Relativistic equation of motion

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ABSTRACT. An alternative theory of relativity, which differs from Einstein's theory of relativity, is applied to modify the Newtonian equation of motion. The new relativistic equation of motion so obtained contains the Newtonian equation of motion as a low-speed limit. Comparisons between the new relativistic equation of motion and Einstein's relativistic equation of motion are given.

RÉSUMÉ. Une autre théorie de la relativité, différant de celle d'Einstein, est appliquée pour modifier les équations du mouvement de Newton. Les équations obtenues contiennent celles de Newton à la limite des faibles vitesses. Il est fait une comparaison entre ces équations et celles d'Einstein.

We start with presenting an alternative theory of relativity which is formulated based on assumptions closely related to well-accepted experiments. The alternative theory of relativity differs from Einstein's theory of relativity primarily in the physical interpretations of space-time and the relativistic transformations. The alternative theory of relativity is then shown to provide a unified scheme to relativistically modify the Newtonian equation of motion. Finally, the relativistic equation of motion so obtained is compared with the relativistic equation of motion in Einstein's theory of relativity.

1. A theory of relativity based upon physical reality.

1.1 The Underlying Assumptions of the Alternative Theory of Relativity.

The basic assumptions on which the alternative theory of relativity rests are: besides the principle of relativity, (1) the invariance of the speed of light, (2) the relativistic (transverse) Doppler shift, and (3) the existence of an instantaneous rest frame.[1] The invariance of the speed of light is an assumption in Einstein's theory of relativity[2], and it is generally accepted as confirmed experimentally. The relativistic Doppler shift between *inertial* frames is a prediction of Einstein's theory of relativity. We take it as another assumption of the alternative theory of relativity, holding even for *noninertial* reference frames with constant velocities between them. The existence of an instantaneous rest frame states that for any particle moving with respect to a reference frame, at any arbitrary instant, there exists a rest frame of the particle in which the particle is at rest, that is, the particle moves with zero velocity, zero acceleration, zero time derivative of acceleration,..., and so on. More generally, there are an infinite number of frames with uniform velocities relative to the rest frame such that, with respect to these frames, the particle moves with uniform velocities, but zero acceleration, zero time derivative of acceleration,..., and so on. This assumption is, in fact, logically more restrictive than Einstein's equivalence principle in describing the motion of a *single* particle in *gravitational* fields.[2,3] In that case, this assumption is accepted by Einstein's theory of general relativity in describing the motion of a *single* particle in *gravitational* fields. However, it should be emphasized that this assumption is applicable to the motion of a single particle in arbitrary force fields. This assumption is not equivalent to the equivalence principle in Einstein's general relativity.[1,4]

1.2 The Invariant Length of the Infinitesimal Four-Dimensional Displacements of a Uniformly Moving Material Particle.

As usual, in describing the motion of particles, a reference frame, which has stipulated measurements for the space unit and the time unit, is to be introduced in the beginning. Unit standards of measurements should be provided to build up the three-dimensional space coordinates and one-dimensional time coordinate; then the coordinate reference frame can be employed for describing the motion of particles. Relativistic transformations relate the values of physical quantities between reference frames. In order to ensure that relativistic transformations of physical quantities among reference frames are physically meaningful, these reference frames must have the same stipulation of the space and time units. Hereafter, all the coordinate reference frames under consideration are presumed to have the same stipulation of the space and time measurements. Furthermore, it is an empirical fact that if a particle moves uniformly with respect to any one of the reference frames, which have relative constant velocities among them, then this particle moves uniformly with respect to the others. If in addition one of them is an inertial frame, then the others are also inertial frames. References frames, however, are not assumed inertial unless explicitly stated to be inertial.

Now, suppose two reference frames, X frame and $\bar{X}$ frame, move with constant velocity with respect to one another. Consider a light-source particle moving with a constant velocity $\mathbf{v}$ with respect to the X frame. Then, this light source moves with a constant velocity $\bar{\mathbf{v}}$ with respect to the $\bar{X}$ frame. Suppose also that the frequency of light emitted by this light source is ν_0 measured with respect to its rest frame. By the assumption of relativistic Doppler shift, the frequency of light emitted by this light source measured with respect to the X frame is

$$\nu = (1 - (v/c)^2)^{1/2}\nu_0, \tag{1}$$

where c is the speed of light. Similarly, the frequency of that light measured with respect to the $\bar{X}$ frame is

$$\bar{\nu} = (1 - (\bar{v}/c)^2)^{1/2}\nu_0. \tag{2}$$

Rewrite Eq. (1) as

$$\Delta t_0 = (1 - (v/c)^2)^{1/2}\Delta t, \tag{3}$$

where $\Delta t_0 \equiv 1/\nu_0$, and $\Delta t \equiv 1/\nu$. Similarly, from Eq. 2, we have

$$\Delta t_0 = (1 - (\bar{v}/c)^2)^{1/2}\Delta \bar{t}, \tag{4}$$

where $\Delta \bar{t} \equiv 1/\bar{\nu}$.

As usual in describing the motion of particles in classical mechanics, during the time interval Δt with respect to the X frame, the particle will move from x^i to $x^i + \Delta x^i \quad (i = 1, 2, 3)$, where $\Delta x^1 = v_x \Delta t$, $\Delta x^2 = v_y \Delta t$ and $\Delta x^3 = v_z \Delta t$. Then, we define $\Delta x^0 \equiv c\Delta t$ and define the length Δs, which is associated with the 4-vector of displacement $\Delta x^\alpha = (\Delta x^0, \Delta x^1, \Delta x^2, \Delta x^3)$ of the moving particle, as

$$\Delta s \equiv ((\Delta x^0)^2 - (\Delta x^1)^2 - (\Delta x^2)^2 - (\Delta x^3)^2)^{1/2}. \tag{5}$$

By the definition of Δx^α $(\alpha = 0, 1, 2, 3)$, we have

$$\Delta s = (1 - (v/c)^2)^{1/2}\Delta x^0. \tag{6}$$

Similarly, during the time interval $\Delta\bar{t}$ with respect to the $\bar{X}$ frame, this particle will move from $\bar{x}^i$ to $\bar{x}^i + \Delta\bar{x}^i$. Consequently, we have

$$\Delta\bar{s} \equiv ((\Delta\bar{x}^0)^2 - (\Delta\bar{x}^1)^2 - (\Delta\bar{x}^2)^2 - (\Delta\bar{x}^3)^2)^{1/2}, \quad \text{and} \tag{7}$$

$$\Delta\bar{s} = (1 - (\bar{v}/c)^2)^{1/2}\Delta\bar{x}^0. \tag{8}$$

From Eqs. (3), (4), (6) and (8), we obtain

$$\Delta s = \Delta\bar{s} = c\Delta t_0. \tag{9}$$

Therefore, we have, for the uniformly moving particle with respect to the X frame and $\bar{X}$ frame,

$$\begin{aligned} \Delta\tau \equiv c\Delta t_0 &= (\eta_{\alpha\beta}\Delta x^\alpha \Delta x^\beta)^{1/2} \\ &= (\eta_{\alpha\beta}\Delta\bar{x}^\alpha \Delta\bar{x}^\beta)^{1/2} \qquad (\alpha, \beta = 0, 1, 2, 3), \end{aligned} \tag{10}$$

where $\eta_{\alpha\beta}$ are defined as

$$\eta_{\alpha\beta} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{11}$$

In our notation, any index, like α and β in Eq. (10), that appears twice, once as a subscript and once as a superscript, is understood to be summed over.

In general, suppose a material particle moves uniformly with respect to the X frame. Thus, it also moves uniformly with respect to the $\bar{X}$ frame. Consider a reference frame X_s moving with this particle. That is, this particle is at rest with respect to its associated rest frame X_s. During an infinitesimal time interval dt_0 with respect to the X_s frame, the particle will move from x^i to $x^i + dx^i$ with respect to the X frame, or from $\bar{x}^i$ to $\bar{x}^i + d\bar{x}^i$ with respect to the $\bar{X}$ frame. From Eq. (10), the invariant length of infinitesimal displacements $d\tau$ is obtained as

$$d\tau = cdt_0 = (\eta_{\alpha\beta}dx^\alpha dx^\beta)^{1/2} = (\eta_{\alpha\beta}d\bar{x}^\alpha d\bar{x}^\beta)^{1/2}. \tag{12}$$

It should be emphasized that, though the formula Eq. (12) is the same as Minkowski's differential space-time distance in Einstein's theory of relativity, the physical interpretations of this formula are different between the alternative theory herein and Einstein's theory of relativity. The dx^α is a four-dimensional infinitesimal displacement vector which characterizes the state of motion. The $d\tau$ is the invariant infinitesimal length of the four-dimensional displacements dx^α. Eq. (12) merely gives a relationship of the physical quantities of motion, rather than a relationship of the space-time structure of coordinate systems, between the two reference frames X and $\bar{X}$.

In classical mechanics, the $dx^\alpha(x)$ and $d\tau(x)$ are functions of the space and time coordinates x of the particle with respect to the frame X. However, according to quantum mechanics, those physical quantities should be considered as functions of time only, due to the impossibility of simultaneously determining the definite values of positions and velocities of particles.[5,6] Hereafter, for convenience, we do not explicitly express the dependent variables of these physical quantities of motion.

1.3 The Relativistic Transformation for the Infinitesimal Four-Dimensional Displacements of Motion.

Consider a reference frame $\bar{X}$ moving with a constant velocity $\mathbf{V}$ with respect to a reference frame X. As usual, it must be postulated that given a velocity $\mathbf{V}$ of the $\bar{X}$ frame relative to the X frame, then when measured relative to the $\bar{X}$ frame, the X frame has velocity $-\mathbf{V}$. Suppose that at an arbitrary instant a material particle has the state of motion dx^α with respect to the frame X; at that instant the material particle has the state of motion $d\bar{x}^\mu$ with respect to the frame $\bar{X}$. Assume that the general functional relationship between the four-dimensional displacements dx^α and $d\bar{x}^\mu$ is linear, that is,

$$dx^\alpha = a^\alpha_\mu(X, \bar{X}) d\bar{x}^\mu \quad (\alpha, \mu = 0, 1, 2, 3). \tag{13}$$

Logically, the coefficients $a^\alpha_\mu(X, \bar{X})$ could depend on the motion of the particle, as well as relationships between the frames X and $\bar{X}$ such as the orientation of coordinate systems and the relative velocity $\mathbf{V}$ between the reference frames. Suppose that there exists a universal transformation which could be applied to particles with arbitrary motion. That is, the coefficients $a^\alpha_\mu(X, \bar{X})$ of the universal transformation depend on the relationship between the two frames, not on the motion of particles. This supposed existing universal transformation of the four-dimensional infinitesimal displacements will be derived as follows.

We simplify the mathematical problem by keeping the corresponding coordinate axes of the frames X and $\bar{X}$ parallel and the relative velocity **V** in the direction of a chosen axis, for example, X^1-axis in the derivation below. For convenience, we also simplify the notation of $a^\alpha_\mu(X, \bar{X})$ as a^α_μ.

(1) If a particle is at rest in the $\bar{X}$ frame, then for this particle $d\bar{x}^1 = d\bar{x}^2 = d\bar{x}^3 = 0$ with respect to the $\bar{X}$ frame and $dx^2 = dx^3 = 0$ with respect to the X frame. Therefore, from the assumed linear equation of transformation Eq. (13), we have $a^2_0 = a^3_0 = 0$ and $a^1_0/a^0_0 = dx^1/dx^0 = V/c \equiv \beta$.

(2) If a particle is at rest in the X frame, then $dx^1 = dx^2 = dx^3 = 0$ and $d\bar{x}^2 = d\bar{x}^3 = 0$. Therefore, we have $a^2_1 = a^3_1 = 0$ and $-a^1_0/a^1_1 = d\bar{x}^1/d\bar{x}^0 = -V/c = -\beta$.

(3) If a particle moves in the $X^1 - X^2$ plane, then $dx^3 = 0$ and $d\bar{x}^3 = 0$. Therefore, we have $a^3_2 = 0$.

(4) If a particle moves in the $X^1 - X^3$ plane, then $dx^2 = 0$ and $d\bar{x}^2 = 0$. Therefore, we have $a^2_3 = 0$.

(5) If a particle moves in the $X^2 - X^3$ plane, then $dx^1 = 0$ and $d\bar{x}^1/d\bar{x}^0 = -\beta$. Therefore, we have $a^1_2(d\bar{x}^2/d\bar{x}^0) + a^1_3(d\bar{x}^3/d\bar{x}^0) = 0$. Because $d\bar{x}^2/d\bar{x}^0$ and $d\bar{x}^3/d\bar{x}^0$ are arbitrary, $a^1_2 = a^1_3 = 0$.

(6) If a particle moves in the $\bar{X}^2 - \bar{X}^3$ plane, then $d\bar{x}^1 = 0$ and $dx^1/dx^0 = \beta$. Therefore, we have $dx^1/dx^0 = a^1_0/(a^0_0 + a^0_2(d\bar{x}^2/d\bar{x}^0) + a^0_3(d\bar{x}^3/d\bar{x}^0)) = \beta$. From (1) and (2), we obtain $a^1_1 = a^0_0$. Since $d\bar{x}^2/d\bar{x}^0$ and $d\bar{x}^3/d\bar{x}^0$ are arbitrary, we have $a^0_2 = a^0_3 = 0$. Now, we have reduced the universal transformation to

$$\begin{cases} dx^0 &= a^0_0 d\bar{x}^0 + a^0_1 d\bar{x}^1 \\ dx^1 &= \beta a^0_0 d\bar{x}^0 + a^0_0 d\bar{x}^1 \\ dx^2 &= a^2_2 d\bar{x}^2 \\ dx^3 &= a^3_3 d\bar{x}^3 \quad . \end{cases} \tag{14}$$

(7) Suppose a particle moves uniformly with respect to the frames X and $\bar{X}$. Then, the reduced transformation Eq. (14) should also satisfy Eq. (12). Consequently, we obtain $a^0_0 = \pm(1-\beta^2)^{-1/2}$, $a^0_1 = \beta a^0_0$, $a^2_2 = \pm 1$ and $a^3_3 = \pm 1$. According to the chosen orientation of coordinate systems between the frames X and $\bar{X}$, we finally obtain the relativistic transformation of the infinitesimal displacements dx^α:

$$\begin{cases} dx^0 &= \gamma(d\bar{x}^0 + \beta d\bar{x}^1) \\ dx^1 &= \gamma(d\bar{x}^1 + \beta d\bar{x}^0) \\ dx^2 &= d\bar{x}^2 \\ dx^3 &= d\bar{x}^3 \quad , \end{cases} \tag{15}$$

where $\gamma \equiv (1-\beta^2)^{-1/2}$.

The mathematical form of the Lorentz transformation of displacements Eq. (15) is close to that of the usual Lorentz transformation of space-time coordinates in Einstein's theory of relativity,

$$\begin{cases} x^0 &= \gamma(\bar{x}^0 + \beta \bar{x}^1) \\ x^1 &= \gamma(\bar{x}^1 + \beta \bar{x}^0) \\ x^2 &= \bar{x}^2 \\ x^3 &= \bar{x}^3 \ . \end{cases} \tag{16}$$

However, the physical meaning of the Lorentz transformation of the displacements of motion in the alternative theory is entirely different from that of the Lorentz transformation of space-time coordinates in Einstein's theory of relativity. The locally generated Lorentz transformation of displacements is considered as a physical rule to instantaneously transform the infinitesimal displacement 4-vector dx^α with respect to one reference frame to another. The infinitesimal 4-vector dx^α characterizes the state of motion of a particle with respect to the frame X; the components are not the units grid of space-time measurements of the frame X. The dx^α are physical quantities of the motion of the particle with respect to the frame X; the units grid of space and time of the frame X should be presumed in the beginning, in order to measure the physical quantity of motion dx^α. The Lorentz transformation of displacements derived herein is not an integrable or global transformation of the space-time structure from one reference frame to another as it is so interpreted in Einstein's theory of relativity. Further distinctions between the Lorentz transformation of motion and the Lorentz transformation of space-time coordinates from the point of view of quantum theory are presented in the paper[6].

1.4 The Relativistic Addition of Velocities and the Relativistic Energy-Momentum of a Free Material Particle.

The equation of *uniform* motion for a free material particle with respect to a reference frame X is

$$\frac{d^2x^\alpha}{d\tau^2} = 0, \quad \text{where} \tag{17}$$

$$d\tau = (\eta_{\alpha\beta} dx^\alpha dx^\beta)^{1/2}. \tag{18}$$

The equation of uniform motion is covariant under this Lorentz transformation of motion Eq. (15). The general equation of motion of a non-uniformly moving material particle shall be introduced in the next section, and be shown to be covariant under this Lorentz transformation of motion.

Now, consider a material particle of rest mass m moving with constant velocity $\mathbf{v}$ with respect to the reference frame X. From Eq. (17), we have

$$\frac{dx^\alpha}{d\tau} = U^\alpha, \tag{19}$$

where $d\tau = (1-(v/c)^2)^{1/2}dx^0$ and U^α are constants. Consequently, we have

$$U^0 = (1-(v/c)^2)^{-1/2} \equiv \gamma_v, \quad \text{and} \quad U^i = \gamma_v \frac{v^i}{c} \quad (i=1,2,3). \tag{20}$$

With respect to the reference frame $\bar{X}$, this particle moves with constant velocity $\bar{\mathbf{v}}$. Similarly, we have

$$\frac{d\bar{x}^\alpha}{d\tau} = \bar{U}^\alpha, \tag{21}$$

where $d\tau = (1-(\bar{v}/c)^2)^{1/2}d\bar{x}^0$ and $\bar{U}^\alpha$ are constants. Hence, we have also

$$\bar{U}^0 = (1-(\bar{v}/c)^2)^{-1/2} \equiv \gamma_{\bar{v}}, \quad \text{and} \quad \bar{U}^i = \gamma_{\bar{v}} \frac{\bar{v}^i}{c} \quad (i=1,2,3). \tag{22}$$

By applying the Lorentz transformation of *motion* alone, Eq. (15), we obtain the formulas for the relativistic addition of velocities,

$$\begin{cases} \gamma_v &= \gamma\gamma_{\bar{v}}(1+\beta\frac{\bar{v}^1}{c}) \\ \frac{v^1}{c} &= \frac{\beta+\frac{\bar{v}^1}{c}}{1+\beta\frac{\bar{v}^1}{c}} \\ \frac{v^2}{c} &= \frac{\frac{\bar{v}^2}{c}}{\gamma(1+\beta\frac{\bar{v}^1}{c})} \\ \frac{v^3}{c} &= \frac{\frac{\bar{v}^3}{c}}{\gamma(1+\beta\frac{\bar{v}^1}{c})} \end{cases}. \tag{23}$$

As usual, we define the energy and momentum for a uniformly moving material particle with respect to the frame X, respectively, as

$$E \equiv \gamma_v mc^2, \quad \text{and} \tag{24}$$

$$\mathbf{p} \equiv \gamma_v m \mathbf{v}. \tag{25}$$

Therefore, we have the relativistic relationship of the energy and momentum for a uniformly moving material particle,

$$E^2 - \mathbf{p}^2 c^2 = m^2 c^4. \tag{26}$$

The relativistic addition of velocities and the relativistic energy-momentum of a free material particle so obtained are the same as those in Einstein's theory of relativity.

1.5 The General Relativistic Equation of Motion of a Material Particle.

Consider a material particle moving with respect to the *inertial* reference frames X and $\bar{X}$, which move uniformly relative to each other. By the assumption of a rest frame, at any arbitrary instant, there exists an *instantaneous* reference frame X_s with respect to which this particle is at rest. It should be noted that the rest frame X_s is not necessarily an inertial frame and is associated with the particle under consideration.

At any arbitrary instant, the equation of motion of this particle with respect to the chosen rest frame X_s is given as

$$dx_s^i = 0 \quad (i = 1, 2, 3), \quad \text{and} \tag{27}$$

$$\frac{d^2 x_s^\alpha}{{dx_s^0}^2} = 0. \tag{28}$$

Or, the equation of motion with respect to the rest frame X_s is

$$\frac{d^2 x_s^\alpha}{d\tau^2} = 0, \quad \text{where} \tag{29}$$

$$d\tau = dx_s^0 = (\eta_{\alpha\beta} dx_s^\alpha dx_s^\beta)^{1/2}. \tag{30}$$

We assume that there exists a linear transformation of the infinitesimal displacements dx^μ of the moving particle with respect to the frame X to the dx_s^α with respect to the rest frame X_s, that is,

$$dx_s^\alpha = A_\mu^\alpha(X_s, X) dx^\mu \quad (\alpha, \mu = 0, 1, 2, 3). \tag{31}$$

The coefficients $A^{\alpha}_{\mu}(X_s, X)$ depend on the relationship between the reference frames X and X_s, that is, the instantaneous motion of the rest frame X_s with respect to the frame X. If the particle moves uniformly with respect to the frame X, then the associated rest frame X_s moves with constant velocity with respect to the frame X. The coefficients $A^{\alpha}_{\mu}(X_s, X)$ are then reduced to the coefficients a^{α}_{μ} of the Lorentz transformation of motion which depend on the relative constant velocity between the frames X and X_s. In general, the coefficients $A^{\alpha}_{\mu}(X_s, X)$ depend not only on the velocity, but also on the acceleration, the time derivative of acceleration,..., of the instantaneous rest frame X_s with respect to the frame X. In other words, the coefficients $A^{\alpha}_{\mu}(X_s, X)$ depend on the velocity, the acceleration, the time derivative of acceleration,..., of the particle at that instant with respect to the frame X. From Eqs. (30) and (31), we obtain

$$d\tau = (\eta_{\alpha\beta} A^{\alpha}_{\mu}(X_s, X) A^{\beta}_{\nu}(X_s, X) dx^{\mu} dx^{\nu})^{1/2}. \tag{32}$$

Or, we have

$$d\tau = (g_{\mu\nu} dx^{\mu} dx^{\nu})^{1/2}, \tag{33}$$

where $g_{\mu\nu}$ are defined as

$$g_{\mu\nu} \equiv \eta_{\alpha\beta} A^{\alpha}_{\mu}(X_s, X) A^{\beta}_{\nu}(X_s, X). \tag{34}$$

From Eqs. (29), (30), (31), (33) and (34), we obtain the *general equation of motion* of a single material particle with respect to the reference frame X,

$$\frac{d^2 x^{\mu}}{d\tau^2} + \Gamma^{\mu}_{\lambda\nu} \frac{dx^{\lambda}}{d\tau} \frac{dx^{\nu}}{d\tau} = 0, \tag{35}$$

where $\Gamma^{\mu}_{\lambda\nu}$ are defined as

$$\Gamma^{\mu}_{\lambda\nu} \equiv \frac{1}{2} G^{\mu\sigma} \{ \frac{\partial g_{\lambda\sigma}}{\partial x^{\nu}} + \frac{\partial g_{\nu\sigma}}{\partial x^{\lambda}} - \frac{\partial g_{\lambda\nu}}{\partial x^{\sigma}} \}, \tag{36}$$

with $G^{\lambda\nu}$ are defined as

$$G^{\lambda\sigma} g_{\sigma\nu} \equiv \delta^{\lambda}_{\nu} \equiv \begin{cases} 1, & \text{if } \lambda = \nu \\ 0, & \text{if } \lambda \neq \nu \end{cases}, \tag{37}$$

where $\lambda, \mu, \nu, \sigma = 0, 1, 2, 3$. Rewriting the general equation of motion explicitly in terms of the coordinates of the reference frame X, we obtain it in an alternative form:[3]

$$\frac{d^2 x^i}{d{x^0}^2} = (\Gamma^0_{\lambda\nu} \frac{dx^i}{dx^0} - \Gamma^i_{\lambda\nu}) \frac{dx^{\lambda}}{dx^0} \frac{dx^{\nu}}{dx^0} \quad (i = 1, 2, 3). \tag{38}$$

By similar arguments to the above, we obtain the general equation of motion with respect to the frame $\bar{X}$,

$$\frac{d^2\bar{x}^\mu}{d\tau^2} + \bar{\Gamma}^\mu_{\lambda\nu}\frac{d\bar{x}^\lambda}{d\tau}\frac{d\bar{x}^\nu}{d\tau} = 0, \quad \text{where} \tag{39}$$

$$d\tau = (\bar{g}_{\mu\nu}d\bar{x}^\mu d\bar{x}^\nu)^{1/2}. \tag{40}$$

It must be emphasized that $g_{\mu\nu}$ and $\Gamma^\mu_{\lambda\nu}$ are not the geometrical elements of the curved space-time as interpreted in Einstein's general relativity. The $g_{\mu\nu}$ and $\Gamma^\mu_{\lambda\nu}$ are the physical quantities associated with the state of motion of the particle, that is, velocity, acceleration,..., and so on. If the particle moves uniformly with respect to the frame X, then the $g_{\mu\nu}$ are reduced to $\eta_{\mu\nu}$ and $\Gamma^\mu_{\lambda\nu} = 0$. The general equation of motion Eq. (35) is then reduced to the equation of uniform motion Eq. (17). Furthermore, it is immediately evident from the definitions of $g_{\mu\nu}$ and $\Gamma^\mu_{\lambda\nu}$ that $g_{\mu\nu}$ and $\Gamma^\mu_{\lambda\nu}$ are covariant under the Lorentz transformation of motion between the frames X and $\bar{X}$. Consequently, the general relativistic equation of motion, Eq. (35), is covariant under the Lorentz transformation of motion.

We can further reduce the equation of the invariant length of the infinitesimal displacement 4-vector, Eq. (33). Suppose that with respect to the reference frame X, at an arbitrary instant t, the velocity of the particle is $\mathbf{v}$. At that instant, consider a reference frame $\bar{X}$ moving with the constant velocity $\mathbf{v}$ with respect to the reference frame X. Then, with respect to the frame $\bar{X}$, at that instant $\bar{t}$, this particle moves with zero velocity, but not necessarily zero acceleration. Therefore, from Eq. (40), we have

$$d\tau = (\bar{g}_{00})^{1/2}d\bar{x}^0. \tag{41}$$

In addition, from the Lorentz transformation of motion between the frames X and $\bar{X}$, as well as $\bar{v}^i/c = d\bar{x}^i/d\bar{x}^0 = 0$ of this particle with respect to the frame $\bar{X}$, we have

$$d\bar{x}^0 = (1 - (v/c)^2)^{1/2}dx^0. \tag{42}$$

Therefore, from Eqs. (41) and (42), we obtain

$$d\tau = h(\mathbf{v}, \mathbf{a}, \dot{\mathbf{a}}, ...)(1 - (v/c)^2)^{1/2}dx^0, \tag{43}$$

where the function $h(\mathbf{v}, \mathbf{a}, \dot{\mathbf{a}}, ...)$ is defined as

$$h(\mathbf{v}, \mathbf{a}, \dot{\mathbf{a}}, ...) \equiv (\bar{g}_{00})^{1/2}. \tag{44}$$

The function $h(\mathbf{v}, \mathbf{a}, \dot{\mathbf{a}}, ...)$ is, in general, a function of velocity, $\mathbf{v}$, acceleration, $\mathbf{a}$, and time derivative of acceleration, $\dot{\mathbf{a}}$,..., of the particle with respect to the frame X at that instant. When the particle moves uniformly with respect to the frame X, Eq. (43) must be reduced to $d\tau = (1 - (v/c)^2)^{1/2} dx^0$. That is, $h(\mathbf{v}, \mathbf{a} = 0, \dot{\mathbf{a}} = 0, ...) = 1$, for any value of $\mathbf{v}$. Hence, the function $h(\mathbf{v}, \mathbf{a}, \dot{\mathbf{a}}, ...)$ must not contain terms that depend solely on velocity. When the particle moves with zero velocity, but not necessarily zero acceleration, Eq. (43) is reduced to $d\tau = h(\mathbf{v} = 0, \mathbf{a}, \dot{\mathbf{a}}, ...) dx^0$. The invariant length $d\tau$ depends not only on velocity, but also on acceleration, time derivative of acceleration,..., and so on. As usual in classical mechanics, $\mathbf{v}$, $\mathbf{a}$, $\dot{\mathbf{a}}$,..., and so on are considered as functions of the location and time of the particle with respect to the frame X, that is, $\mathbf{v}(\mathbf{x}, t)$, $\mathbf{a}(\mathbf{x}, t)$, $\dot{\mathbf{a}}(\mathbf{x}, t)$,..., and so on. Therefore, Eq. (43) can be written as

$$d\tau = h(x)(1 - (v(x)/c)^2)^{1/2} dx^0, \tag{45}$$

where x represents the space and time coordinates of the moving particle with respect to the frame X.

The quantities $h(\mathbf{v}, \mathbf{a}, \dot{\mathbf{a}}, ...)$ and $(1 - (v/c)^2)^{1/2}$ are physical quantities related to the state of motion. In order to measure these physical quantities of motion, $\mathbf{v}$, $\mathbf{a}$, $\dot{\mathbf{a}}$,..., and so on, the units of space and time coordinates should be presumed in the beginning. Eq. (45) should not be thought of as a metric equation of the curved space-time coordinates in Einstein's general relativity such as the Schwarzschild solution of Einstein's gravitational field equations. *The theory presented here is not a metric theory as are both Einstein's special relativity and his general relativity of gravitation.*

2. Relativistic modification of Newton's law of motion.

2.1 The Unified Method Modifying the Newtonian Equation of Motion.

Consider a material particle of rest mass m moving under the influence of a given force field in an inertial frame X. It should be noted that the given force field is predefined by Newton's law of motion

$$\mathbf{F} = m\mathbf{a}. \tag{46}$$

In general, force fields can vary in time, but herein we will consider only time-constant (static) force fields. From the general relativistic equation of motion Eq. (38), in the low-speed limit $v \ll c$, we obtain

$$\frac{d^2x^i}{dt^2} = -c^2\Gamma^i_{00}. \tag{47}$$

This equation, Eq. (47), is the equation of motion for particles whose speeds are not comparable with the speed of light. Therefore, for a particle moving in the given force field $\mathbf{F}(\mathbf{x})$, Eq. (47) should be equivalent to the Newtonian equation of motion, which holds for particles whose speeds are not comparable with the speed of light,

$$\frac{d^2x^i}{dt^2} = F^i/m, \text{ or } \mathbf{a} = \mathbf{F}/m\,. \tag{48}$$

Returning to the general case of arbitrary speed, from Eq. (43) we have

$$d\tau^2 = h^2(x)\,((dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2). \tag{49}$$

Then, by Eqs. (33) and (49), we have

$$g_{\mu\nu} = h^2(x)\,\eta_{\mu\nu}\,. \tag{50}$$

From Eqs. (36), (37) and (50), we obtain

$$\Gamma^i_{00}(x) = \frac{-1}{2}\frac{\partial\phi(x)}{\partial x^i}, \tag{51}$$

where $\phi(x)$ is defined by

$$h(x) \equiv Exp[-\phi(x)/2]. \tag{52}$$

Then, from Eqs. (47), (48) and (51), we obtain

$$\frac{\partial\phi(x)}{\partial x^i} = \frac{2F^i(\mathbf{x})}{mc^2}. \tag{53}$$

From Eqs. (36), (37), (50), (51) and (53), we obtain

$$\begin{cases} \Gamma^0_{i0} &= \Gamma^i_{00} = \Gamma^i_{ii} = \Gamma^j_{ji} = -F^i/mc^2 \\ \Gamma^i_{jj} &= F^i/mc^2 \quad (j \neq i; \quad i,j = 1,2,3) \\ \Gamma^\lambda_{\mu\nu} &= 0 \quad (\text{otherwise})\;. \end{cases} \tag{54}$$

Since the force field $\mathbf{F}$ does not vary with time, the function $\phi(x)$ does not *explicitly* depend on time, that is, $\partial\phi(x)/\partial x^0 = 0$. Substituting $\Gamma^\lambda_{\mu\nu}$ of Eq. (54) into the general form of relativistic equation of motion Eq. (38), we finally obtain the new relativistic equation of motion for a material particle under an action of force $\mathbf{F}$

$$\frac{d^2\mathbf{x}}{dt^2} = \mathbf{a} = (\mathbf{F}/m)\,(1 - (v/c)^2). \tag{55}$$

The new relativistic equation of motion Eq. (55) gives a simple correction factor $1 - (v/c)^2$ to the Newtonian equation of motion Eq. (48). The accelerations of particles due to a given force are in the same direction as that of the given force. The new relativistic equation of motion contains Newton's law of motion as a low-speed limit. The accelerations of material particles by a given force decrease as the speeds of particles increase. No material particles can be accelerated to the speed of light. In the limit case that $v \to c$, the new relativistic equation of motion predicts that photons can not be accelerated by any force. Detailed applications of the alternative theory of relativity in describing motion of material particles in the electromagnetic field, or the gravitational field, are presented in the papers[4,7].

2.2 Comparisons between the New Relativistic Equation of Motion and Einstein's Relativistic Equation of Motion.

According to Einstein's theory of relativity, the definition for the force $\mathbf{f}$ acting on a particle of rest mass m and velocity $\mathbf{v}$ is given as,[2,8,9]

$$\mathbf{f} = d\mathbf{p}/dt\,, \tag{56}$$

where $\mathbf{p} \equiv \gamma m\mathbf{v}$, with $\gamma \equiv (1 - (v/c)^2)^{-1/2}$. Consequently, Einstein's relativistic equation of motion is obtained from Eq. (56) as[9,10]

$$\gamma m\mathbf{a} = \mathbf{f} - \mathbf{v}(\mathbf{v}\cdot\mathbf{f})/c^2. \tag{57}$$

From Eq. (57), we see that $\mathbf{a}$, $\mathbf{f}$, and $\mathbf{v}$ are coplanar; but in general the force vector and its associated acceleration vector will *not* be in the same direction as was the case in Newtonian mechanics. The force vector and its associated acceleration vector are parallel only in the following cases: (1) $v = 0$, (2) $\mathbf{f}$ is parallel to $\mathbf{v}$, or (3) $\mathbf{f}$ is perpendicular to $\mathbf{v}$. For these three extreme cases, we have $m\mathbf{a} = \mathbf{f}$ for the case (1), $\gamma^3 m\mathbf{a} = \mathbf{f}$ for the case (2), and $\gamma m\mathbf{a} = \mathbf{f}$ for the case (3).

In contrast, the new relativistic equation of motion Eq. (55) contains a simple scalar factor $1-(v/c)^2$ modifying Newton's law of motion. Thus, the direction of accleration is always along the direction of the force that causes the particle to accelerate. Also, the modification factor on the magnitude of acceleration does not depend on the orientation between the directions of velocity and force. It should be emphasized that the force $\mathbf{F}$ given in Eq. (55) of the alternative theory of relativity is the force predefined by Newton's law of motion Eq. (46). In contrast, the force $\mathbf{f}$ defined by Eq. (56) in Einstein's theory of relativity is not necessarily the same as the force $\mathbf{F}$ defined by Eq. (46) in Newtonian mechanics. The alternative theory provides a unified scheme to modify Newton's law of motion for particles moving in arbitrary force fields, for example, the electromagnetic field, or the gravitational field, as long as in those force fields Newton's law of motion is empirically valid in the nonrelativistic region. On the contrary, for particles moving in the electromagnetic field, Einstein's relativistic equation of motion is given in accordance with the Lorentz force law in Einstein's special relativity;[11] whereas for particles moving in the gravitational field, Einstein's relativistic equation of motion is given in accordance with Einstein's gravitational field equations in Einstein's general relativity.[2,3]

3. Conclusions.

To the best of this author's knowledge, no experimental evidence is reported for *directly* testing Einstein's relativistic equation of motion in the relativistic region.[12–14] Most physicists may think that many high-precision experiments in high energy physics, which are based on the Lorentz force law, have so far been performed without any discrepancy being found. Therefore, most physicists may be content to assume that the Lorentz force law is confirmed experimentally, and thus Einstein's relativistic equation of motion could be considered to be confirmed experimentally. However, we reexamined the muon $g-2$ experiments[15–17], which are often acclaimed as one of the high-precision experiments in high energy physics, and concluded, contrary to common belief, that the Lorentz force law is not convincingly confirmed by those experiments.[18]

The Lorentz force law is one of the most important laws in Einstein's theory. Many high-precision experiments, and thus theoretical models, in high energy physics depend upon the veracity of the Lorentz force law. Thus, even a slight discrepancy between the Lorentz force law and the *actual* electromagnetic force law in the relativistic region would cause serious conceptual problems for modern physics. Therefore, inquiries on the extent of empirical accuracy of the Lorentz force law, and thus questions on the validity of Einstein's relativistic equation of motion, should be greatly welcome.

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