circ}{R}$, as it is scalar); it reads

$$\begin{aligned}\widehat{\overset{\circ}{R}} =& \overset{\circ}{R}{}^{(4)} - \frac{1}{4}\phi F_{\rho\mu}{}^i\, F^{\rho\mu}{}_i + \frac{N}{4}\phi^{-1} \\ & - N\widehat{\overset{\circ}{\nabla}}{}^\mu \widehat{\overset{\circ}{\nabla}}_\mu \ln\phi + \frac{N(1-N)}{4}\left(\widehat{\overset{\circ}{\nabla}}{}^\mu \ln\phi\right)\left(\widehat{\overset{\circ}{\nabla}}_\mu \ln\phi\right)\end{aligned} \tag{20}$$

where $\overset{\circ}{R}{}^{(4)}$ is the curvature scalar in the usual 4- dimensional Riemannian spacetime and $\widehat{\overset{\circ}{\nabla}}$ is the Riemannian covariant derivative in the lifted system.

Since $\phi = \phi(x)$, in the right-hand side of eq.(20) there is only one term showing a dependence on the internal dimensions: it is the quadratic term in F . Actually, however, one can show [15] that F is decomposable in terms of a "Yang-Mills field strength" $\mathcal{F}$ and the elements of the adjoint representation matrix of the group G, as follows $(x \in V_4,\, y \in G)$:

$$F_{\mu\nu}{}^k(x,y) = \mathcal{F}_{\mu\nu}{}^n(x)\, D^{-1}{}_n{}^k(y) \tag{21}$$

If one also consider that

$$\delta_{kn} = D^i_k\, D^j_n\, \delta_{ij} \tag{22}$$

then the quadratic term in F can be replaced by

$$-\frac{1}{4}\phi\, \mathcal{F}_{\mu\nu}{}^k\, \mathcal{F}^{\mu\nu m}\, \delta_{km} \tag{23}$$

As a consequence of the application of the Kaluza constraint, the curvature scalar $\overset{\circ}{R}$ depends solely on $x \in V_4$ and the action (12) results dimensionally reduced. In fact

$$\det \widehat{\gamma}_{MN} = \phi^N \det g_{\mu\nu} \tag{24}$$

and, if the ipervolume of the internal space G is finite, $\overset{\circ}{A}_g$ reads as

$$\overset{\circ}{A}_g = const' \int \Big[\overset{\circ}{R}{}^{(4)} - \frac{1}{4}\phi \mathcal{F}_{\mu\nu}{}^k \mathcal{F}^{\mu\nu n}\delta_{kn} + \frac{N}{4}\phi^{-1} - N \widehat{\overset{\circ}{\nabla}}{}^\mu \widehat{\overset{\circ}{\nabla}}_\mu \ln\phi$$
$$+ \frac{N(1-N)}{4}\left(\widehat{\overset{\circ}{\nabla}}{}^\mu \ln\phi\right)\left(\widehat{\overset{\circ}{\nabla}}_\mu \ln\phi\right)\Big] \phi^{\frac{N}{2}} (\det g_{\mu\nu})^{\frac{1}{2}} \, d^4x \tag{25}$$

4. The generalized Kaluza constraint and the dimensional reduction of a theory with non-vanishing torsion.

In this section we consider a Riemann-Cartan multidimensional manifold endowed with the structures of principle fibre bundle and connection. So we deal with (U_D, π, U_4, G) and a multidimensional metric of a Jordan- Thiry type, like (19), that is we impose the Kaluza constraint on the metric. Moreover, we choose a totally antisymmetric torsion on the manifold U_D; in such a way

$$K_{MN}{}^P = -S_{MN}{}^P \tag{26}$$

The geometry of our manifold is completely determined by these choices; in particular the relation (2) holds and the curvature scalar $\widehat{R}(=R)$ reads as

$$\widehat{R} = \overset{\circ}{R} - \widehat{K}_P{}^{MN} \widehat{K}_{MN}{}^P \tag{27}$$

Let us go from geometry to physics! It is natural to maintain a first order action principle in our Riemann-Cartan framework, through the choice of

$$A_g = const. \int R \, (\det \gamma_{MN})^{\frac{1}{2}} d^D z \tag{28}$$

for the extended spacetime continuum and

$$A_m = const. \int \mathcal{L}_m(\psi, \partial\psi, \gamma, \partial\gamma, S) d^D z \tag{29}$$

for a matter field ψ interacting with the Riemann- Cartan spacetime background. In a standard way, by varying $A_g + A_m$ with respect to metric and torsion one can derive the field equations of the theory outlined. In the following we do not consider this problem, nor particular matter fields which can make the dimensional reduction of A_m straightforward. On the contrary, we dwell upon the research of a general condition for the dimensional reduction of A_g.

Let us now come to the essential point. In the previous section we have seen that the application of the Kaluza constraint on the multidimensional metric leads to its independence from the internal coordinates and to the dimensional reduction of A_g. Now, as a formal analogy, we consider a "Kaluza" constraint on the D- dimensional contortion, requiring

$$\mathcal{L}_{V_i^*}\mathbf{K} = 0 \qquad (i = 5, \ldots, D = 4 + N), \tag{30}$$

and testing the physical benefits of its application in our framework.

If $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$ are vector fields defined on U_D, from the Leibniz rule in the definition of Lie derivative for fields [5] the following decomposition holds:

$$\begin{aligned}\mathcal{L}_{V_i^*}\mathbf{K} =& \mathbf{V}_i^*(\mathbf{K}(\mathbf{X},\,\mathbf{Y},\,\mathbf{Z})) - \mathbf{K}\left([\mathbf{V}_i^*,\,\mathbf{X}]\,,\,\mathbf{Y},\,\mathbf{Z}\right)\\ & -\mathbf{K}\left(\mathbf{X},\,[\mathbf{V}_i^*,\,\mathbf{Y}]\,,\,\mathbf{Z}\right) - \mathbf{K}\left(\mathbf{X},\,\mathbf{Y},\,[\mathbf{V}_i^*,\,\mathbf{Z}]\right) = 0\\ & (i = 5, \ldots, D)\end{aligned} \tag{31}$$

Substituting the horizontal liftings $\widehat{\mathbf{V}}_\mu$ and/or the fundamental fields $\mathbf{V}_i^*$ to $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ in (31), and applying the commutation rules (10-11), one obtains:

$$\mathbf{V}_i^*\,\widehat{K}_{\mu\nu\rho} = 0, \tag{32}$$

$$\mathbf{V}_i^*\,\widehat{K}_{\mu\nu j} - f_{ij}{}^k\,\widehat{K}_{\mu\nu k} = 0, \tag{33}$$

$$\mathbf{V}_i^*\,\widehat{K}_{\mu jk} - f_{ij}{}^n\,\widehat{K}_{\mu nk} - f_{ik}{}^n\,\widehat{K}_{\mu jn} = 0, \tag{34}$$

$$\mathbf{V}_i^*\,\widehat{K}_{jkl} - f_{ij}{}^n\,\widehat{K}_{nkl} - f_{ik}{}^n\,\widehat{K}_{jnl} - f_{il}{}^n\,\widehat{K}_{jkn} = 0, \tag{35}$$

If one keeps in mind the fundamental relation [15,5]

$$\mathbf{V}_i^* \, D_j^n = f_{ij}{}^k \, D_k^n, \tag{36}$$

it is not difficult to verify that general solutions of eqs.(32-35) are respectively ($x \in U_4$, $y \in G$):

$$\widehat{K}_{\mu\nu\rho} = \widehat{K}_{\mu\nu\rho}(x), \tag{37}$$
$$\widehat{K}_{\mu\nu j} = h_{\mu\nu p}(x) \, D_j^p(y), \tag{38}$$
$$\widehat{K}_{\mu jk} = h'_{\mu pq}(x) \, D_j^p(y) \, D_k^q(y), \tag{39}$$
$$\widehat{K}_{jkl} = h"_{pqr}(x) \, D_j^p(y) \, D_k^q(y) \, D_l^r(y). \tag{40}$$

Here D_i^j are the usual elements of the adjoint representation matrix.

The first remark on eqs.(37-40) is that, as the Kaluza constraint on γ uniquely determines the 4-dimensional metric $\mathbf{g}$, analogously here the Kaluza constraint on K_{MNP} uniquely determines the contortion $K_{\mu\nu\rho} = K_{\mu\nu\rho}(x)$ in the ordinary spacetime.

In general, however, the other components of contortion depend on $y \in G$ too. Only if G is Abelian (so that its adjoint representation is the identity), then $\widehat{K}_{MNP} = \widehat{K}_{MNP}(x) \quad \forall M, N, P = 1, \ldots, D$.

Therefore, in the case of real physical relevance (non-Abelian G) the Kaluza constraint on contortion does not lead to its total independence from internal coordinates.

Nevertheless, we can obtain an important and non- trivial result as a consequence of the application of the constraint (30). In fact, keeping in mind the fundamental formula (36) and the property of bi- invariance of the vertical metric, it is not difficult to prove that

$$\mathbf{V}_i^* (\widehat{K}_{MNP} \widehat{K}^{MNP}) = 0 \qquad \begin{array}{l} \forall M, N, P = 1, \ldots, D \\ \forall i = 5, \ldots, D. \end{array} \tag{41}$$

Eq.(41) shows that terms square in contortion depend only on $x \in U_4$, being constant on the fibre of $x \quad \forall p \in \pi^{-1}(x) \subset U_D$.

In such a way the curvature scalar in eq.(27) depends only on $x \in U_4$, thanks to the properties of $\overset{\circ}{R}$ and to the further constraint (30). As a direct consequence, the action (28) is automatically dimensionally reduced. Actually, the set of the Kaluza constraints (17) and (30), added to the condition (18) and to the bi-invariance of γ_{ij}, forms a "generalized Kaluza constraint", representing a sufficient condition for the dimensional reduction of first order multidimensional actions with torsion for the extended spacetime continuum.

5. Comments.

In this paper we have adopted a very general and explicit formalism. In this framework, from a pure formal analogy with the classical Kaluza constraint on the metric, the dimensional reduction of A_g is directly obtained. At this stage, the role of the generalized Kaluza constraint is especially formal and euristic: for example, it must be tested in cases of matter fields of physical relevance (the general case of field equations derived varying first order A_g with torsion, independent of the internal dimensions, has already been treated elsewhere [33]). Moreover, also the possibility that our geometrical model provides the background for new compactification mechanisms is under investigation.

It is interesting to point out two other features of the geometrical scheme previously outlined that make it well founded. First of all, relevant phisics can be implemented in our scheme, because it reveals a basic property of a physical theory, namely the metricity condition is valid; in fact

$$\begin{aligned} \nabla_P \gamma_{MN} &= \overset{\circ}{\nabla}_P \gamma_{MN} + K_{PM}{}^R \gamma_{RN} + K_{PN}{}^R \gamma_{MR} \\ &= K_{PMN} + K_{PNM} = 0. \end{aligned} \tag{42}$$

Notice that the result (42) is critically derived from the total antisymmetry of the contortion (torsion) tensor.

Moreover, even if one finds in literature assertions about the incompatibility between the presence of torsion in the 4-dimensional spacetime and the validity of the equivalence principle (see ref.[39] for a recent example), one can show that in general this statement is not true and that in particular our geometrical scheme allows the validity of the equivalence principle in the ordinary spacetime. The general problem is faced by von der Heyde [40] and Gogala [38]: their treatment shows that the incompatibility exists if one considers a coordinate basis, and not a more general anholonomic basis, on the spacetime. Actually, the restriction of the equivalence principle to holonomic bases takes a "non locality" in the geometrical scheme, which is contrary to the local character of the principle [40].

Obviously, we can consider an anholonomic basis on U_4 in our scheme, simply through a change in the commutation rule (9), which becomes

$$\left[\widehat{\mathbf{V}}_\mu, \widehat{\mathbf{V}}_\nu\right] = c_{\mu\nu}{}^{\rho}\widehat{\mathbf{V}}_\rho - F_{\mu\nu}{}^{k}\,\mathbf{V}^*_k. \tag{43}$$

As a consequence, $\overset{\circ}{\Gamma}{}^{\rho(4)}_{\mu\nu}$ looses its symmetry in the lower indices, but the decomposition formulae of the geometrical quantities characterizing our extended spacetime do not change. In particular eqs.(20) and (27) hold. Moreover, in a coordinate basis, the totally antisymmetric torsion introduced in the geometrical scheme previously described does not modify the geodesic equation.

In conclusion, the general formalism outlined in sections $2-3$ has induced to introduce a Kaluza constraint on the contortion in a natural way. We have shown the direct geometrical and physical consequences and the potentialities of this constraint and, in general, of this geometrical scheme.

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