A trajectory interpretation of tunneling

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ABSTRACT.

A trajectory description of tunneling through a rectangular barrier is examined. A deterministic trajectory that tunnels through the barrier is developed in closed form. Its equations of motion are generated from Hamilton's characteristic function for quantum motion. We develop from the trajectory description an associated wave function that has in the incident domain a compound modulation in both amplitude and conjugate momentum. This wave function with compound modulation is an eigenfunction of the Schrödinger representation. Neither assigning a probability amplitude to the wave function by the Copenhagen interpretation, nor Bohm's stochasticy, are needed to describe tunneling.

RÉSUMÉ.

Une description de la trajectoire de traversée d'un tunnel à travers une barrière rectangulaire est examinée. Une trajectoire déterminée à travers la barrière est développée sous forme analytique. L'équation du mouvement est engendrée par la fonction caractéristique de Hamilton du mouvement quantique. A partir de la description de la trajectoire, nous développons une fonction d'onde associée qui a, dans le domaine d'incidence, une modulation composée avec l'amplitude et la quantité conjuguée du mouvement. Cette fonction d'onde de modulation composée est une fonction caractéristique de la représentation de Schrödinger. Ni la probabilité d'amplitude à la fonction d'onde, ni la représentation stochastique de Bohm ne nous sont nécessaires pour décrire la traversée du tunnel.

1. Introduction.

The contemporary Schrödinger representation of quantum mechanics investigates tunneling phenomenon by a particle with sub-barrier energy as an incident wave function that the barrier divides into reflected and transmitted wave functions. The Copenhagen interpretation assigns a probability amplitude to the Schrödinger wave function to predict statistics for an ensemble undergoing an event[1]. For the same particle, classical mechanics predicts only reflection, albeit Cohn and Rabinowitz [1] describe the possibilities of barrier hopping by a classical particle whose length sufficiently exceeds the width of a finite barrier[2]. This phenomenological description of tunneling has been one of the basic triumphs of quantum mechanics. The Copenhagen interpretation deduces the probability for tunneling and reflection of an ensemble of identical wave functions from the probability amplitudes of the transmitted and reflected waves respectively. Herein, we examine barrier tunneling using a deterministic, phenomenological trajectory description couched in a Hamilton-Jacobi representation and show that replicable particles, specifiable a priori, with sub-barrier energy may tunnel with certainty. We consider a particle encountering a finite rectangular barrier with normal incidence. The Hamilton-Jacobi equation for continuous quantum motion is a nonlinear third-order differential equation. The first and second derivatives of Hamilton's characteristic function for quantum continuous motion are continuous across the interfaces of the barrier. On the other hand, Hamilton's characteristic function for classical motion has a discontinuous first derivatives at the barrier. We show that Hamilton's characteristic function, a generator of the motion, for an individual trajectory of a single event contains all the information needed to deduce the corresponding Schrödinger wave function. We determine the Hamilton's

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[1] We distinguish between "Schrödinger representation" and "Copenhagen interpretation" of quantum mechanics. Lest we forget, Schrödinger opposed the Copenhagen interpretation of his wave function. The principal element of the Copenhagen interpretation under examination herein is the assignment of a probability amplitude to the Schrödinger wave function.

[2] While we investigate quantum mechanical tunneling herein, classical tunneling of Cohn and Rabinowitz [1] has bearing. Our barrier has a finite height and a finite thickness. On the incident side of the barrier, our stationary state for the time-independent Schrödinger equation has semi-infinite length. Thus, the classical tunneling criteria of Cohn and Rabinowitz are met. We comment later on differences between this classical tunneling and our trajectory results regarding velocity while tunneling through the barrier.

characteristic function for a particle with sub-barrier energy to tunnel through the barrier with certainty and to generate the corresponding wave function. In the incident domain, the corresponding wave function for tunneling with certainty has compound modulation in both amplitude and conjugate momentum. Since Hamilton's characteristic function and the Schrödinger representation are mutually consistent and since a deterministic trajectory interpretation of a single event suffices to describe the wave function, the assignment of a probability amplitude for an ensemble of events to the Schrödinger wave function is unnecessary.

As a target of opportunity, we also investigate briefly velocity while tunneling. We describe the trajectory inside the barrier for an individual particle with sub-barrier energy.

We have chosen to examine tunneling through a rectangular barrier for conceptual and computational tractability. Analysis by the contemporary Schrödinger representation is familiar. We may apply separation of the variables. Normal incidence reduces the problem to one dimension. The Schrödinger and trajectory representations may be described in closed form. For piecewise continuous potentials, the trajectories are described by linear, trigonometric, inverse trigonometric, and hyperbolic functions while the contemporary Schrödinger representation may be described by exponential functions.

Dewdney and Hiley have investigated scattering of normally incident trajectories by a rectangular potential barrier based upon Bohm's quantum potential [2]. Consequently, Dewdney and Hiley used the same Hamilton-Jacobi equation for quantum continuous motion that we have used herein.

Nevertheless, their analyses differ from ours. Instead of solving the Hamilton-Jacobi equation for the generator of the motion, Dewdney and Hiley assumed that the conjugate momentum would be the mechanical momentum (i.e;, the product of mass and velocity) and that the gradient of Bohm's quantum potential would produce a quantum force that, in turn, would produce a proportional rate of change in mechanical momentum [2].

Bohm's quantum potential is a function of the Schrödinger wave function. Dewdney and Hiley followed Bohm's procedure for introducing stochasticy by assigning separate variables for the argument of the wave function and the position of the particle [3]. Subsequently, Dewdney and Hiley assumed that the equations of motion for the trajectory would be established by integrating the conjugate momentum. Dewdney and Hiley investigated an ensemble of numerically generated trajectories for a wave packet to present graphically densities of the transmitted and reflected trajectories while presenting neither analytic nor numerical coefficients for transmitted or reflected wave functions.

However, Halpern noted that Bohm's quantum potential is dependent upon energy [4]. As such, Bohm's quantum potential is dependent upon path in phase space. Hence, its gradient does not produce a conservative force [5]. Consequently, the conjugate momentum is not the mechanical momentum [6]. Yet, there is a standard recipe for resolving motion. Hamilton's characteristic function is still a generator of the motion. The equations of motion for the particle trajectory are still established by the Hamilton-Jacobi transformation equations for constant coordinates analogous to classical mechanics. Time, t, as a function of particle position is specified by the Hamilton-Jacobi transformation equation for the constant coordinate τ as $t - \tau = \partial W/\partial E$ where W is Hamilton's characteristic function and E is energy.

Herein, we solve the Hamilton-Jacobi equation for quantum continuous motion as a third order differential equation [7] with continuous first and second derivatives and present our results quantitatively in closed form for an individual trajectory of a single event where the conjugate momentum is not the mechanical momentum. The trajectory is deterministic rather than stochastic. Its Hamilton's characteristic function contains all the information necessary to describe scattering analytically consistent with the Schrödinger representation without introducing stochasticy.

In Section 2, we develop a trajectory description of a quantum particle with sub-barrier energy that tunnels through a rectangular barrier with normal incidence. We develop the generator of the motion for this trajectory and subsequently the equations of motion in the classically forbidden region inside the barrier. In Section 3, we develop the corresponding wave function that tunnels with certainty. This wave function has compound modulation in both amplitude and wave number in specified regions. In Section 4, we show that this wave function with compound modulation in specified regions is consistent with the unmodulated incident and reflected wave functions of the contemporary Schrödinger representation. This wave function with compound modulation is shown to be an eigenfunction by the principle of superposition.

In Section 5, we discuss the various interpretations and show that the trajectory interpretation offers a new deterministic resolution to scattering. While the trajectory and Schrödinger representations are mutually consistent, the assignment of a probability amplitude to the Schrödinger wave function is unnecessary. In the Appendix, the unmodulated plane waves of the contemporary Schrödinger representation are synthesized from a set of compoundly modulated waves by the superposition principle.

While we develop a trajectory presentation herein, the reader without any familiarity with the trajectory representation but familiar with the Schrödinger representation may choose to read Sections 4, 3 and 2 in that order.

2. Trajectory description.

Consider a plane wave progressing in the positive x-direction with sub-barrier energy, E, that tunnels, with normal incidence, through a barrier given by the potential

$$V = \begin{cases} 0, & x < -q \\ U > E, & -q \le x \le q \\ 0, & x > q \; . \end{cases} \tag{1}$$

The Hamilton-Jacobi equation for quantum continuous motion is given by [7]

$$\frac{(W')^2}{2m} + V - E = -\frac{\hbar^2}{4m}\Big(\frac{W'''}{W'} - \frac{3}{2}\big(\frac{W''}{W'}\big)^2\Big) \tag{2}$$

where W is Hamilton's characteristic function and the generator of the motion, m is the particle mass, and $\hbar$ is Planck's constant. The energy may be expressed as $E = (\hbar k)^2/(2m)$ where k is the wave number. The left side of Eq.(2) represents the classical Hamilton-Jacobi equation while the right side of Eq.(2) contains the quantum terms that raise the Hamilton-Jacobi equation from a first-order differential equation for classical motion to a third-order differential equation for quantum continuous motion.

W' is the conjugate momentum (not the mechanical momentum in contrast to Dewdney and Hiley [2]) given by [7]

$$W' = (2m)^{1/2}(a\phi^2 + b\theta^2 + c\phi\theta)^{-1} \tag{3}$$

where (ϕ, θ) is a set of scaled independent solutions of the associated time-independent Schrödinger equation, $-\hbar^2\psi''/(2m) + (V-E)\psi = 0$ with a Wronskian, W, normalized by $W^2(\phi,\theta) = 2m/\big(\hbar^2(ab-c^2/4)\big)$. Substituting Eq.(3) for W' into Eq.(2) renders the Schrödinger equation [7]. The subsequent general solution for W is given by [6]

$$W = \hbar \arctan\Big(\frac{b(\theta/\phi)+c/2}{(ab-c^2/4)^{1/2}}\Big) + K \tag{4}$$

where K is the additive constant of integration.

As Eq.(2) is third order and does not explicitly contain W, specifying W' and W'' at some initial (or terminal) point determines a unique solution for W within the arbitrary additive constant, K. Including W in the initial conditions specifies K. The terminal conditions that represent a trajectory of a running wave that has been transmitted through the barrier are given by Huygen's principle [8] at $x=\infty$ where $W'(x=\infty)=k$ and $W''(x=\infty)=0$. In addition, W and its first two derivatives with respect to x must be continuous across the barrier interfaces at $x=-q,q$.

The terminal conditions and the continuity conditions along with the specified normalization of the Wronskian are sufficient to determine the coefficients $\big(a,b,c\big)$ and a continuous (to degree two) Hamilton's characteristic function (a generator of the motion) in units of Planck's constant for all x as

$$W = \begin{cases} \hbar k(x-q), & x>q \\ \hbar \arctan\big(\frac{k}{\kappa}\tanh\big(\kappa(x-q)\big)\big), & -q \le x \le q \\ \hbar \arctan\big(\dfrac{\mathcal{N}}{\mathcal{D}}\big) & x<-q \end{cases} \tag{5}$$

where

$$\mathcal{N} = \frac{k}{\kappa}\sinh(-2\kappa q)\cos\big(k(x+q)\big) + \cosh(-2\kappa q)\sin\big(k(x+q)\big)$$

$$\mathcal{D} = \cosh(-2\kappa q)\cos\big(k(x+q)\big) + \frac{\kappa}{k}\sinh(-2\kappa q)\sin\big(k(x+q)\big)$$

$$\kappa = \frac{\big(2m(U-E)\big)^{1/2}}{\hbar}$$

and where we have picked the integration constant K so that

$$W(x=q) = 0\ .$$

We apply the trajectory interpretation to examine the particle velocity while tunneling through the barrier. Different models including Wigner trajectories and hydrodynamic models have been used to determine or, from dwell times, infer velocities for wave packets in the classically forbidden regions and have been discussed in literature [9, 10]. We now have a generator of the motion, Eq.(5), that is valid for an individual monochromatic particle in both the classically forbidden and the classically allowed regions. The equation of motion for the trajectory, which differs from that of Dewdney and Hiley [2], is the Hamilton-Jacobi transformation equation $t-\tau = \partial W/\partial E$. Thus, the temporal behavior of the trajectory for $x > -q$ (the temporal behavior before tunneling, $x < -q$, is represented regrettably by a very long expression, albeit a closed-form expression, whose significance will soon be reported separately) is given by differentiating Eq.(5) with respect to E or

$$t-\tau = \begin{cases} \dfrac{m}{\hbar k}(x-q), & \text{for } x > q \\ \dfrac{m/\hbar}{1+(k/\kappa)^2\tanh^2\big(\kappa(x-q)\big)}\Big(\big(\dfrac{1}{\kappa k}+\dfrac{k}{\kappa^3}\big)\tanh\big(\kappa(x-q)\big) & \\ \quad -\dfrac{k(x-q)}{\kappa^2}\operatorname{sech}^2\big(\kappa(x-q)\big)\Big), & \text{for } -q \le x \le q\ . \end{cases} \tag{6}$$

Note that Eq.(6) describes the trajectory in the classically forbidden region, $-q \le x \le q$. The particle velocity, x, is given by differentiating Eq.(6). We report that only for $x > q$ does $\dot{x} = W'/m$. This is a manifestation that W' is generally not the mechanical momentum [11] (i.e., $W' \neq m\dot{x}$), but rather it is related to the phase velocity [6] that is given by Park [12] for matter as $W'/(2m)$. Meanwhile, particle velocity has been shown [6] to be related to group velocity for Eq.(6) is a Hamilton-Jacobi transformation equation. We have from Eq.(6) that in the classically forbidden region, $\dot{x} > \hbar k/m$ consistent with Olkhovsky and Recami [13] and the previous findings of Floyd [11] that showed the velocity increased in the region beyond the WKB turning point for bound states but different from the findings of Hartmann [14] and Fletcher [15] who found that $\dot{x} < \hbar k/m$ for thin barriers. Nevertheless, the velocity increases with barrier thickness consistent with the Hartmann-Fletcher [14, 15] effect for thick barriers. Tunneling dwell times, $\big(t(x=q) - t(x=-q)\big)$, by Eq.(6) decrease with increasing κ consistent with Barton [16]. However, our results differ from those of Dewdney and Hiley [2] who graphically showed that the magnitude of the velocity was less while tunneling in the classically forbidden region. Our results also differ with the classical analogy of Cohn and Rabinowitz [1] where the magnitude of velocity is less while hopping over the barrier.

3. Tunneling with certainty.

Hamilton's characteristic function as presented by Eq.(5) is the generator of motion for an individual deterministic trajectory that tunnels through the barrier with certainty. We now develop the corresponding Schrödinger wave function for this trajectory that tunnels with certainty. This wave function has a continuous logarithmic derivative across the barrier interfaces at $-q, q$.

While Eq.(3) gives the relationship between the conjugate momentum W' and the solution set of independent wave functions (ϕ, θ), an inverse relationship is given by [7]

$$\psi = \frac{\exp\big(iW/\hbar\big)}{(W')^{1/2}} \tag{7}$$

For $x > q$, we have easily from Eqs.(5) and (7) a transmitted, unmodulated running wave given by

$$\psi = (\hbar k)^{-1/2} \exp\big(ik(x-q)\big), \quad x > q \tag{8a}$$

where the integration constant, K, has been chosen so that phase is zero at the barrier interface $x = q$.

For $-q \leq x \leq q$ and from Eqs.(3) – (5) and (7), the Schrödinger wave function is

$$\begin{aligned} \psi = &\Big(\frac{(\kappa/k)\cosh^2(\kappa x) + (k/\kappa)\sinh^2(\kappa x)}{\hbar\kappa}\Big)^{1/2} \\ &\cdot \exp\Big(i \arctan\big(\frac{k}{\kappa}\tanh(\kappa(x-q))\big)\Big), \\ &-q \leq x \leq q \end{aligned} \tag{8b}$$

where in Eqs.(3) and (4) $\phi = \cosh\big(\kappa(x-q)\big)$, $\theta = \sinh\big(\kappa(x-q)\big)$, $a = \big((2m)^{1/2}/(\hbar\kappa)\big)(\kappa/k)$, $b = \big((2m)^{1/2}/(\hbar\kappa)\big)(k/\kappa)$, and $c = 0$. This Schrödinger wave function represented by Eqs.(8a) and (8b) has a continuous logarithmic derivative across the barrier interface $x = q$. As

Eq.(8b) manifests a wave running in the positive x-direction, there is no reflection at the barrier interface $x = q$.

For $x < -q$ and from Eqs.(3) – (5) and (7), the Schrödinger wave function is presented in the form that facilitates further manipulations in Sect. 4 as

$$\psi = \frac{1}{(\hbar k)^{1/2}}\Bigg(\cosh^2(-2\kappa q) + \frac{1}{2}\big(\frac{\kappa}{k}+\frac{k}{\kappa}\big)\sinh(-4\kappa q)\sin\big(2k(x+q)\big)$$
$$+\sinh^2(-2\kappa q)\Big(\big(\frac{\kappa}{k}\sin\big(k(x+q)\big)\big)^2 + \big(\frac{k}{\kappa}\cos\big(k(x+q)\big)\big)^2\Big)\Bigg)^{1/2}$$
$$\cdot \exp\Big(i\arctan(a)\Big)$$
$$\text{for } x < -q \tag{8c}$$

where

$$a = \frac{\frac{k}{\kappa}\sinh(-2\kappa q)\cos\big(k(x+q)\big) + \cosh(-2\kappa q)\sin\big(k(x+q)\big)}{\cosh(-2\kappa q)\cos\big(k(x+q)\big) + \frac{\kappa}{k}\sinh(-2\kappa q)\sin\big(k(x+q)\big)}$$

and where $\phi = \cos\big(k(x+q)\big)$ and $\theta = \sin\big(k(x+q)\big)$

The Schrödinger wave function as represented by Eqs.(8b) and (8c) has a continuous logarithmic derivative across the barrier interface at $x = -q$. Similar to the situation at the barrier interface at $x = q$, Eq. (8c) manifests a compoundly modulated wave progressing in the positive x-direction, so there is no reflection of this wave at the interface $x = -q$ either.

The Schrödinger wave function represented by Eqs.(8a) – (8c) manifests a wave progressing in the x-direction everywhere. There is no reflection. This Schrödinger wave function is an eigenfunction for energy E and for our given rectangular barrier. Hence, this eigenfunction represents a particle with sub-barrier energy that tunnels through the barrier with certainty.

Only recently did we recognize that eigenfunctions for a constant potential could be wave functions with compound modulation in amplitude and wavenumber [6]. While one could confirm that the wave function represented by Eqs.(8a) – (8c) is an eigenfunction by brute force through substituting this wave function into the Schrödinger equation, we suggest waiting for the analyses of the next section.

4. Consistency with the contemporary Schrödinger representation.

We now analyze the wave function represented by Eqs. (8a) – (8c) into familiar, unmodulated plane waves and hyperbolic functions of the contemporary Schrödinger representation in the allowed and forbidden regions respectively for a piecewise constant potential. First, it is convenient to combine the trigonometric relationships [7]

$$(a\phi^2+b\theta^2+c\phi\theta)^{1/2}\cos\Big(\arctan\big(\frac{b\frac{\theta}{\phi}+\frac{c}{2}}{(ab-\frac{c^2}{4})^{1/2}}\big)\Big)=\big(a-\frac{c^2}{4b}\big)^{1/2}\phi$$

and

$$(a\phi^2+b\theta^2+c\phi\theta)^{1/2}\sin\Big(\arctan\frac{b\frac{\theta}{\phi}+\frac{c}{2}}{(ab-\frac{c^2}{4})^{1/2}}\Big)=b^{1/2}\theta+\frac{c\phi}{2b^{1/2}}$$

into an exponential relationship

$$\begin{aligned}(a\phi^2+b\theta^2+c\phi\theta)^{1/2}\exp\Big(i\arctan\frac{b\frac{\theta}{\phi}+\frac{c}{2}}{(ab-\frac{c^2}{4})^{1/2}}\Big)\\ =\Big(\big(a-\frac{c^2}{4b}\big)^{1/2}+i\frac{c}{2b^{1/2}}\Big)\phi+ib^{1/2}\theta\end{aligned} \qquad (9)$$

For $x>q$, Eq.(8a) is trivially the familiar, transmitted unmodulated wave function $(\hbar k)^{-1/2}\exp\big(ik(x-q)\big)$ of the contemporary Schrödinger representation.

We find from Eqs.(3), (4) and (9) that the wave function represented by Eq.(8b) in the classically forbidden region $-q\le x\le q$ may also be expressed in an alternative form as

$$\psi=\frac{1}{(\hbar k)^{1/2}}\Big(\cosh\big(\kappa(x-q)\big)+i\frac{k}{\kappa}\sinh\big(\kappa(x-q)\big)\Big),\quad -q\le x\le q \qquad (10)$$

This representation of Eq.(10) in hyperbolic functions is the very representation of contemporary Schrödinger representation that matches the logarithmic derivative of the transmitted wave $(\hbar k)^{-1/2}\exp\big(ik(x-q)\big)$ at the barrier interface $x=q$. Thus, the trajectory representation accounts for the continuity requirements for the Schrödinger wave function to have a continuous logarithmic derivative across an interface. The compoundly modulated wave function given by Eq.(8b) is equivalent to the contemporary Schrödinger representation and does indeed represent the eigenfunction for sub-barrier energy E in the classically forbidden region. We note that we could have also shown consistency between for this case by using a procedure suggested by Bohm [17] where, knowing Eq.(10), we specify $W=\hbar\arctan\big(\Im(\psi)/\Re(\psi)\big)$ and $(W')^{-1}=\big(\Re(\psi)\big)^2+\big(\Im(\psi)\big)^2$ to deduce Eq.(8b).

For $x<-q$, we have before the barrier that

$$\begin{aligned}\psi =&(\hbar k)^{-1/2}\Big(\big(\cosh(-2\kappa q)+i(k/\kappa)\sinh(-2\kappa q)\big)\cos\big(k(x+q)\big)\\ &+\big((\kappa/k)\sinh(-2\kappa q)+i\cosh(-2\kappa q)\big)\sin\big(k(x+q)\big)\Big),\\ &\qquad\text{for all } x<-q\end{aligned} \tag{11}$$

Like the case at $x=q$, Eq.(11) is the solution of the contemporary Schrödinger representation that has a continuous logarithmic derivative at the barrier interface $x=-q$ with the solution represented by Eq. (10). We may represent this wave function for $x<-q$ in terms of unmodulated running waves as

$$\begin{aligned}\psi =& \frac{1}{(\hbar k)^{1/2}}\bigg(\Big(\cosh(-2\kappa q)+\frac{i}{2}\big(\frac{k}{\kappa}-\frac{\kappa}{k}\big)\sinh(-2\kappa q)\Big)\exp\big(ik(x+q)\big)\\ &+\big(\frac{i}{2}\big(\frac{k}{\kappa}-\frac{\kappa}{k}\big)\sinh(-2\kappa q)\big)\exp\big(-ik(x+q)\big)\bigg),\\ &\qquad\text{for all } x<-q\end{aligned} \tag{12}$$

The terms in the first set of mid-size parenthesis on the right side of Eq.(12) manifest the coefficient for the incident running wave while the terms within the second set of mid-size parenthesis manifest the coefficient for the reflected running wave of the contemporary Schrödinger representation for scatter from a rectangular barrier. Hence, the trajectory representation is consistent with the contemporary Schrödinger representation for scattering by a rectangular barrier.

From a mathematical viewpoint, for any solution set (ϕ,θ) to the Schrödinger equation, we can always form another set (ζ,ξ) of independent solutions by the principle of superposition for linear differential equations so that $\zeta = A\phi + B\theta$ and $\xi = C\phi + D\theta$ where the coefficients obey the relationship $(AD - BC) \neq 0$. Thus, Eq.(9) manifests an application of the superposition principle that reversibly maps the wave functions represented by Eqs.(10) and (11) into the synthetic wave functions represented by Eqs.(8b) and (8c). We present in the Appendix the inverse mappings for synthesizing unmodulated plane waves from compoundly modulated waves.

Thus, the compoundly modulated wave represented by Eqs.(8a), (8c) is an eigenfunction of the Schrödinger equation for sub-barrier energy E that tunnels through the barrier with certainty.

5. Discussion and interpretation.

All work has been done exactly. The trajectory described by Hamilton's characteristic function, Eq.(5), tunnels through the barrier. This Hamilton's characteristic function specifies the Schrödinger wave function as described by Eqs.(8a) - (8c) where a compoundly modulated wave tunnels through the barrier to generate an unmodulated transmitted wave. This wave function is an eigenfunction of sub-barrier energy E as shown in Section 4 by Eqs.(10) and (12). Hence, the compoundly modulated wave function with sub-barrier energy eigenvalue E tunnels through the barrier with certainty.

A spectral resolution of the compoundly modulated wave function in the domain $\big(-\infty, -q\big)$ before the barrier would reveal a two component spectrum ($+k$ and $-k$) in consonance with Eq.(12). This spectral resolution only demonstrates that the trajectory and contemporary Schrödinger representations are consistent by the principle of superposition. Heretofore, the contemporary Schrödinger representation has in practice solved the tunneling problem by analysis into intuitive, unmodulated (incident, reflected and transmitted) wave functions.

Now, the trajectory representation, which can synthesize the spectral resolution into a compoundly modulated wave in the incident domain, offers us other possible choices of eigenfunction to describe tunneling. The principle of superposition for linear differential equations has been well understood mathematically; to wit it gives us the mapping between standing and running wave functions. Now, we can extract physical insight from the principle of superposition for two (incident and reflected) unmodulated plane wave function of unequal amplitudes can be synthesized into a single incident wave function with compound modulation.

Hence, the synthetic wave function is the eigenfunction that represents the ensemble, and the associated Hamilton's characteristic function is the generator of motion for the unpartitioned ensemble. In an ensemble of sufficiently large number, N, of particles, there is flexibility in the actual partitioning among the possible eigenfunctions. The ensemble may be composed entirely of unmodulated incident and reflected waves in the region before the barrier as visualized by the contemporary Schrödinger representation. But the ensemble for large N is not restricted to such a distribution. Other distributions are possible. On the other hand, the ensemble size can be reduced to a single particle. Then, only the normalization of the wave function for the ensemble would be changed in the Schrödinger representation. For $N = 1$, the tunneling particle is described by the wave function given by Eqs.(8a) - (8c).

We recognize that time-independence is an idealization. Our wave packet for a particle has an infinitely long wave train. Before incidence, this wave train is compoundly modulated and has spectral components running in opposite directions. While we examine such an idealization in this exposition, we can nevertheless comment on the integrity of a wave packet consisting of a finitely long, compoundly modulated wave train during its temporal evolution before incidence. We can put to rest any reservations that its spectral components running in opposite directions might induce a spontaneous splitting of such a wave packet. Let us consider the familiar wave packet consisting of just a plane wave train. As shown in the Appendix, a plane wave can be mapped into compoundly modulated waves that run in opposite directions. Therefore, the integrity of the familiar plane wave packet infers the integrity of the compoundly modulated wave packet. By the superposition principle, the integrity of either wave packet infers the integrity of the other.

In an experiment, if N particles each with a compoundly modulated wave function described by Eq.(8c) are incident to the rectangular barrier described by Eq.(1), then all N particles will tunnel through the barrier with certainty to emerge as unmodulated wave functions as described by Eq.(8a). Inside the barrier, the wave function for each particle is described by Eq.(8b). This ensemble is composed of N identically prepared particles. N may be any positive integer. Note that in the trajectory representation, a generator of the motion, such as given by Eq.(5), also specifies this experiment for an individual particle.

The trajectory representation is deterministic, even in the classically forbidden region. By precept, it should be so. If the initial conditions (W

does not appear explicitly in Eq.(2)) for W' and W'' for some $x < -q$ are such to determine a solution to the third-order Hamilton-Jacobi equation for quantum motion that describes W' as a positive constant for $x > q$, then that trajectory shall tunnel through the barrier to generate a transmitted unmodulated running wave. The corollary is that the initial conditions for W' and W'' for some $x < -q$, along with an arbitrary additive constant K, establish a unique solution for W everywhere and consequently establish the spectral components of the wave function everywhere. These initial conditions are the necessary and sufficient set of variables [18] that determine the behavior of the particle. As W itself does not appear in the Hamilton-Jacobi equation for quantum continuous motion, Eq.(2), the initial conditions W' and W'' along with the normalization of the Wronskian are sufficient to specify the set of coefficients (a, b, c) at the initial point.

The deterministic trajectory from first principles describes propagation for an individual monochromatic particle in the forbidden regions. It can confirm other findings [11, 14, 16] with respect to tunneling by wave packets. However, these other findings assumed a contemporary Schrödinger representation in describing the wave function so they a priori assumed the incident wave packet was split by the barrier into transmitted and reflected components. (Barton [16] did construct a packet that was so severely spread that spreading masked reflection). Hence, these other findings are applicable to a different physical situation where tunneling is only possible, so not all findings need be confirmed.

The Copenhagen interpretation renders statistical expectations for an ensemble of particles. It does not describe the trajectories of individual particles. On the other hand, the trajectory representation is deterministic for the individual particle.

As the deterministic trajectory and Schrödinger representations are mutually consistent, we find that assigning a probability amplitude to the Schrödinger wave function by the Copenhagen interpretation is unnecessary. Also, from the consistency between the trajectory and Schrödinger representations, we find that the introduction of stochasticy by the Bohm school in a trajectory interpretation is unnecessary.

The Copenhagen interpretation sets epistemological limits to knowledge. Before collapsing the wave function, only the probability that an individual particle will tunnel successfully could be known. No property of the wave function could predetermine the tunnelling outcome. Meanwhile, the trajectory representation is deterministic. Tunnelling outcomes are predetermined by Hamilton's characteristic function. As the trajectory representation is applicable to an individual trajectory, there is no need to collapse the wave function to determine its successful outcome in the trajectory interpretation.

As we reject the Copenhagen interpretation, can we give another interpretation to the wave function ? As the Schrödinger and trajectory representations are consistent for the ensemble and as the wave function and Hamilton's characteristic function mutually infer each other, we conclude that the phase of the Schrödinger wave function for an ensemble is also a generator of the motion for that unpartitioned ensemble.

Appendix – Inverse mappings.

The incident wave function with compound modulation, Eq.(8c), can be synthesized under the superposition principal from two spectral components running in opposite direction as shown by Eq.(12). Likewise, an unmodulated plane wave running in one direction can be synthesized from two waves with compound modulation running in the opposite direction for mappings under the superposition principle are reversible.

As a heuristic example, consider analyzing the unmodulated plane waves (eigenfunctions for the free particle with energy E) into the solution set (ζ, ξ) where

$$\zeta = \frac{1}{(\hbar k)^{1/2}}\Big(\cosh^2(-2\kappa q) + \frac{1}{2}(\frac{\kappa}{k} + \frac{k}{\kappa})\sinh(-4\kappa q)\sin\big(2k(x+q)\big)$$
$$+ \sinh^2(-2\kappa q)\big((\frac{\kappa}{k}\sin\big(k(x+q)\big)\big)^2 + (\frac{k}{\kappa}\cos\big(k(x+q)\big)\big)^2\big)\big)^{1/2}$$
$$\cdot \exp\Big(+i\arctan$$
$$(\frac{\frac{k}{\kappa}\sinh(-2\kappa q)\cos\big(k(x+q)\big) + \cosh(-2\kappa q)\sin\big(k(x+q)\big)}{\cosh(-2\kappa q)\cos\big(k(x+q)\big) + \frac{\kappa}{k}\sinh(-2\kappa q)\sin\big(k(x+q)\big)})\Big)$$

and

$$\xi = \frac{1}{(\hbar k)^{1/2}}\Big(\cosh^2(-2\kappa q) + \frac{1}{2}(\frac{\kappa}{k} + \frac{k}{\kappa})\sinh(-4\kappa q)\sin\big(2k(x+q)\big)$$
$$+ \sinh^2(-2\kappa q)\big((\frac{\kappa}{k}\sin\big(k(x+q)\big)\big)^2 + (\frac{k}{\kappa}\cos\big(k(x+q)\big)\big)^2\big)\Big)^{1/2}$$
$$\cdot \exp\Big(-i\arctan$$
$$(\frac{\frac{k}{\kappa}\sinh(-2\kappa q)\cos\big(k(x+q)\big) + \cosh(-2\kappa q)\sin\big(k(x+q)\big)}{\cosh(-2\kappa q)\cos\big(k(x+q)\big) + \frac{\kappa}{k}\sinh(-2\kappa q)\sin\big(k(x+q)\big)})\Big) .$$

We now consider the mappings

$$\begin{aligned}\frac{1}{(\hbar k)^{1/2}}&\big(\cosh(-2\kappa q)+\frac{i}{2}(\frac{k}{\kappa}-\frac{\kappa}{k})\sinh(-2\kappa q)\big)\exp i\big(k(x+q)\big)\\ &=\big(\cosh^2(-\kappa q)+\frac{1}{4}(\frac{k}{\kappa}-\frac{\kappa}{k})^2\sinh^2(-2\kappa q)\big)\zeta\\ &-\big(\cosh(-\kappa q)+\frac{i}{2}(\frac{k}{\kappa}-\frac{\kappa}{k})\sinh(-2\kappa q)\big)\\ &\cdot\big(\frac{i}{2}(\frac{k}{\kappa}-\frac{\kappa}{k})\sinh(-2\kappa q)\big)\xi\end{aligned}\tag{A1}$$

and

$$\begin{aligned}\frac{1}{(\hbar k)^{1/2}}&\big(\frac{i}{2}(\frac{k}{\kappa}+\frac{\kappa}{k})\sinh(-2\kappa q)\big)\exp -i\big(k(x+q)\big)\\ &=-\big(\frac{1}{4}(\frac{k}{\kappa}+\frac{\kappa}{k})^2\sinh^2(-2\kappa q)\big)\zeta\\ &+\big(\cosh(-\kappa q)+\frac{i}{2}(\frac{k}{\kappa}-\frac{\kappa}{k})\sinh(-2\kappa q)\big)\\ &\cdot\big(\frac{i}{2}(\frac{k}{\kappa}-\frac{\kappa}{k})\sinh(-2\kappa q)\big)\xi\ .\end{aligned}\tag{A2}$$

Equations (A1) and (A2) respectively map the incident unmodulated plane wave and the reflected unmodulated plane wave of the contemporary Schrödinger representation (cf. Eq.(12)) into set (ζ,ξ) of compoundly modulated eigenfunctions that run in opposite directions. We note that Eqs.(A1) and (A2) sum to ζ, which manifests the incident wave with compound modulations, Eq.(8c), as expected.

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