Smooth Vortex-Like de Broglie Particle/Waves

T. Waite

13778 Shablow Avenue,
Sylmar, CA 91342, USA

ABSTRACT. The necessary and sufficient field equations are derived for all systems of well behaved fields which obey the relativistic conservation laws for mass (energy) and momentum, and include an electromagnetic field with sources, and which, in every Lorentz frame, include a $3D$ pseudovector field whose field lines are ($P1$) pairwise disjoint, ($P2$) closed or endless, ($P3$) conserved, and ($P4$) embedded in and moving point by point with the charge/current density $(\rho\,,\mathbf{j})$ at velocity $v = \mathbf{j}/\rho$. The conserved flux is analogous to, but different from a perfectly conserved vorticity field, and causes the charge/current density to segregate into pairwise disjoint droplets, with vortex-like integrity and permanence. The simplest such field theory imposes only the equation given by Einstein and by Pauli as necessary and sufficient for the existence of a Purely Electromagnetic Particle (PEP) with internal charge/current density. Contrary to physics dogma, there exist, and are presented here, a myriad of solutions to that equation with solitons, each centered on a droplet of vortex-like recirculating charge/current density. A hierarchy of other field equations is described, with example equations and solutions having similar properties. The soliton fieds in each level of the hierarchy have properties which suggest that the different views held by de Broglie, by Schroedinger and by the Copenhagen school may, in their essential features, be mutually consistent.

RESUME. On établit les équations de champs, nécessaires et suffisantes pour tous les systèmes qui obéissent à la conservation relativiste de la masse (énergie), et de l'impulsion, et qui comportent un champ électromagnétique interne, avec sources. Dans tout repère de Lorentz, ces équations comprennent un champ 3D pseudo-vectoriel ayant certaines propriétés bien définies. La conservation du flux apparaît semblable, mais diffère cependant de la conservation usuelle *d'un champ de vortex, ce qui entraîne l'existence de paires de particules chargées. La forme la plus simple de cette théorie donne les conditions d'existence de Particules Electromagnétiques Pures (PEP) ayant une densité interne de charge et de courant, selon une équation d'Einstein et Pauli. Contrairement à certains dogmes physiques, il existe un grand nombre de solutions à ces équations, s'apparentant à des solitons dans l'esprit de Schrödinger et de de Broglie.*

1. Introduction

Lochak [1] has compared the de Broglie and Schroedinger ideas regarding particle waves. Lochak recalls that "...de Broglie believed in a particle represented as a localized bump in the wave. He knew that such a stable feature can occur only in certain non-linear equations, and he often quoted as an example the *solitary waves*: Actually his bump-like wave was a soliton."

According to Schroedinger, the particle is the de Broglie wave with wavelength : $\lambda \approx \hbar/p$. Lochak [1] points out that, "For Schroedinger, a particle is not a permanent aspect of matter, but rather a type of response given by some experimental devices,..." In the bound states, emphasized by Schroedinger's work, particle integrity and permanence arise out of localization of the bound "standing wave". Schroedinger [2] attempted to describe the electron bound in stationary atomic states as a time independent charge/current density $(\rho, \mathbf{j})$ given by $\psi^*\psi$ and $\psi^*\nabla\psi - \psi\nabla\psi^*$. But with free particles, for which Schroedinger's time dependent wave equation gives probabilities for an ensemble of Hamilton-Jacobi trajectories, the linear wave lacks the integrity and permanence of non-linear solitons.

The Copenhagen school, and most physicists, adopt as their own, the methods and result of de Broglie and Schroedinger, but reject their interpretations in terms of physically real waves, because "the wave" can not be measured directly. To the Copenhagen School, only the particle properties, which Schroedinger attributed to the measuring process and measuring instruments, are physically real. To the Copenhagen school, the measurable properties are the point particles, and the wave function is a non-real (unmeasurable) probability amplitude.

The point particle concept is further supported by the fact that no direct experimental measurement of the charge/current density, or of any fields, inside a particle where charge/current density is non-zero, has ever been made. In spite of this, Barut [3,4], Enz [5], Rodrigues, et.al [6], Bostick [7], Honig [8] and O'Connell [9] are among those who have continued to investigate the consequences of assigning physical reality to the $3D$ particle/waves.

Barut [3] has considered an ensemble of wave packets, each with slightly different initial fields. Each obeys the de Broglie relationship, and each is consistent with initial particle measurements. If the wave packets are propagated one at a time by a deterministic wave equation through either a double slit or a Stern Gerlack field, they give ensemble statistics consistent with experiment. The fit to experiment is even better than that obtained by a single broad Schroedinger wave and the Hamilton Jacobi (diffraction) computation of probable trajectories. If this result holds true when non-linear solitons replace Barut's linear wave packets, then the system of soliton fields presented here may provide an alternate method of computing quantum probabilities and relating them to results obtained by direct quantization.

The purpose of this work is to present a new class of soliton field equations (with example equations and solutions) for vortex-like relativistic solitons. These fit de Broglie's soliton concept because there is a smooth $3D$ "bump" of conserved flux embedded in and defining a smooth droplet of charge/current density. This droplet is analogous to the smoke in a smoke ring vortex with its embedded vorticity field. The charge/current density in the vortex-like droplet generates an extended electromagnetic self field which falls to zero only at infinity. This self field is analogous to the Biot Savart velocity field generated by the localized vorticity field of a smoke ring vortex. The mass (energy) and momentum of the vortex-like solitons are in the wave-like self fields, so that propagation is wave-like. But particle-like integrity and permanence follows from the vorticity-like conserved flux embedded in, and foliating the droplet of charge/current density.

These vortex-like solitons occur in systems of fields which :

(F1) at each point are finite, single valued, and continuous functions, differentiable as many times as required for the field equations.

(F2) obey the conservation laws of relativistic mechanics (fields and equations invariant under the Poincaré group),

(F3) include a gauge invariant electromagnetic field with charge/current density sources (gauge invariant fields and field equations),

(F4) are invariant under the CPT transformation, (F5) include, in every Lorentz frame, a permaflux field which is a dynamic $3D$ pseudovector field whose field lines are :

(P1) pairwise disjoint,

(P2) closed or endless,

(P3) conserved absolutely,

(P4) embedded in, and moving point by point with, the charge/ current density $(\rho, \mathbf{j})$ at velocity $\mathbf{v} = \mathbf{j}/\rho$, which causes the charge/ current density to segregate into pairwise disjoint droplets, each with vortex-like integrity and permanence.

This work defines the necessary and sufficient field equations for any and all systems of fields with properties $(F1) - (F5)$. Those field equations are an underdetermined set. Auxiliary equations must be imposed to define well posed problems with unique solutions.

The auxiliary equations must define the non-electromagnetic part of the total system Stress Energy Momentum (SEM) tensor in terms of the fields; and they must provide field equations adequate to determine any non-electromagnetic fields introduced. The non-electromagnetic part of the SEM tensor must couple to the electromagnetic part only throught he Lorentz force density, which gives the rate of conversion of electromagnetic mass (energy) and momentum into non-electromagnetic mass (energy) and momentum. The determination of these auxiliary equations is equivalent to determination of the non-electromagnetic part of the Lagrangian for the pre-quantized fields in quantum field theory, except for the imposition of properties $(F1)$ and $(F5)$.

Property $(F1)$ eliminates point particles and $1D$ strings, because they must exist at infinite valued singularities inf the fields. But field property $(F5)$ and the permaflux field with properties $(P1) - (P4)$ result in vortex-like solitons with natural approximations as $1D$ strings. Solitons approximated as closed strings have further approximations as point particles. The point particles or strings approximations may then be quantized, if their coupled fields are simultaneously quantized as boson fields. The quantization restores soliton wave properties to the particles, and restores soliton particle properties to the boson fields.

But quantization does something more. Quantization defines a limited set of observables which correspond to integrals or averages over a single soliton, such as soliton total charge, total mass and average (mass or charge) centroid position. Quantization introduces the fundamental measurement errors imposed by the uncertainty principle. Quantization imposes indeterminacy, and makes it possible to predict only probabilities for an ensemble of experiments, each of which has the same initial measured values for observables within uncertainty principle errors.

These vortex-like soliton fields are excellent candidates for direct quantization and might eliminate the infinite self-energy problem arising from quantization of their string and point particle approximations. Of greater interest here, these soliton fields present some hope for unifying de Broglie and Schroedinger's ideas about the particle/wave, and further hope for using those ideas to understand the uncertainty and indeterminacy axioms of the Copenhagen School.

The simplest system of fields and field equations with properties $(F1) - (F5)$ arise if the only fields are the electromagnetic fields. The non-electromagnetic SEM tensor is then set equal to zero, indicating the absence of non-electromagnetic mass (energy), momentum and forces. The resulting fied equation is that given by Einstein [10] in 1919, and by Pauli [11] in 1921 as necessary and sufficient for the existence of a Purely Electromagnetic Particle (PEP) with internal charge/current density. For more than eighty years, physics dogma has held that no such PEP solutions can exist. But there is an exception to the assumption that the charge/current density is a time-like vector field, as assumed in the Poincaré/Ehrenfest PEP non-existence theorems (for a review, see ref.[11]). Sections (3) through (5) below present a myriad of solutions to the Einstein/Pauli PEP equation which include vortex-like solitons with internal charge/current density undergoing superphotonic flow. The component of flow parallel to the embedded permaflux field lines is always superphotonic; and it is always a pure recirculation along these closed or endless field lines, so that it does not change the charge/current density as a function of time. As a result, that component of flow does not violate causality and it is not damped by radiating. The component of flow perpendicular to the embedded permaflux field lines is always subphotonic (causal) and is responsible for all translation, rotation, expansion, contraction and deformation of the charge/current density droplet.

Section (4) gives an exact analytic solution to the Einstein/Pauli PEP equation for an isolated PEP charge/current density droplet stationary in a uniform magnetic field. It may be Lorentz transformed to give a PEP droplet propagating through orthogonal uniform magnetic and electric fields. As such, it is clearly de Broglie's "bump in the wave". The wave is the self field of the droplet. All mass (energy) and momentum possessed by the propagating PEP soliton are in its self fields as $\int\int\int(1/2)(E^2_{self}+B^2_{self})dV$ and as $\int\int\int(\mathbf{E}_{self}\times\mathbf{B}_{self})dV$. Therefore, the PEP droplet propagates as a wave so long as the Einstein/Pauli PEP equation is obeyed.

Bound state PEP solitons analogous to Schroedinger's standing waves exist, also. Section (5) gives an exact analytic solution to the Einstein/Pauli PEP equation for time independent PEP solitons, symmetrically distributed about a point magnetic dipole in a uniform magnetic field. The infinite mass of the binding or "bottle" potential seems to be essential to the existence of a localized time independent PEP droplet. The self field of the droplet is analogous to Schroedinger's standing wave, but the charge/current density is localized in a spherical shell droplet with vortex-like recirculation around the binding point dipole.

Section (10) illustrates the essential topology of the most elementary PEP charge/current density droplet bound in a point coulomb potential. It describes how such solutions to the Einstein/Pauli PEP equation might be generated by computer analysis. The ratio of PEP soliton total charge : $Q=\int\int\int\rho dV$, to the PEP total angular momentum about its center of mass : $\Omega=\int\int\int\mathbf{r}\times(\mathbf{E}_{self}\times\mathbf{B}_{self})dV$, is a dimensionless scalar (or topological invariant) of each such axially symmetric, time independent solution. For the simplest solution, it would be consistent, but shocking, if $Q^2/(2\Omega)$ proves to be equal to $e^2/\hbar\cong(137)^{-1}$.

The Einstein/Pauli equation is derived in Section (2) as the simplest possible system of fields with properties $(F1)-(F4)$, ignoring property $(F5)$. It was in an evaluation of the properties of the vortex-like solutions to the Einstein/Pauli PEP equation that the concept of a Permaflux field essential to field property $(F5)$ was discovered. Section (8) derives the Permaflux Theorem which gives the necessary and sufficient field equation for a permaflux field. Section (9) gives the Vorton Conservation Theorem with vortons defined as vortex-like particles or solitons. This theorem provides the equation and concepts needed to define the PEP Hierarchy of systems of fields with properties $(F1)-(F5)$. The field equations for the three lowest levels of that hierarchy are given in Section (9), with examples of exact analytic solutions at each level. All systems of fields in this hierarchy possess the qualitative properties given above, which suggest a possible unification of the de Broglie and Schroedinger viewpoints.

The charge/current density droplets in solutions to the field equations at each level in this hierarchy are pairwise disjoint in accordance with field property ($F5$) and properties ($P1$) and ($P4$) of the permaflux field. Assume for a moment that there exist Intra System Observers (ISO) in such a system. The ISO, their measuring instruments and their probe particles are all comprised of vortons, the vortex-like solitons. The ISO can not possibly measure any fields inside the charge/current droplet of any vorton, because the measured droplet is topologically pairwise disjoint from the measuring or probe vorton droplet. The ISO can only measure integrals and averages over the vorton droplet, such as total charge, total mass, centroid (mass or chage) average position, etc. To the ISO, each vorton droplet is a point particle which propagates as a wave. It has only a limited set of measurable "observables", and there are fundamental uncertainties in those integrals and averages.

The ISO can not use the deterministic field equations to predict future values of observables from past measurements, because the initial fields inside the vorton droplets could not have been measured as required for time integration. One must integrate the deterministic field equations for each member of an ensemble of systems. Each member has initial internal fields consistent with the initially measured integrals and averages, and their fundamental uncertainties. If the vorton solitons obey the de Broglie relationship, one might expect those ensemble statistics, in accordance with the analyses of Barut [3], to agree with experiment. A single broadened wave, typical of the ensemble, may, after propagation through the experiment, give Hamilton Jacobi vorton trajectories in agreement with the soliton ensemble. If it does, then quantum field theory might follow as a Bohr correspondence principle limit, by approximating the ensemble of solitons with a single typical wave. Since there are an infinity of levels in the PEP hierarchy, each with different field equations, not every level can give the desired result. However, because of the fundamental nature of field properties $(F1) - (F4)$, and the natural approximations as point particles and strings arising from property ($F5$), it seems likely that at least one level in the hierarchy, if it contains solutions corresponding to an ISO, will, from the ISO viewpoint, be in agreement with the Copenhagen school postulate and quantum field theory. The qualitative similarity between ISO measurement limitations and the Copenhagen school axioms, further suggests that these theories with hidden (ISO unmeasurable) subspaces may be an exception to Bell's hidden variable theorem.

2. PEP as the Base of the Hierarchy

There exists a hierarchy of different field equations each describing a system of fields with field properties $(F1)-(F5)$. Initially only properties

$(F1)-(F4)$ are considered. Property $(F5)$ is considered in Sections (8)-(11).

Property $(F2)$ requires the existence of a symmetric, second rank $4D$ tensor : $S^{jk} = S^{kj}$, with : $S^{44} \geq 0$, whose divergence vanishes :

$$S^{jk}_{,j} = 0 \tag{1}$$

Then S^{jk} may be defined as the total system SEM tensor. Generally, S^{jk} may be a function of any or all other fields in the system. Here indices take on values $(1, 2, 3, 4)$, the metric signature is $(---+)$, terms are summed on repeated indices, and indices after a comma indicate covariant derivatives.

One may then define an angular momentum density tensor:

$$M^{ijk} = S^{ij}x^k - S^{ik}x^j$$

which is divergenceless by its definition. Mass (energy) and momentum are conserved by virtue of these definitions.

All of electromagnetic theory, as required by property $(F3)$, follows from the existence of a $4D$ vector potential A_j, and a series of gauge invariant definitions. The electromagnetic field is defined as: $F_{jk} \equiv A_{kj} - A_{j,k}$. The charge current density is: $J^k \equiv F^{jk}_{,j}$. The electromagnetic SEM tensor density is: $T^j_k \equiv F^{jm}F_{mk} - (1/4)(F^{mn}F_{nm})\delta^j_k$. The non-electromagnetic SEM tensor density is: $W^j_k \equiv S^j_k - T^j_k$. Free space is defined as a region where $J^j = 0$.

By virtue of these definitions, Maxwell's first set of equations: $\epsilon^{jklm}F_{kl,m} = 0$ are obeyed everywhere, and Maxwell's free space equation: $F^{jk}_{,k} = 0$ is obeyed in free space where: $J^j = 0$. Charge is conserved $J^j_{,j} = 0$ by virtue of the definition of J^j. None of these definitions imposes any constraint on, or equation to determine, the electromagnetic potential A^j and its sources J^j.

By virtue of the conservation law in Eq.(1) and the definitions of T_{jk} and F_{jk}, the non-electromagnetic fields couple to the electromagnetic fields only where $J^j \neq 0$, and only through the Lorentz force density:

$$W^j_{k,j} = S^j_{k,j} - T^j_{k,j} = J^j F_{jk}$$

The rate of conversion of electromagnetic mass (energy) and momentum into non-electromagnetic mass (energy) and momentum is:

$W^j_{k,j} = J^j F_{jk}$. The field W^j_k, like S^j_k, may be a function of any and all fields in the system, so long as: $W^j_{k,j} = J^j F_{jk}$.

The above abstract analysis has required only the existence of S^{jk} and A^j and fields defined in terms of them, and only the equations $S^{jk} = S^{kj}$ and $S^{jk}_{,j} = 0$. The fields and equations satisfy all of properties $(F1) - (F4)$.

The simplest theoretical model based on the fields S^{jk} and A_j, and on the field equations $[S^{jk} = S^{kj}]$ and $[S^{jk}_{,j} = 0]$, is defined by using the simplest possible function for the non-electromagnetic SEM tensor $[W^j_k]$. That simplest function is $[W^j_k = 0]$. This simplest system has no non-electromagnetic masses (energy), momentum or forces. The single field equation is the conservation law :

$$S^j_{k,j} = T^j_{k,j} = -J^j F_{jk} = (A^{j,m}_{,m} - A^{mj}_{,m})(A_{j,k} - A_{k,j}) = 0 \qquad (2)$$

It is a four component equation, only three of which are independent, and it determines the four components of A_j to within the gradient of an unknown scalar gauge field.

Eq.(2) is satisfied identically by solutions to Maxwell's free space equations in "free space" where $[J^k = 0]$. The solutions of interest here are those with regions in which $[J^k \neq 0]$.

Eq.(2) is the equation given by Einstein [10] and by Pauli [11] as necessary and sufficient for existence of a Purely Electromagnetic Particle (PEP) with internal charge/current density. Pauli [11], in reviewing the history of the relativistic electron, considered some properties of a localized charge/current density $[\mathbf{j}\rho]$. These included total charge: $[Q = \int\int\int \rho dV]$ and average velocity: $[\mathbf{V} = Q^{-1} \int\int\int j dV]$ as well as total mass (energy) and momentum:

$$\begin{aligned} M &= \int\int\int (1/2)(B^2_{self} + E^2_{self}) dV \\ \mathbf{P} &= \int\int\int (\mathbf{E}_{self} \times \mathbf{B}_{self}) dV \end{aligned} \qquad (3)$$

where $\mathbf{E}_{self}$ and $\mathbf{B}_{self}$ are the self fields generated by $[\mathbf{j}, \rho]$. Pauli then imposed external fields $[\mathbf{E}_{ext}, \mathbf{B}_{ext}]$ and he showed, without regard to stability and spreading of the distribution $[\mathbf{j}, \rho]$, that:

$$\dot{\mathbf{P}} = Q[\mathbf{V} \times \mathbf{B}_{ext} + \mathbf{E}_{ext}]$$

if and only if:

$$\mathbf{j} \times \mathbf{B} + \rho\mathbf{E} = \mathbf{j} \times (\mathbf{B}_{self} + \mathbf{B}_{ext}) + \rho(\mathbf{E}_{self} + \mathbf{E}_{ext}) = 0 \qquad (4)$$

This is the Pauli/Einstein PEP Eq.(2), omitting the dependent fourth component $[\mathbf{j} \cdot (\mathbf{E}_{self} + \mathbf{E}_{ext}) = 0]$. In this system, the electromagnetic mass (energy) and momentum in the self fields of the charge/current density provide the mechanical mass (energy) and momentum ordinarily identified as "non-electromagnetic".

Within sentences of presenting the PEP Eq.(2), Einstein and Pauli each concurred in the earlier conclusions of Poincaré and Ehrenfest (for a review see ref.[11]) that a PEP with non-zero charge/current density cannot exist. That conclusion was based on the assumption that $[det|F_{jk}| \neq 0]$, or that one need consider only the electric field term in the PEP Eq.(2) or (4). The term $[\mathbf{j} \times \mathbf{B}]$ was ignored on the assumption that radiation damping prohibits a steady recirculating current j; or the assumption that the current is time-like $[j^2 < \rho^2]$ so that there exists a Lorentz frame appropriate to each point in which $[\mathbf{j} = 0, \rho \neq 0]$ in Eq.(4) at that point. In spite of the plausibility of these negative assumptions, Sections (3) - (5) present a myriad of exact analytic solutions to the PEP Eq.(2) with regions in which $det|F_{jk}| = 0$ and $[\rho^2 \leq j^2 \neq 0]$ without violations of causality and without radiation damping of the charge/current density. Each solution is a generalized soliton with each soliton centered on a compact $3D$ subspace filled with a vortex-like recirculating charge current density.

The PEP fields, outside the vortex-like compact $3D$ subspace, are electromagnetic fields obeying Maxwell's linear free space equations. Quite generally, those external fields are well approximated if the compact $3D$ subspace is replaced by fields with infinite valued singularities on $1D$ strings or on singular points within the subspace. This eliminates the regions where the field equations are non-linear. These strings or point particles may then be assumed to inherit the properties of the vortex-like compact subspace and of the self fields generated by the sources in it. This correspondence principle limiting approximation replaces the density distributions in the compact subspaces where $[J^k \neq 0]$, by integrated and average soliton properties, such as Q, M and $\mathbf{P}$ in Eqs.(3), which are then identified as properties of the point particles or strings. But it introduces infinite self energy due to the coupling of the singular point particles or strings as sources to the fields at the infinite valued singularities.

The existence of the steady state vortex-like solutions illustrated in Sections (3)-(5) below suggests that the PEP Eq.(2) supports the existence of a conserved flux field and vortex-like solitons in it, analogous to perfectly conserved fluid vorticity. In Sections (8) and (9), this is proven to be the case, with general theorems which support the generation of a hierarchy of fields and field equations with the same remarkable properties. That hierarchy of systems is more completely described in Section (11) with examples from other levels.

3. Cylindrical PEP Solitons

The simplest illustration of solutions with vortex-like character is in cylindrical coordinates $[R, \Phi, Z, t]$. Consider those solutions which, in one singular Lorentz frame are independent of Φ, Z and t, being functions of R only, with charge/current density zero outside of some infinitely long cylinder of finite radius R^*. The charge/current density must then be a vortex-like flow inside this cylinder, around and along the axis of the cylinder.

For $R \leq R^*$, with $j_R = B_R = E_\Phi = E_z = 0$ the single constraining equation is $J_\Phi B_z - j_z B_\Phi + \rho E_R = 0$. This is easily converted to the linear equation:

$$\frac{d}{dR}(B_z^2) + R^{-2}\frac{d}{dR}(R^2 B_\Phi^2 - R^2 E_R^2) = 0 \tag{5}$$

for the three new variables (B_z^2), $(R^2 B_\Phi^2)$ and $(R^2 E_R^2)$. Maxwell's equations $\nabla \cdot \mathbf{B} = 0$, and $\nabla \times \mathbf{E} = 0$, and the other components of the PEP Eq.(4), are satisfied identically. If one imposes $B_z = 0$ at $R = R^*$, then the only fields present are the self fields generated by the charge/current density inside the cylinder of radius R^*. The fields for $R \geq R^*$ are: $j = 0$, $\rho = 0$, $B_R = B_z = E_\Phi = E_z = 0$ with $E_R = aR^{-1}$ and $B_\Phi = bR^{-1}$, where a and b are selected to match the internal fields at $R = R^*$. The field outside of R^* is exactly that of a charged $1D$ string of supercurrent along the Z axis at $R = 0$, irrespective of the actual charge/current distribution for $R \leq R^*$. The $1D$ string approximation for these highly symmetric solitons is completely general.

This highly symmetric system is under-determined, and rather arbitrary solutions for $R \leq R^*$ are possible. Except for simple null currents $(j^2 = \rho^2)$, they are chiral and vortex-like. This motivates one to inquire as to whether the conservation laws which give particle-like integrity to solitons in fluid vorticity fields may have an analog in this system. That proves to be the case where $[J^k \neq 0]$ as proven in Sections (8) and (9) below.

One simple solution to Eq.(5) is useful in studying properties of PEP fields. For $R \leq R^*$ where R^* is at the first zero of $J_0(\alpha R)$, the charge/current density is:

$$\begin{aligned} j_R &= 0 \\ j_\Phi &= \chi b J_1(\alpha R) \\ j_z &= b(1-v^2)^{-0.5} J_0(\alpha R) \\ \rho &= bv(1-v^2)^{-0.5} J_0(\alpha R) \end{aligned} \tag{6}$$

where $J_n(\alpha R)$ is the cylindrical Bessel function of order n and $\chi = \pm 1$ is the chirality of the helical fields. This charge/current distribution, with its self generated fields, is an exact solution to the PEP Eq.(2) for all constant values of the three parameters b, $R^*(\alpha)$ and v. The parameter v corresponds to Lorentz transforms of the $[v = 0]$ case to velocity v along the z axis. The limit $[v^2 \to 1]$, which is inaccessible, relates to the limit $[\rho^2 \to j^2]$ and field chirality (screw sense of helicity) switching from $[\chi = \pm 1]$ to $[\chi = 0]$, which is impossible.

4. Toroidal PEP Solitons

The above PEP soliton analogs to linear vortices are infinite in one dimension. One may generate exact analytic solutions to the PEP Eq.(2) which are analogous to ring-like vortices and their localized generalizations.

Consider only those solutions to the PEP Eq.(4) which, in one singular Lorentz frame, are time independent and have $\mathbf{E} = 0$ everywhere. Then $[\mathbf{B} \times \nabla \times \mathbf{B} = 0]$, which is equivalent to $[\nabla \times \mathbf{B} = f\mathbf{B}]$ with $[\mathbf{B} \cdot \nabla f = 0]$. This equation arises in other types of systems [12] and a complete set of exact analytic solutions for f constant have been given in spherical coordinates [13,14].

The solutions vanishing at infinity with constant f are re-derived here in a different form to emphasize the relationship of that set of solutions to the symmetry group of $[\mathbf{B} \times \nabla \times \mathbf{B} = 0]$. With f constant, one has:

$$\nabla \times \mathbf{B} = \chi \kappa \mathbf{B} \tag{7}$$

where $\chi = \pm 1$ is chirality representing the parity group, and $0 < \kappa < \infty$ is the soliton wave number which represents the length scale invariance. To find solutions representing the $3D$ rotation group, take the curl of Eq.(7) and substitute from Eq.(7) into the result, giving

$$\nabla \times \nabla \times \mathbf{B} = \kappa^2 \mathbf{B} \tag{8}$$

This is necessary but not sufficient for Eq.(7). It's familiar solutions are the spatial parts of the vector multipole fields. They are eigenfunctions of parity which occur in pairs of opposite parity. The curl of one gives $(\pm i\kappa)$ times the other member of the pair, so they do not satisfy Eq.(7). But the solutions to Eq.(7) are their chiral sum or difference:

$$\mathbf{B}^{\chi}_{nm} = \mathfrak{B}_{nm}(\kappa r, \theta, \phi) - i\chi \mathfrak{E}_{nm}(\kappa r, \theta, \phi) \tag{9}$$

where the magnetic multipole fields are given (cf. Jackson [16]) by:

$$\begin{aligned} \mathfrak{B}_{nm} &= i\kappa^{-1} \nabla \times \mathfrak{E}_{nm} \\ \mathfrak{E}_{nm} &= j_n(\kappa r) \mathbf{L} Y_{nm}(\theta, \phi) \end{aligned}$$

Here, $j_n(\kappa r)$ is the spherical (half order) Bessel function of order n; and $Y_{nm}(\theta, \phi)$ is the spherical harmonic of degree n and order m representing the $3D$ rotation group. The operator $\mathbf{L}$ is $[\mathbf{L} = (-ir \times \nabla)]$. The parameters $[\chi, n, m, \kappa]$ in $B^{\chi}_{nm}(\kappa r, \theta, \phi)$ may be used as a representation of the symmetry group of Eq.(7).

All solutions to the linear Eq.(7) are exact solutions to the non-linear PEP Eq.(2). Any linear combination of the solutions to Eq.(7) with the same values for κ and χ remain exact solutions to Eq.(7) and, therefore, to the PEP Eq.(2). Any Lorentz transform of any of these solutions becomes time dependent, the field $\mathbf{E}$ is non-zero and non-uniform, and the charge density is non-zero and non-uniform, but it remains an exact solution to the Lorentz invariant PEP Eq.(2).

The simplest solution of the above type, for the case $n = 1$, $m = 1$ in Eq.(9) is:

$$\begin{aligned} B_r &= 2bB^* \{(\kappa r)^{-1} j_1(\kappa r)\} \cos\theta \\ B_\theta &= bB^* \{(\kappa r)^{-1} j_1(\kappa r) - j_0(\kappa r)\} \sin\theta \\ B_\phi &= b\chi B^* \{j_1(\kappa r)\} \sin\theta \end{aligned} \tag{10}$$

with $\mathbf{E} = 0$, $\mathbf{j} = \chi\kappa\mathbf{B}$ and $\rho = 0$. The factor b emphasizes the scale invariance of the fields and may be adjusted to fit boundary conditions for isolated solitons.

The magnetic field line components in a meridional plane are illustrated in Figure (1). They are the familiar standing wave dipole radiation field lines, but here, the magnetic fied is time independent and has an azimuthal (ϕ) component, also. The solid circles in Figure (1) have radii R_n for which $j_1(\kappa R_n) = 0$. When Figure (1) is rotated around its polar axis, these circles form spheres which divide space into a central $3D$ sphere and concentric $3D$ spherical shells, each of which is a *vorton subspace.* The radial component of $\mathbf{B}$ and $\mathbf{j}$ vanish on the $2D$ sphere surfaces at $r = R_n$. Each closed or endless magnetic field line lies only in one *vorton subspace* and there is no current flow across the boundaries of a *vorton subspace.*

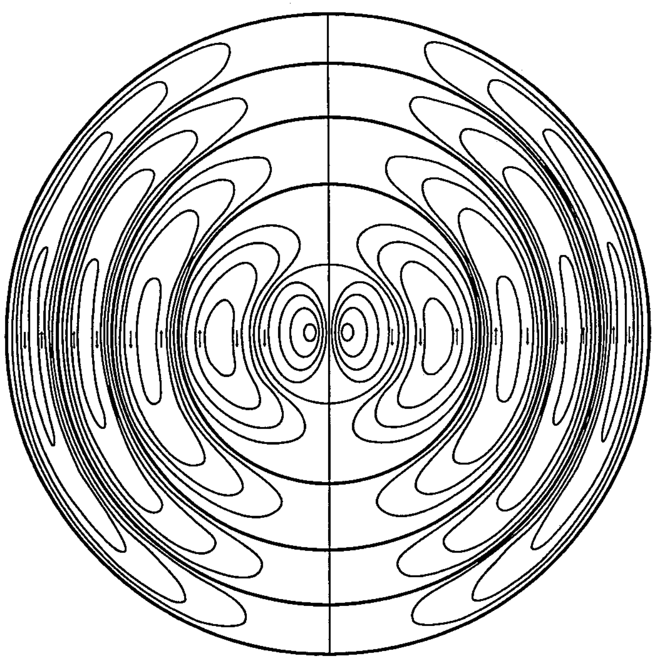

Figure 1. The meridional plane components of the magnetic field in the meridional plane, for the array of PEP solitons given by Eq.(10).

When Figure (1) is rotated around its polar axis, each set of nested ellipse-like closed curves sweeps out a set of nested $2D$ toroidal surfaces within each *vorton subspace.*

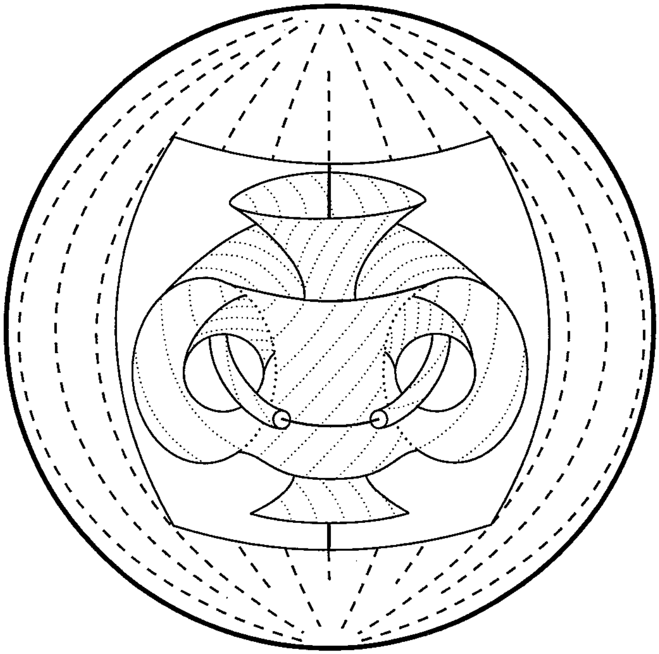

Figure 2. Schematic of magnetic field lines foliating the PEP in the central sphere of the array in Fig.(1) and Eq.(10), with some of the outer field lines fiolating $2D$ tori cut away to expose some of those inside in $3D$ perspective.

Figure (2) is a schematic of some of the magnetic field lines inside the central $3D$ sphere. They foliate and define the set of nested $2D$ toroidal surfaces in the central sphere. The $2D$ toroidal surfaces in turn fill and foliate the $3D$ sphere. In Fig.(2), portions of the $2D$ sphere and $2D$ toroidal surfaces are cut away to expose those inside. The winding number of field lines on each $2D$ toroid is constant, since field lines on it do not intersect. But winding number varies from infinity on the innermost nested toroid (degenerate as a $1D$ circle), to zero on the outermost (the sphere with its polar diameter as the degenerate toroid hole). Each magnetic field line within that $3D$ sphere links with every other within it. This is typical of a soliton *chiral vorton subspace.* Each concentric $3D$

spherical shell is a *chiral vorton subspace* with similar field line topology and interlinking. A *vorton subspace* and its *chirality* are defined more precisely and more generally in Sections (8) and (9) below.

One may extract the central $3D$ sphere vorton subspace, or any one of the $3D$ spherical shell vorton subspaces from the array, and generate an exact solution to the PEP Eq.(2) for an isolated soliton centered on the single vorton subspace. One simply replaces the remainder of the array by a solution to Maxwell's free space equations which matches the fields at the boundaries of the isolated vorton subspace retained. That free space field, in each case, is the self field of the sources in the retained vorton subspace, plus an externally applied "electromagnetic bottle" field. The "bottle" field, in the interior of the retained vorton subspace, replaces the external self fields of all the other vorton subspaces removed from the original array. The "bottle" field has sources at infinity or on a singular point at the origin.

The external free space field required to match the central $3D$ sphere *vorton subspace* of Fig.(1), for $r \geq R^* \cong 4,49/\kappa$ is:

$$\begin{aligned} B_r &= B^*[1 - (R^*/r)^3] \cos\theta \\ B_\theta &= B^*[1 + \frac{1}{2}(R^*/r)^3] \sin\theta \end{aligned} \tag{11}$$

with $B_\phi = 0$, $\mathbf{E} = 0$, $\mathbf{j} = 0$, $\rho = 0$. The solution given in Eq.(10) above for the array now applies only for $r \leq R^*$ with $b \cong 6.9$ to match Eq.(11) at $R = R^*$, the first zero of $j_1(\kappa r)$. The meridional plane components of some of the magnetic fields lines are illustrated schematically in Fig.(3).

The externally applied "electromagnetic bottle" field is a uniform magnetic field of magnitude B^* parallel to the soliton polar axis. The external self field of the sources inside the vorton subspace is identically that of a magnetic point dipole at the origin. The self field inside the vorton subspace generated by the retained vorton subspace sources is the field given in Eq.(10) for $[r \leq R_0]$, minus the applied uniform external field. Any Lorentz transform of this solution is an isolated soliton with non-uniform charge density propagating in orthogonal, uniform magnetic and electric fields.

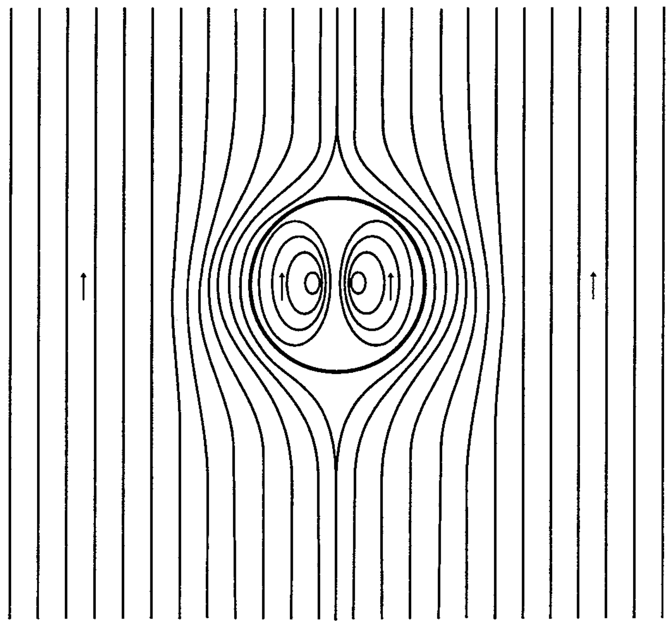

Figure 3. Schematic of meridional plane components of the magnetic field in the meridional plane of the central sphere PEP of Fig.(2) and Eq.(10), after that PEP has been removed from the array and placed in a uniform external magnetic field.

The total soliton consists of: (1) the vorton subspace containing the sources; and (2) the self fields which fill the vorton subspace and extend outward falling to zero only at infinity. The soliton mass given by Eq.(3), and momentum given by Eq.(3) exist entirely in this wave-like self field which determines soliton propagation. However, the Permaflux Theorem proven in Section (8) below, when applied to the PEP Eq.(2), assures that these PEP solitons with vorton subspaces have a particle-like topological integrity and permanence in random interactions of any magnitude, so long as the PEP Eq.(2) is obeyed. A soliton chiral vorton remains particle-like even for solutions unstable in the conventional sense, and even if they become chaotic. This is proved in Sections (8) and (9) below.

Because this isolated stationary soliton has an external self field equal to that of a point magnetic dipole, this soliton has a natural approximation as a point particle with magnetic point dipole moment (or as a circular closed string of supercurrent). That approximation linearizes the field theory, but adds the mechanics of relativistic point particles coupled to the field. Both the point particle or closed string approximations are entitled to inherit from the soliton its: (1) particle-like integrity and permanence; (2) its chirality; (3) its rest mass; and (4) magnetic moment. But the infinite energy in the field of the point particle approximation must be ignored [or re-normalized] in both the classical and quantized systems.

5. PEP Bound States

Consider a massive point particle with a point magnetic dipole moment μ^* in a uniform magnetic field B^* parallel to its axis. The above solution in Eq.(10) makes it possible to define discrete bound states for purely electromagnetic solitons bound to that massive point dipole. Let x_n be the solutions to $j_1(x_n) = 0$, with $x_0 = 0$. For $(2m+1)$ bound solitons, use the solution of Eq.(10) for the region $R_{2n} \leq r \leq R_{2n-2m-1}$ where $R_n = x_n/\kappa$. For the region $r \geq R_{2n}$ use the solution of Eq.(11) with $R^* = R_{2n}$. For the region $r \leq R_{2n-2m-1}$ use :

$$\begin{aligned} B_r &= \mu^*(R_{2n-2m-1})^{-3}[1 - (R_{2n-2m-1}/r)^3] \cos\theta \\ B_\theta &= \mu^*(R_{2n-2m-1})^{-3}[1 + (1/2)(R_{2n-2m-1}/r)^3] \sin\theta \end{aligned} \tag{12}$$

with $B_\phi = 0$, $\mathbf{E} = 0$, $\mathbf{j} = 0$, $\rho = 0$. One may then adjust b and κ (which also adjusts R_{2n} and $R_{2n-2m-1}$) to make the fields continuous at R_{2n} and $R_{2n-2m-1}$.

These discrete bound states for a bound soliton (or any odd integer number of bound solitons) are time independent. They are non-radiating because they have no time dependent multipole moment. These solutions are not intended to represent anything in the real world. But they do establish that time independent bound states for purely electromagnetic solitons are possible, simply by permitting the soliton to form a steady state, charge/current distribution symmetrically around the binding source. The PEP Eq.(2) requires that distribution to take a form in which self field forces plus external binding forces cancel everywhere. Rather interesting flowfield topology problems result.

It is no longer possible to find a plausible approximate solution for the soliton as a non-radiating bound point particle. The assumption that radiation damping prevents permanent recirculation of a point particle in the Bohr Rutherford atom is valid. But there may be truth in Schroedinger's [2] conjecture, that the wave-particle duality of a soliton permits such permanent recirculation. Schroedinger [2] noted that mixing degenerate atomic eigenfunctions with complex coefficients describes a "stationary current distribution" in non-radiating atoms, and commented, "...we may in a certain sense speak of a return to electrostatic and magnetostatic atomic models...". If states of opposite parity are mixed, as in Eq.(9) above, the charge/current density is chiral and vortex-like. If any such steady state intra-atomic charge/current densities exist, obey Maxwell's equation, and obey the conservation laws of relativistic mechanics, then they must, in the absence of non-electromagnetic forces, obey the PEP equation for a time independent charged PEP soliton in a point coulomb field as "electromagnetic bottle". For the time independent case, one may express the self fields in Eq.(4) as integrals over the steady state charge and current densities to obtain, in the presence of a point coulomb field:

$$\mathbf{j}\times\int\left[\mathbf{r}_{12}\times\mathbf{j}(r_2)/r_{12}^3\right]dV_2+\rho\left\{Q\mathbf{r}/r^3+\int[\mathbf{r}_{12}\rho(r_2)/r_{12}^3]dV_2\right\}=0 \quad (13)$$

Here $r_{12}=(r_2-r)$ and $\int\rho dV=-Q$ for a neutral system. The fields are required to vanish as $r\to\infty$.

One may define a $3D$-vector field $v=j/\rho$ and the normalized scalar density $\rho^*=\rho/Q$. Then one may replace ρ by $(Q\rho^*)$ and j by $(Q\rho^* v)$ in Eq.(10), giving an equation for the field v and the normalized density ρ^*. The equation for this time independent neutral system is now independent of the scale of Q and is invariant to scale changes in the coordinates. It is also invariant to rotations on $3D$ space and to parity transformations. One desires to find the solutions which vanish at infinity. The topology of the simplest PEP vorton with total charge, total mass and total spin angular momentum equal to those of an electron is described in Section (10) below. The simplest steady state bound state solution to Eq.(13) should be a PEP vorton of that structure centered about the binding coulomb point charge.

The PEP Eq.(2) for the time independent purely magnetic case (in that one Lorentz frame), when no external magnetic fields are present, can be written in the form of Eq.(13) with $\rho=0$ in the second term. That case, described in Sections (4) and (5) above, was also invariant to the scale of the fields, to the scale of the coordinates, and to rotations in $3D$ space and to the parity transformation. Consequently its solutions were a representation of these groups. Each solution had two continuous scale parameters for field and length scale invariances [the field magnitude B^* and the wave number $\kappa = 4.49/R_0$ in Eqs.(9) and (10)]. There were two solutions of opposite chirality corresponding to each spherical harmonic representing the $3D$ rotation group. The two solutions of opposite chirality were permuted by the parity operator, but were not eigenfunctions of parity in this non-linear theory.

Eq.(13) with the coulomb field is invariant under these same groups, and is the same equation with one more term and one more dependent variable. One expects that its solutions will be representations of the same groups, with two continuous scale parameters [Q and κ], with pairs of solutions with opposite chirality, and with solutions for each spherical harmonic. However, Eq.(13) has one important change from the purely magnetic case. A fourth variable ρ has been introduced without increasing the number of equations. The system may be underdetermined with a plethora of solutions, absent some additional constraint.

Any solution to Eq.(13), if such exist, will have at least one physically significant dimensionless topological invariant. It will be $Q^2\{|\int r \times (\mathbf{E} \times \mathbf{B}) dV|\}^{-1}$. It is the ratio of total charge squared to total angular momentum, and, for real world quantized systems, is related to $[e^2/\hbar]$. In less symmetric solutions, the scalar angular momentum must be computed from the conserved scalar $[\Omega^2 = \Omega^{jk}\Omega_{jk}]$ where $[\Omega^{jk} = \int\int\int M^{4jk} dV]$ and $[M^{ijk} = T^{ij}x^k - T^{ik}x^j]$.

6. PEP Wave Equations

The solutions presented above suggest that there may be alternate forms of the PEP equation which are more readily solved or which more clearly relate to Schroedinger's conjecture. Since $[\mathbf{E} \cdot \mathbf{B} = 0]$ in regions where $[J^j \neq 0]$, one may resolve $\mathbf{j}$ onto the moving orthogonal triad $[\mathbf{E}, \mathbf{B}, \mathbf{E} \times \mathbf{B}]$. The result is:

$$\begin{aligned} \mathbf{j} = \nabla \times \mathbf{B} - \dot{\mathbf{E}} &= \chi\kappa\mathbf{B} + \rho\mathbf{E} \times \mathbf{B}/B^2 \\ \mathbf{E} \cdot \mathbf{B} &= 0 \\ \chi &= \pm 1 \\ \kappa &= (\sqrt{j^2B^2 - \rho^2E^2}/B^2 \end{aligned} \tag{14}$$

Here, χ in $[\chi\kappa = \mathbf{B}\cdot\mathbf{j}/B^2]$ indicates whether $\mathbf{B}$ has a component parallel or antiparallel to $\mathbf{j}$. Where $[\mathbf{j} = \nabla\times\mathbf{B} - \dot{\mathbf{E}} \neq 0]$, usually $\nabla\times\mathbf{B} \neq 0$. Then $[\chi\kappa = \mathbf{j}\cdot\mathbf{B}/B^2 \neq 0]$ implies $[\mathbf{B}\cdot\nabla\times\mathbf{B} \neq 0]$, so that the magnetic field has circulation about its vector direction, giving its field lines helicity with chirality χ.

If one takes the curl of Eq.(14), one obtains:

$$\begin{aligned}\nabla\times\nabla\times\mathbf{B} + \ddot{\mathbf{B}} = \kappa^2\mathbf{B} + \chi(\nabla\kappa)\times\mathbf{B} + \chi\kappa\dot{\mathbf{E}} \\ + \chi\kappa\rho\mathbf{E}\times\mathbf{B}/B^2 + \nabla\times[\rho\mathbf{E}\times\mathbf{B}/B^2]\end{aligned} \tag{15}$$

If one takes the time derivative of Eq.(14), one obtains:

$$\nabla\times\nabla\times\mathbf{E} + \ddot{\mathbf{E}} = \chi\kappa\nabla\times\mathbf{E} - \chi\dot{\kappa}\mathbf{B} - \frac{\partial}{\partial t}(\rho\mathbf{E}\times\mathbf{B}/B^2) \tag{16}$$

Time independent solutions (such as the bound states discussed above) with $[B^2 \gg E^2]$ present almost linear wave equations for $\mathbf{B}$ and $\mathbf{E}$, except for the troublesome topological constraints of $[\mathbf{E}\cdot\mathbf{B} = 0]$, which, with Eqs.(14) implies $[\mathbf{E}\cdot\mathbf{j} = 0]$, also.

The time independent solutions given by the chiral sum of vector multipole fields in Eq.(9) which satisfy Eq.(7) in one singular Lorentz frame, satisfy $(\chi\kappa)^{-1}\nabla\times\mathbf{B} = \mathbf{B} = \nabla\times\mathbf{A}$ and Eq.(15) with only the $\chi^2\mathbf{B}$ term on the right. Therefore, with proper choice of gauge, the potential A_j, which satisfies Eq.(7) in one singular Lorentz frame, satisfies:

$$A^k_{j,k} = \kappa^2 A_j \tag{17}$$

in every Lorentz frame in the Lorentz gauge $[A^j_{,j} = 0]$. This is the relativistic Schroedinger equation for a $4D$ vector particle [Proca Equation] with rest mass κ. It is necessary but not sufficient for the PEP solitons satisfying Eq.(7).

It seems plausible that the PEP self generated self interfering electromagnetic field, even if unmeasurable in regions where $[j^j \neq 0]$, may be related to a self interfering wave field which is approximated by Schroedinger's wave function or a relativistic analog. The more complex systems of fields above the PEP system in this hierarchy, as described in Section (11), present other possibilities for a similar interpretation.

7. The PEP Existence Theorem

Although a variety of exact analytic solutions to the PEP Eq.(2) have been presented, it would be encouraging if an existence theorem for more general solutions with conserved solitons could be established.

One may apply the Cauchy Kowalevsky existence theorem to the PEP Eq.(4), since it can be solved for the time derivatives of the fields in terms of the fields and their spatial derivatives. In regions where $[J^j = 0]$, Maxwell's free space equations satisfy the PEP Eq.(2) so that $[\dot{\mathbf{E}} = \nabla \times \mathbf{B}]$ and $[\dot{\mathbf{B}} = -\nabla \times \mathbf{E}]$ with $[\nabla \cdot \mathbf{B} = 0]$ and $[\nabla \cdot \mathbf{E} = 0]$ as initial conditions.

In regions where $[J^j \neq 0]$, $[\dot{\mathbf{B}} = -\nabla \times \mathbf{E}]$ still applies. However, $\dot{\mathbf{E}}$ must be derived from the PEP equation $[\mathbf{j} \times \mathbf{B} + \rho\mathbf{E} = 0]$ and $[\mathbf{j} \cdot \mathbf{E} = 0]$. Since $[\mathbf{E} \cdot \mathbf{B} = 0]$, one may resolve $\dot{\mathbf{E}}$ onto the moving triad $[\mathbf{E}, \mathbf{B}, \mathbf{E} \times \mathbf{B}]$. Rotating the triple product in $[\mathbf{E} \cdot (\mathbf{j} \times \mathbf{B} + \rho\mathbf{E}) = 0]$ one obtains the component of $\dot{\mathbf{E}}$ parallel to $\mathbf{E} \times \mathbf{B}$. The component of $\dot{\mathbf{E}}$ along $\mathbf{E}$, is given directly from $[\mathbf{j} \cdot \mathbf{E} = 0]$. Since $[\mathbf{E} \cdot \mathbf{B} = 0]$ is independent of time, $[\dot{\mathbf{E}} \cdot \mathbf{B} = -\dot{\mathbf{B}} \cdot \mathbf{E} = \mathbf{E} \cdot \nabla \times \mathbf{E}]$ gives the component of $\mathbf{E}$ parallel to $\mathbf{B}$. Therefore, $\dot{\mathbf{E}}$ is given by:

$$\begin{aligned} \dot{\mathbf{E}} = &-\left[\frac{\mathbf{E} \cdot (\nabla \times \mathbf{B})}{E^2}\right]\mathbf{E} + \left[\frac{\mathbf{E} \cdot (\nabla \times \mathbf{E})}{B^2}\right]\mathbf{B} \\ &+ \left[\frac{\mathbf{E} \cdot [\mathbf{B} \times (\nabla \times \mathbf{B})] - E^2(\nabla \times \mathbf{E})}{E^2 B^2}\right](\mathbf{E} \times \mathbf{B}) \end{aligned} \tag{18}$$

The required initial conditions are now $\nabla \cdot \mathbf{B} = 0$ and $\mathbf{E} \cdot \mathbf{B} = 0$. One may prove that Eq.(18) is sufficient for the PEP Eq.(4) by substituting $\dot{\mathbf{E}}$ from Eq.(18) into Eq.(4) and using Maxwell's equations in the presence of sources. By the Cauchy Kowalevsky theorem, there will exist a time integrated solution to the PEP Eq.(2) for initial fields satisfying $[\nabla \cdot \mathbf{B} = 0]$ and $[\mathbf{E} \cdot \mathbf{B} = 0]$ at all points where $[\dot{\mathbf{B}} = -\nabla \times \mathbf{E}]$ and $\dot{\mathbf{E}}$ (as given by Eq.[18]) are real analytic functions.

Since $[\dot{\mathbf{E}} = \nabla \times \mathbf{B}]$ where $[J^j = 0]$, but $\dot{\mathbf{E}}$ is given by Eq.(18) where $[J^j \neq 0]$, one must identify the vorton subspace boundaries in the initial fields. Those boundaries must be moved point by point at the velocity $[v_N = (\mathbf{j}/\rho)_N = (\mathbf{E} \times \mathbf{B})_N/B^2]$ where the subscript N indicates the component normal to the boundary, while $\mathbf{j}$ and ρ are given inside the vorton subspace where $J^j \neq 0$. This boundary velocity follows from the flux conservation theorem proven below.

8. Flux Conservation

The steady state vortex-like solutions to the PEP equation strongly suggest an analog with classical vorticity conservation. That proves to be the case, except that, in the PEP system, flux conservation is perfect, there being no analog to viscosity, nor to the deviations from conservation caused by Euler's equation itself. The PEP conserved flux may be Lorentz transformed and is conserved in every Lorentz frame. This is established by the theorem in $4D$ differential geometry stated and proved in Appendix A, which was discovered in the evaluation of the PEP equation and its solutions. However, the meaning of the flux conservation equations are physically clearer, and they are more readily compared to vorticity, when presented as an integration of a formula derived by Sommerfeld [15].

A *permaflux field* with the topological properties of a perfectly conserved vorticity field *is defined* as a dynamic $3D$ pseudovector field whose field lines are : (P1) pairwise disjoint; (P2) closed or endless; (P3) conserved; and (P4) moving point by point at velocity v with a conserved entity σ which obeys $[\dot{\sigma} + \nabla \cdot (\sigma v) = 0]$.

A *field line representation* of an N-dimensional vector field is *defined* as a variable density foliation of N-dimensional space with 1-dimensional curves whose tangent at each point gives the direction of the vector field, and whose flux density at each point is proportional to the magnitude of the vector field.

Property (P1) of a permaflux field $\mathbf{G}$ follows because $\mathbf{G}$ is well behaved. It cannot point in two directions from a single point when $\mathbf{G} \neq 0$, as would be required for two field lines of $\mathbf{G}$ to intersect, even for a moment. One permaflux field line cannot move so as to cross another, as would be required to link or unlink. Property (P2) requires that $\nabla \cdot \mathbf{G} = 0$.

Sommerfeld [15], in his textbook "The Dynamics of Deformable Bodies", derives a geometric/kinematic formula which he attributes to Helmholtz, then uses it to derive necessary and sufficient equations for a permaflux field. Consider two time dependent $3D$ vector fields, $\mathbf{G}$ and $\mathbf{v}$. In a field line representation of $\mathbf{G}$, the flux of $\mathbf{G}$ (total number of G field lines) through an arbitrary $2D$ surface increment Σ, bounded by a closed $1D$ curve C, is given by:

$$N = \int\int [\mathbf{G}]_n d\Sigma \tag{19}$$

where the subscript n indicates the vector component normal to Σ at each point. If the $2D$ surface increment Σ moves point by point at velocity v, and maintains its continuity, Helmholtz's formula (derived by Sommerfeld [15]) gives the time rate of change of N due to motion and deformation of Σ, as well as due to space and time variations in $\mathbf{G}$. That rate is:

$$\delta N/\delta t = \int\int [\dot{\mathbf{G}} - \nabla \times (\mathbf{v} \times \mathbf{G}) + \mathbf{v}\nabla \cdot \mathbf{G}]_n d\Sigma \tag{20}$$

The continuity of Σ as it moves point by point at velocity v is assured by associating v with the flowfield of a conserved entity σ which obeys:

$$\dot{\sigma} + \nabla \cdot (\sigma v) = 0 \quad , \quad \sigma v \neq 0 \tag{21}$$

Eqs.(19)-(21) neither require nor imply any functional relationship between the fields $\mathbf{G}$ and $\mathbf{v}$.

Sommerfeld [15] considered only $\mathbf{G}$ fields with closed or endless field lines so that $\nabla \cdot \mathbf{G} = 0$. He noted that $(\delta N/\delta t = 0)$ everywhere for all arbitrary $2D$ surface increments Σ, moving point by point at velocity $\mathbf{v}$, *if and only if*, each and every field line of $\mathbf{G}$ is conserved through time, and moves point by point at the velocity $\mathbf{v}$ of earch arbitrary Σ which it penetrates. Therefore, a permaflux field $\mathbf{G}$ has properties (P2), (P3) and (P4), *if and only if*,

$$\nabla \cdot \mathbf{G} = 0 \tag{22}$$

$$\dot{\mathbf{G}} - \nabla \times (\mathbf{v} \times \mathbf{G}) = 0 \tag{23}$$

The field $\mathbf{G}$ is a permaflux field conserved and "dragged" by the flow field ($\sigma\mathbf{v}$, σ), if and and only if the functional relationships of Eqs.(21)-(23) are obeyed and $\mathbf{G} \neq 0$, $\sigma\mathbf{v} \neq 0$.

The symmetry group of Eqs.(21)-(23) includes a Lorentz group. Define the field:

$$\mathbf{F} \equiv -\mathbf{v} \times \mathbf{G} \tag{24}$$

and substitute into Eq.(23). Then Eqs.(22) and (23), analogous to Maxwell's equations, are the necessary and sufficient conditions for the existence of fields $\mathbf{Q}$ and q for which:

$$\nabla \times \mathbf{Q} = \mathbf{G} \tag{25}$$

$$-\dot{\mathbf{Q}} - \nabla q = \mathbf{F} = -\mathbf{v} \times \mathbf{G} \tag{26}$$

Taking the scalar product of Eq.(26) with ($\mathbf{v}\cdot$), one obtains:

$$\mathbf{v} \cdot (-\dot{\mathbf{Q}} - \nabla q) = \mathbf{v} \cdot \mathbf{F} = 0 \tag{27}$$

If one multiplies Eqs.(26) and (27) by σ, together they take the manifestly Lorentz invariant form:

$$P^j G_{jk} = 0 \quad , \quad G_{jk} \equiv (Q_{j,k} - Q_{k,j}) \tag{28}$$

where $P^j \Rightarrow (\sigma v, \sigma)$ and $Q^j \Rightarrow (Q, q)$ in each Lorentz frame. Here indices take the values $(1, 2, 3, 4)$; the metric signature is $(- - -+)$; expressions are summed on repeated indices; and indices after a comma indicate covariant derivatives. Then Eq.(21) gives:

$$P^j_{,j} = 0 \tag{29}$$

One may derive Eqs.(21), (22) and (23) from Eqs.(28) and (29). However, one must assure that the Σ surfaces, used to relate Eqs.(21), (22) and (23) to flux conservation, remain continuous as they move, and that they move forward in time in every Lorentz frame. With Eqs.(28) and (29), it is sufficient to require:

$$P^j \neq 0 \quad , \quad G_{jk} \neq 0 \quad , \quad G^{jk} G_{jk} \geq 0 \tag{30}$$

in the region of interest, since zero eigenvalue eigenvectors of a $4D$ antisymmetric $2D$ tensor always occur in pairs. Therefore, Eqs.(28) and (30) assure that one zero eigenvalue eigenvector of G_{jk} is time-like. That is, given Eq.(28), with $P_j \neq 0$, it follows that: $det|G_{jk}| = 0$, and that G_{jk} and its dual are simple bivectors:

$$G_{jk} = K_j L_k - L_j K_k = (1/2)\epsilon_{jklm}(S^l T^m - T^l S^m) \tag{31}$$

The vectors K_j, L_j,S_j and T_j may be chosen to be orthogonal, with T^j time-like or null: $T^j T_j \geq 0$, since: $G_{jk} G^{jk} \geq 0$. Obviously, $S^j G_{jk} = 0$ and $T^j G_{jk} = 0$ so that P^j lies in the plane of S^j and T^j. The vector orthogonal to P^j in that plane is: $R^j = \epsilon^{jklm} J_k G_{lm}$. Every vector of the form $U^j = \alpha S^j + \beta T^j$ satisfies $U^j G_{jk} = 0$. If one treats $U^j_{,j} = 0$ as a partial differential equation for α and β under the constraint that U^j be time-like or null, then non-unique solutions for α and β invariably exist such that $[U^j_{,j} = 0]$; $[U^j U_j \geq 0]$ and $[U^j G_{jk} = 0]$. If P^j in Eqs.(28) and (29) is space-like, one may replace it by U_j which is time-like or null. This assures that Sommerfeld's Σ surfaces remain continuous and move forward in time in every Lorentz frame if $G^{jk}G_{jk} \geq 0$ of Eq.(30) is satisfied.

In any Lorentz frame with the identifications $U^j \rightarrow (\sigma u, \sigma)$ and $G_{jk} \Rightarrow (\mathbf{F}, \mathbf{G})$, one has

$$\sigma \mathbf{u} \times \mathbf{G} + \sigma \mathbf{F} = 0 \qquad \sigma \mathbf{u} \cdot \mathbf{F} = 0$$

from which it follows that:

$$\mathbf{u} = (\mathbf{u} \cdot \mathbf{G}/G^2)\mathbf{G} + \mathbf{F} \times \mathbf{G}/G^2$$

The velocity field $\mathbf{u}$ gives the point by point velocity of the field lines of $\mathbf{G}$. Since velocity of a closed or endless field line of G everywhere parallel to itself does not displace the field line, the actual displacement velocity of the field lines of $\mathbf{G}$ is $(\mathbf{F} \times \mathbf{G}/G^2)$. This component of $\mathbf{u}$ is independent of the choice of α and β in $U^j = \alpha S^j + \beta T^j$, so that every $4D$ vector in the plane of S^j and T^j leads to the same displacement velocity $(\mathbf{F} \times \mathbf{G}/G^2)$ for the field lines of $\mathbf{G}$. But $G^{jk}G_{jk} \geq 0$ in Eq.(30) assures that $G^2 \geq F^2$ in every Lorentz frame, so that $|\mathbf{F} \times \mathbf{G}/G^2| \geq 1$ in every Lorentz frame. Disturbances can not propagate through the field G with velocity greater than one, the Lorentz limiting velocity. Since this is true in every Lorentz frame, disturbances can not propagate through the fields $\mathbf{F}$ or G_{jk} faster than the Lorentz limit. It follows that the condition $G^{jk}G_{jk} \geq 0$ in Eq.(30) is the causality condition, applicable even if P^j in Eq.(28) is space-like.

The above analysis has proven the theorem required to establish the field equations for property (F5) for the desired hierarchy of fields. It is: **The Permaflux Theorem** : GIVEN the existence of the $4D$ vector fields P_j and Q_j with the defined field $G_{jk} \equiv Q_{kj} - Q_{j,k}$ which in some region satisfy:

$$P^j G_{jk} = P^j (Q_{k,j} - Q_{j,k}) = 0 \tag{32}$$

$$P^j_{,j} = 0 \tag{33}$$

$$P^j \neq 0 \quad , \quad G_{jk} \neq 0 \quad , \quad G^{jk}G_{jk} \geq 0 \tag{34}$$

THEN in that region, with the definitions $P^j \Rightarrow (\sigma v, \sigma)$; $Q^j \Rightarrow (\mathbf{Q}, q)$; $G_{jk} \Rightarrow (\mathbf{F}, \mathbf{G})$, $\mathbf{G} \equiv \nabla \times \mathbf{Q}$ and $\mathbf{F} \equiv -\dot{\mathbf{Q}} - \nabla q$, in every Lorentz frame, the $3D$ pseudovector field $\mathbf{G}$ is a permaflux field whose field lines are:

(P1) pairwise disjoint, (P2) closed or endless, (P3) conserved, and (P4) moving point by point with a conserved entity σ at velocity $\mathbf{v}$ satisfying:

$$\mathbf{v} = \alpha\mathbf{G} + \mathbf{F} \times \mathbf{G}/G^2 \tag{35}$$

$$|\mathbf{F} \times \mathbf{G}/G^2| \leq 1 \tag{36}$$

$$\dot{\sigma} + \nabla \cdot (\sigma\mathbf{v}) = 0 \tag{37}$$

This theorem is proved again in Appendix A by $4D$ differential geometry. That proof emphasizes the $4D$ geometry and establishes a corollary which assures that there exists a one-to-one correspondence between permaflux field lines before and after Lorentz transformation and/or propagation through $4D$ space time.

Equations (32), (33) and (34) which are sufficient for the existence of a permaflux field are an under-determined set. An auxiliary equation is required to define well posed problems. Sommerfeld overlooked the Lorentz invariance, and he used only Euler's equation as the auxiliary equation with his Eqs.(22) and (23) in order to determine conditions under which perfect vorticity conservation occurs.

The above details regarding the $4D$ vector and tensor fields are important in applying the above results to the PEP system because the $4D$ current vector J^k is invariably space-like, or null, or zero in the PEP system. The auxiliary equation which must be imposed with the permaflux field Eqs.(32) and (33) in order to define the PEP system is:

$$P^j = Q^{j,m}_{,m} - Q^{m,j}_{,m} \tag{38}$$

which causes Eq.(32) to take the form

$$(Q^{j,m}_{,m} - Q^{m,j}_{,m})(Q_{j,k} - Q_{k,j}) = 0$$

This is the PEP Eq.(2) with $Q_j = A_j$. Then by the auxiliary Eq.(38), $P^j = J^j$. That is, the PEP system is a permaflux field given by Eqs. (32) and (33) with the auxiliary Eq.(38). In each Lorentz frame the other permaflux fields corresponding to electromagnetic entities are $G_{jk} = F_{jk}$, $\mathbf{F} = \mathbf{E}$, $\mathbf{G} = \mathbf{B}$, $\sigma\mathbf{v} = \mathbf{j}$ and $\sigma = \rho$.

All of the above results for permaflux fields apply to the PEP system in regions where Eq.(34) is satisfied corresponding to $[J^j \neq 0]$, $[F_{jk} \neq 0]$ and $[F^{jk}F_{jk} \geq 0]$ or $[B^2 \geq E^2]$. In such regions, Maxwell's equations continue to be obeyed, but in addition, the magnetic field lines **B** embedded in the charge/current density $(\mathbf{j}, \rho)$ are permaflux field lines $(\mathbf{B} = \mathbf{G})$ with properties (P1) to (P4) in the permaflux definition above. The field lines of **B** embedded in the charge/current density move with the charge density flow at velocity $(\mathbf{v} = \mathbf{j}/\rho)$. That velocity is greater than the velocity of light, but it is of the form:

$$\mathbf{v} = (\mathbf{j}/\rho) = (\mathbf{j} \cdot \mathbf{B}/B^2)\mathbf{B} + \mathbf{E} \times \mathbf{B}/B^2$$

with only the component parallel to **B** superphotonic. The component of **B** field line velocity parallel to **B** is a field line non-motion, and disturbances propagate through the **B** and **E** fields only at subluminal velocities because $[F^{jk}F_{jk} \geq 0]$ where $[J^k \neq 0]$. The component of charge flow parallel to **B** is a pure recirculation along the closed or endless field lines of **B**. There can be no net displacement of the charge/current density at superphotonic velocities because it would carry with it a disturbance in the **B** and **E** fields at superphotonic velocities which is prohibited by $[F^{jk}F_{jk} \geq 0]$ where $[J^k \neq 0]$. The superphotonic component of PEP current **j** parallel to **B**, is a non-motion which generates the magnetic forces that cancel the electric self repulsion which Poincaré and Ehrenfest used to prove their irrelevant PEP non-existence theorems. In regions outside the charge/current density, where $J^j = 0$, the condition $B^2 \geq E^2$ may reverse itself because the theory admits electric charge but no magnetic monopole density.

9. Vorton Conservation

The existence of a permaflux field with properties (P1) to (P4) in the permaflux definition causes solitons to be inherent in the solutions to the field equations. With each permaflux field there is the fluid-like entity σ, which is conserved by $[\dot{\sigma} + \nabla \cdot (\sigma v) = 0]$, but which is not required to obey fluid mechanics. It would have arbitrary structureless flow, except for the permaflux field lines embedded in it, which move point by point with σ at velocity v. As a result, σ seems to be composed of long, thin, flexible but unbreakable molecules which are the permaflux field lines. There is even more structure because these long thin molecules are $1D$ curves which are a foliation of $3D$ space. If two field lines (molecules) have infinitesimal separation at one point, they are almost congruent, and have infinitesimal separation and near perfect congruence along their entire closed or endless lengths. Any two such field lines may move, stretch and deform with the flow, but they remain nearly congruent in that flow. The flowfield $[\sigma v, \sigma]$ takes on structure and even a degree of local rigidity due to the embedded permaflux field.

Solitons are intimately associated with permaflux field line linking, and the chirality of permaflux field lines. If two closed or endless field lines are linked, they define a screw sense or chirality. But if two field lines are linked in a space foliated by field lines, continuity requires a continuum of linked field lines. It forces a torsion onto each individual field line in order that the two linked field lines may be part of one continuous foliation by linked field lines. The chirality of an individual permaflux field line is the sign of its average torsion, averaged over its closed or endless length. If that average has any vlaue other than the singular value zero, the field line cannot close on itself without winding around and linking with the field lines infinitesimally close to itself in the foliation; nor could it continue endlessly (if not closed) without winding around and linking with its neighbors. If the average torsion and chirality of permaflux field lines do not have the singular value zero, those field lines form chiral wreaths of twisted and mutually interlinked field lines. Since the permaflux field lines are conserved and cannot unlink [by property (P3) in the permaflux definition], these chiral wreaths are topologically indestructible, and each forms a vorton or vortex-like soliton/particle. Each such chiral wreath fills, foliates and defines a $3D$ compact subspace called a $3D$ vorton subspace.

A chiral vorton subspace is defined as a compact $3D$ subspace bounded by a closed oriented $2D$ surface which is foliated by a permaflux field obeying Eqs.(32), (33) and (34), such that: (B1) the component of the permaflux field normal to the $2D$ boundary surface is everywhere zero on that surface; (B2) the permaflux field lines inside the vorton subspace do not link with any permaflux field lines outside the vorton subspace; and (B3) each and every permaflux field line inside the vorton subspace links directly or in chains with every other permaflux field line inside the vorton subspace.

The existence of such vorton subspaces is proven by example in the PEP system. Equation (10) describes an array of such subspaces illustrated in cross section in Fig.(1). A single vorton subspace with a schematic illustration of some of the linked permaflux field lines is given in Fig.(2). All of the PEP system solutions such as those given by Eq.(7), which, in one singular Lorentz frame satisfy $[\nabla \times \mathbf{B} = \chi\kappa\mathbf{B}]$ and $[\mathbf{E} = 0]$ with $[\chi\kappa \neq 0]$, are segregated into such vorton subspaces. These chiral vorton subspaces have topological integrity and permanence as established by the following theorem:

Vorton Conservation Theorem: GIVEN that a $3D$ chiral vorton subspace exists, THEN (V1) the permaflux field lines within it remain forever within it; and no permaflux field line from outside may enter; (V2) the topology of the knotting and linking of the permaflux field lines inside never changes; (V3) the chirality of the vorton is invariant; (V4) no increment of the conserved entity σ defined by $P^j \Rightarrow (\sigma v, \sigma)$ may enter or leave the vorton subspace; and (V5) the vorton subspace is pairwise disjoint with respect to all other vorton subspaces. PROOF: Property (V1) follows from (P3) the conservation of permaflux field lines and that they cannot unlink to escape because by (P2) they are closed or endless and by (P1) they are pairwise disjoint. Similarity, no new permaflux field lines may be created, and no unlinked field line may penetrate from the outside without producing a discontinuity in the permaflux field. Property (V2) follows because the field lines are closed or endless and pairwise disjoint. Property (V3) follows because linking defines chirality and linking can not change. Property (V4) follows because the entity σ moves point by point with the permaflux field lines and none of them may enter or leave the vorton subspace. Property (V5) follows because the permaflux field lines inside the vorton subspace are pairwise disjoint with those outside.

In principle, any finite, single valued and continuous fields obeying the permaflux Eqs.(32), (33) and (34), in regions where chirality (and average field line torsion) are not zero may be divided into discrete vorton subspaces. One simply begins by selecting a single permaflux field line and includes in the same vorton subspace with it all permaflux field lines linked to it, directly or in chains. If there are any permaflux field lines left outside, the process is repeated as often as necessary.

The boundary of a $3D$ vorton subspace is a closed oriented $2D$ surface and therefore is, topologically, a sphere with $2p$-holes connected by p-handles. A sphere has $p = 0$ and the torus has $p = 1$, etc. Since that $2D$ surface is foliated by the divergenceless permaflux field lines with no component normal to the surface, each field line in the surface must lie in that surface over its entire closed or endless length. The only closed oriented $2D$ surface which will support such a divergenceless field is the torus. The $2D$ boundary of every soliton $3D$ vorton subspace must be a $2D$ torus or disjoint linked $2D$ toruses. The torus may have infinite radius corresponding to an infinitely long $2D$ cylindrical surface.

If one admits zero valued topological singularities, contrary to Eq.(34), boundary surfaces with genus member ($p \neq 1$) may be possible. It may, or may not, be possible to decompose those vortons into systems of degenerate, knotted and linked toroidal vortons.

Because each soliton $3D$ vorton subspace is conserved topologically, it persists through time in the Lorentz frame in which it is defined, sweeping out a compact $4D$ subspace with endless time-like extent on $4D$ space-time. The closed or endless permaflux field lines foliating the vorton subspace persists through time, also, and sweep out closed or endless $2D$ surfaces, each with endless time-like extent, foliating the $4D$ vorton subspace. Because the permaflux field lines foliating the $3D$ vorton subspace are pairwise disjoint, the permaflux $2D$ surfaces foliating the $4D$ vorton subspace are pairwise disjoint and do not intersect. Because the permaflux field lines were interlinked and because the permaflux $2D$ surfaces have endless time-like extent, the permaflux $2D$ surfaces foliating the $4D$ vorton subspace are interlinked.

In any Lorentz frame, the $3D$ space orthogonal to the time axis at any time (t) intersects the $4D$ vorton subspace and the $2D$ surfaces foliating it as a $3D$ vorton subspace foliated by permaflux $1D$ field lines. If a $3D$ vorton subspace exists at any one time in any one Lorentz frame, it exists in all Lorentz frames at all times so long as Eqs.(32), (33) and (34) are obeyed. Because the $2D$ surfaces on $4D$ Lorentz space time are closed or endless in all space-like and time-like directions, and because they are pairwise disjoint (non-intersecting), the topology of the permaflux $1D$ field line knotting and linking is preserved on Lorentz transformation or on propagation through space-time.

In Appendix A it is proven that the $2D$ surfaces on $4D$ Lorentz space-time which correspond to the permaflux field lines are well defined Lorentz invariant $2D$ surfaces. At each point those $2D$ surfaces are tangent to P^j and $[\epsilon^{jkmn} P_k G_{mn}]$ where P_j and $G_{jk} = Q_{j,k} - Q_{k,j}$ are the fields appearing in the permaflux field Eqs.(32), (33) and (34).

The $4D$ vorton subspace on $4D$ space-time suggests the topological possibility that the entire $4D$ vorton subspace, and the $2D$ surfaces foliating it, may make a U-turn with respect to the time axis. This cannot occur so long as all of Eqs. (32), (33) and (34) are obeyed exactly. If, however, the time-like zero eigenvector of G_{jk} goes through zero everywhere on some $3D$ cross section of the $4D$ vorton subspace at the U-turn, the permaflux conservation is not violated, it is simply not applicable at the U-turn because Eq.(34) is not satisfied. In the PEP system, for example, the $4D$ current J^k is space-like, which may reverse its orientation with respect to the time axis without the singularity of passing through the light cone. This reversal simply changes the sign of the time component of $[j, \rho]$ which is the sign of the charge, corresponding to charge conjugation. On the other hand, the field lines of the time-like vector $[\epsilon^{jkmn} J_k F_{mn}]$ serve as the world lines of the individual conserved field lines of **B** embedded in and moving with the charge/current density. This time-like vector can pass from the forward light cone to the past light cone at the U-turn only by passing through the light cone. In order to do so, each field line of $[\epsilon^{jkmn} J_k F_{mn}]$ must pass through a point at the U-turn where this field has the value zero. This corresponds to time reversal of the soliton $4D$ vorton subspace. The PCT theorem of electromagnetic theory admits a parity transformation and charge conjugation with the time reversal, as well. This reverses the chirality of the conserved flux field, the magnetic field, in regions where $J^k \neq 0$. Whether or not the chiral wreaths of conserved flux in two PEP subspaces, such as that in Fig.(2), can be annihilated or created in pairs, if they have opposite chiralities and charge densities is not determined here. But it is not prohibited by permaflux conservation and vorton subspace conservation if Eqs.(32), (33) and (34) are not all satisfied exactly at the U-turn with respect to time.

The pairwise disjoint property of the vorton subspaces, in property (V5) of the Vorton Conservation Theorem, establishes that point by point fields inside a vorton subspace cannot be measured under the constraint that fields are measured only by probe soliton/particles. Probe solitons cannot penetrate or overlap the measured vorton subspace. Under that measurement constraint, only soliton properties which are integrated or averaged over the vorton subspace can be measured. Vorton subspace centroids cannot be measured exactly. Soliton futures cannot be determined by integrating the field equations, because initial values of the fields inside the vorton subspace are unknown. This establishes a qualitative correspondence with the limited set of particle observables, the uncertainty and the indeterminacy in modern quantum theories. The use of ensembles and statistics seems necessary in the analysis of permaflux soliton systems, unless one can establish a quantitative correspondence principle between such ensemble statistics and the quantum probabilities obtained by quantizing point particle, string or soliton approximations to the permaflux fields. Establishment of such a quantitative correspondence principle is not attempted here. Nor is there any attempt to determine whether these hidden (unmeasurable) vorton subspace theoreis are an exception to Bell's hidden variable theorem.

10. Electron-Like PEP Vorton

The existence theorem for PEP solutions given in Section (7) above, becomes an existence theorem for PEP solitons when combined with the Vorton Conservation Theorem proved in Section (9) above. If a vorton subspace is constructed into initial fields satisfying the existence theorem, then a time integrated solution exists for a soliton with that topologically indestructible vorton subspace conserved through time.

In order to illustrate the topology of the fields which one must build into the initial fields to assure the existence of a time integrated soliton with charge, rest mass, spin angular momentum and magnetic dipole moment (gyromagnetic ratio) which are initially equal to that of an electron, an approximate solution is described briefly. From the infinitely long cylinder of charge/current density given by Eq.(6), cut a length L, bend it into a loop, and adjust the charge/current density cross section to assure current conservation in the flow. Compute the self fields of this long, thin, chiral charge/current loop and adjust the parameters b, R^*, v and L so that $[e = \int \rho dV]$; $[m_0 = \int (1/2)(E^2+B^2)dV]$; $[\hbar/2 = |\mathbf{r}\times(\mathbf{E}\times\mathbf{B})dV|]$, and so that the gyromagnetic ratio is 2, where e and m_0 are the charge and mass of the electron and $\hbar$ is Planck's constant. This long, thin, chiral charge/current loop and its self field are no longer an exact solution to the PEP Eq.(4), but the magnetic forces cancel the electric forces with a residue orders of magnitude smaller than either; and several orders of magnitude smaller than in the Lorentz/Abraham electron model. Exact solutions with topology similar to this would certainly change with time without some externally imposed "electromagnetic bottle". For example, the coulomb field, as in Eq.(13) might provide such a "bottle" field.

This loop with diameter of the order of the Compton wave length is not adequate as a model of an electron in most circumstances. But it is a conserved vorton whose conserved magnetic field lines are a permaflux field with properties (P1)-(P4). Heuristically, one might conjecture that the conserved magnetic flux moving point by point with the charge/current density [established in Sections (8) and (9) above] causes magnetic field lines to act as pucker strings, permitting the charge current distribution in which it is embedded to expand, contract and deform to fit its environment. However, for any intra-system observers who are part of the system, such conjectures must remain forever heuristic, because as proven in Section (9), it is topologically, and therefore experimentally, impossible for intra-system observers to make direct, point by point measurements of fields in regions where $[J^k \neq 0]$.

11. The PEP Hierarchy

The PEP system was derived as the simplest possible system with properties (F1)-(F4). Property (F5) was actually discovered in the evaluation of the inherent properties of that simplest system. In that simplest system, all fields are derived from a single $4D$ vector field A_j; the field equations are polynomials of second degree in A_j and its derivatives; and the second order derivative of A_j is the highest order derivative in the field equations.

One could, in a variety of ways, generate a hierarchy of increasingly more complex systems with properties (F1)-(F5) using the PEP system as the lowest level in the hierarchy. One quickly arrives at a stage of complexity which discourages pursuit of the more complex. But some of the simpler systems are interesting.

Consider only those systems in which all fields may be expressed as functions of a single vector field V^j and its derivatives. Consider only field equations which are polynomials no higher than second degree in V^j and its derivatives. This permits only pairwise forces between solitons. Then permit field equations which, at each succeeding level in the hierarchy, may include derivatives of V^j of one higher order, with the constraint that properties (F1)-(F5) be obeyed in all cases. This requires that P^j and Q_j, in Eqs.(32), (33) and (34), each be linear functions of V^j and its derivatives, and that S^{jk} be a quadratic function of V^j and its derivatives. It further requires that the equation $[S^j_{k,j} = 0]$ and Eq.(32) must be identical after each are expressed in terms of V^j and its derivatives.

The only case limited to first order derivatives of V^j is $P^j = Q^j = V^j$ with:

$$S^j_k = V^j V_k - (1/2) V^m V_m \delta^j_k \tag{39}$$

Then Eq.(33) becomes $V^j_{,j} = 0$ and the conservation law is:

$$S^j_{k,j} = V^j (V_{j,k} - V_{k,j}) = 0 \tag{40}$$

Eq.(40) is identical in form with Eq.(32). It is well known that S_k^j given by Eq.(39) is the SEM tensor of the relativistic superfluid with velocity of sound equal to the velocity of light. It is a compressible fluid $[p = \sigma c^2 = \sigma]$ with compressibility $[c^{-2} = 1]$. It is generally believed that this superfluid has conserved vorticity only when σ in $V^j \Rightarrow (\sigma v, \sigma)$ is constant and uniform, and only in the Lorentz frame where σ is uniform. That is true if vorticity is defined as $[\nabla \times v]$. If one defines *relativistic fluid vorticity* as $[\nabla \times (\sigma v)]$, then it is a conserved flux in every solution to Eq.(40) in every Lorentz frame.

In any Lorentz frame, using the conventional fluid notation $V^j \Rightarrow (\sigma v, \sigma)$, the three independent components of Eq.(40) take the form:

$$(\sigma v) \times [\nabla \times (\sigma v)] + \sigma[-\partial(\sigma v)/\partial t) - \nabla\sigma] = 0$$

Consider only those solutions which in one singular Lorentz frame are time independent and have uniform, constant σ so that $\sigma v \times [\nabla \times (\sigma v)] = 0$. Obviously, the solutions given in Section (4) for Eq.(7) are solutions to the relativistic superfluid Eq.(40) with $\mathbf{B}$ replaced by $(\sigma \mathbf{v})$. Figure (1) then represents the parallel vorticity and flow field lines in an ordered array of ring vortices described by Eq.(10) with $\mathbf{B}$ replaced by $(\sigma \mathbf{v})$. The central sphere in Fig.(2) is a smoke ring vortex in which the toroidal region of smoke and vorticity has a degenerate central hole squeezed down onto a diameter of the sphere.

With the rest of the array removed and replaced by a circulation free flowfield, as in Eq.(11), the central sphere represents that smoke ring vortex stationary in the flowing bulk fluid as in Fig. (3). It has spin angular momentum Ω due to the ϕ component of flow about its symmetry axis. If the bulk fluid is Lorentz transformed to rest, the smoke ring vortex propagates through the bulk fluid with linear momentum P inside the central sphere (ellipse). The spin angular momentum Ω is chirally oriented with respect to propagation through the bulk fluid with the same chirality as the vorticity field inside the sphere. The Biot Savart velocity field outside the sphere generated by the sphere (ellipse) of vorticity, is a wave propagating with the vortex which moves bulk fluid out from in front of the sphere, around the sphere, and in behind it. Its wavelength is approximately twice the spin angular momentum (2Ω) divided by the linear momentum P, satisfying a de Broglie relationship with Planck's constant replaced by twice the spin angular momentum. But again, this level of the hierarchy is not gauge invariant and any electromagnetic interpretation is clumsy.

The field $[G_{jk} = V_{j,k} - V_{k,j}]$ is a Maxwellian field. In order to interpret it as an electromagnetic field as required for property (F3), one must interpret V_j as the electromagnetic potential $[V_j = A_j]$, then define and interpret $[T^j_k = G^{jm}G_{mk} - (1/4)(G^{mn}G_{nm})\delta^j_k]$ as the electromagnetic SEM tensor, which is included in S^j_k of Eq. (39) by implication. Then one must interpret $[W^j_k = S^j_k - T^j_k]$ as a non-electromagnetic SEM tensor. There is no reasonable physical interpretation for $(W_{jk} = S_{jk} - T_{jk})$ in this superfluid. It strains the imagination to interpret $(V_{j,k} - V_{k,j})$ in this system as an electromagnetic field. But it is interesting to compare the electromagnetic interpretation of $(V_{j,k} - V_{k,j})$ with the hydrodynamic interpretation. The magnetic field is the conserved relativistic vorticity $[\mathbf{B} = \nabla \times (\sigma\mathbf{v})]$. The electric field times the σ density gives that portion of the virtual force of fluid acceleration not caused by the conservative force arising from the pressure gradient with $[p = \sigma c^2]$. That "non-conservative" force $[-\partial(\sigma v)/\partial t - \nabla\sigma]$ is strongly dependent on the Lorentz coordinate system. It is equal to $[v \times \nabla \times (\sigma v)]$ and, at any point, it vanishes in coordinate systems in which the fluid flow velocity is zero $[v = 0]$ at that point. If one interprets $[\nabla \times \sigma v]$ as local circulation, this force is like a coriolis force, except that it is caused by rotary circulation in a uniformly moving coordinate system, rather than by a uniform motion in a rotating coordinate system. But this system does not satisfy properties (F1)-(F5) for another reason. It is not gauge invariant. This relativistic superfluid is interesting in its own right, and it establishes the analog between vortices and vortons in general. But it is not gauge invariant as required by property (F3).

If one admits derivatives of V^j to second order, the only total system SEM tensor permitted is:

$$S^j_k = a[V^jV_k - (1/2)V^mV_m\delta^j_k] + b[F^{jm}F_{mk} - (1/4)(F^{mn}F_{nm})\delta^j_k] \quad (41)$$

with a and b positive constants. Then, with $[V^j_{,j} = 0]$, the conservation law is:

$$S^j_{k,j} = aV^j(V_{k,j} - V^{j,k}) + b(V^{j,m}_{,m} - V^{m,j}_{,m})(V_{k,j} - V_{j,k}) = 0 \quad (42)$$

This is the form of the permaflux Eq.(32) if one defines $[Q_j = V_j]$ and $[P^j = aV^j + b(V^{j,m}_m - V^{m,j}_m)]$. With $a = 1$ and $b = 0$ this is the superfluid defined by Eqs.(39) and (40). With $a = 0$ and $b = 1$ it is the PEP system. With $b = 1$ and arbitrary a, one may consistently interpret

V_j as the electromagnetic potential $[V_j = A_j]$. The non-electromagnetic SEM tensor W^j_k has a physically meaningful interpretation in terms of a superfluid energy and momentum. In every Lorentz frame, using the identification $V^j = A^j \Rightarrow (\mathbf{A}, \phi)$, and conventional electromagnetic notation, Eq.(42) takes the form:

$$(a\mathbf{A} + b\mathbf{j}) \times \mathbf{B} + (a\phi + b\rho)\mathbf{E} = 0$$

Considering only those solutions which in one singular Lorentz frame are time independent with $\mathbf{E} = 0$ everywhere, this takes the form $(a\nabla \times \mathbf{A} + b\nabla \times \nabla \times \mathbf{A}) \times (\nabla \times \mathbf{A}) = 0$. Obviously the solutions for $\mathbf{B} = \nabla \times \mathbf{A}$ given by Eqs. (7)-(10) satisfy Eq. (42). But the field given by Eq.(11) outside the central sphere and illustrated by Fig.(3) is not a solution. This electromagnetic fluid ether with $[a \neq 0, b \neq 0]$ has soliton solutions with permaflux vorton subspaces, but the electromagnetic field in free space where $[J^k = 0]$, is severely constrained by Eq.(42). With $a \neq 0$, Eqs.(41) and (42) are not gauge invariant as required by property (F3), further limiting the electromagnetic interpretation and violating property (F3) of the hierarchy.

If one admits derivatives of V^j to third order in the field equations, the SEM tensor is:

$$S^{jk} = a[J^j J_k - (1/2)(J^m J_m)\delta^j_k] + b[F^{jm} F_{mj} - (1/4)(F^{mn} F_{nm})\delta^j_k] \quad (43)$$

The conservation law is:

$$S^j_{kj} = aJ^j(J_{k,j} - J_{j,k}) + bJ^j F_{kj} = 0 \quad (44)$$

This is identical to Eqs.(32) and (33) with $[P^j = J^j]$ and $[Q_j = aJ_j + bA_j]$. It has been assumed that V_j is the electromagnetic potential $[V_j = A_j]$ and the electromagnetic definitions of F_{jk} and J^j and J^j have been used. With $a = 0$ and $b = 1$ it is the PEP theory. With $b = 1$ and a an arbitrary positive constant, this becomes a theory of electromagnetic conserved flux solitons in which the vorton subspaces, where $[J^k \neq 0]$, includes hydrodynamic masses, momenta and forces. The charge current density is also an inertial superfluid flowfield with velocity of sound equal to the velocity of light.

In each Lorentz frame, using conventional electromagnetic notation, so that $P^j = J^j \Rightarrow (\mathbf{j}, \rho)$ and $Q^j = A^j + aJ^j \Rightarrow (\mathbf{A} + a\mathbf{j}, \phi + b\rho)$, the three independent components of Eq.(44) are:

$$j \times (\mathbf{B} + a\nabla \times \mathbf{j}) + \rho(\mathbf{E} - a\dot{\mathbf{j}} - a\nabla\rho) = 0$$

Consider only those solutions which, in one singular Lorentz frame, are time independent with $\mathbf{E} = 0$ everywhere. This equation becomes $(\nabla \times \mathbf{B}) \times (\mathbf{B} \times a\nabla \times \nabla \times \mathbf{B}) = 0$. Obviously the solutions to Eq.(7), are solutions to Eq.(44) in this singular Lorentz frame. The solutions for the isolated vorton given in Section (4) and Fig.(3) is a solution, also.

This model has interesting possibilities because a may be positive or negative, and the constraint $S^{44} \geq 0$ may be relaxed inside the charge/current density (where physical measurements by intra-system observers are impossible), so long as the integral of S^{44} over each vorton droplet is positive. If a is negative, the hydrodynamic analogue to a pressure field in Euler's equation is replaced by the analogue of a gravitational potential in Euler's equation, which is generated by the fluid itself. However, this conjecture has not been examined for more basic inconsistencies, and this level of the PEP based hierarchy has received very little attention.

The characteristic topology of chiral permaflux fields, the existence of chiral vortons at all levels of the PEP hierarchy, and the analogy with vortices in the relativistic superfluid are supported by the fact that the potential fields $(\mathbf{A}, \phi)$ given by:

$$\mathbf{A}^{\chi}_{mn}(\kappa r) = \mathfrak{B}_{nm}(\kappa r) - i\chi\mathfrak{C}_{nm}(\kappa r) \tag{45}$$

with $[\phi = 0]$, are exact analytic solutions to each of Eqs.(40),(42) and (44) in one singular Lorentz frame. Here, $\mathfrak{B}_{nm}$ and $\mathfrak{C}_{nm}$ are the vector multipole fields used in Eq.(9) above. Linear combinations of these fields with the same values for χ and κ are also exact analytic solutions to each of the above systems. Lorentz transforms of any of these solutions remain solutions to each of the above systems. These solutions to Eq.(45) satisfy the permaflux field Eqs.(32),(33) and (34) with a variety of auxiliary equations, suggesting that the topologies of the linked field lines in the chiral vorton subspaces inherent in these solutions is fundamental to a permaflux field.

One could extend the hierarchy above. One could produce branches in the hierarchy by admitting field equations with polynomials of succeedingly higher degree in V^j and its derivatives. One could add further complexity by introducing fields other than V^j with auxiliary equations as required to determine the added fields.

Eqs.(32),(33) and (34) for P^j and Q^j are underdetermined and are compatible with the conservation law $[S^j_{k,j} = 0]$ when S^j_k is almost any arbitrary symmetric tensor function of P^j, Q^j and their derivatives. One could build a hierarchy by starting with S^j_k as polynomials of some maximum degree in P^j, Q^j and their derivatives of some maximum order. One could introduce additional fields $\tilde{P}^j$ and $\tilde{Q}^j$ which independently satisfy equations of the form of Eqs.(32),(33) and (34), also. Then S^{jk} may be a function of P^j, Q^j, $\tilde{P}^j$ and $\tilde{Q}^j$ and their derivatives with two types of permaflux vorton subspaces which may be coupled through the definition of S^{jk}.

One may include spinor fields in S^j_k, also, if one provides auxiliary equations to determine those fields. There are standard methodes for converting vectors such as P^j or Q^i, and tensors such as $(Q_{j,k} - Q_{k,j})$, into spinor form and one may define spinor fields which correspond to permaflux solitons and vorton subspaces.

All of these methods of constructing more complex fields can be carried out so as to generate only systems with properties (F1)-(F5) above. However, as indicated, one quickly arrives at a level of complexity which discourages pursuit of the more complex. This is amplified by the fact that the simplest system is defined by the intractable non-linear Einstein/Pauli PEP equation.

Summary and Discussion

Einstein in 1919 and Pauli in 1921 each gave the PEP Eq.(2) as the necessary and sufficient condition for the existence of a Purely Electromagnetic Particle (PEP). That PEP Eq.(2) was derived here as the simplest in a hierarchy of theoretical models which include only finite continuous fields which obey the conservation laws of relativistic mechanics and include an electromagnetic field with sources. All levels of the hierarchy share these properties, and have, in each Lorentz frame, a conserved flux field whose field lines are closed or endless, conserved and pairwise disjoint. These form solitons centered on vortex-like vorton subspaces with topological integrity and permanence. A myriad of exact analytic solutions to the PEP Eq.(2) with these properties are presented. An existence theorem for more general PEP solutions is proven. The solitons centered on vortex-like vorton subspaces have natural approximations as point particles or strings. The particles or strings may inherit the solitons topological permanence and integrity, its conserved integrals of charge, mass (energy), linear momentum and spin angular momentum, and its field chirality and higher order electric and magnetic multipole moments, all of which are intrinsic properties of these vortex-like solitons. The points and strings obey linear mechanics and are bilinearly coupled to the linear electromagnetic field.

The PEP model was obtained by accepting electromagnetic theory for macroscopic systems in free space, where slowly varying fields may be measured accurately with probe particles, then extrapolating it down to the microscopic, where probe particle measurements are impossible, while requiring the electromagnetic field to be finite, single valued and continuous. The extrapolation is made more plausible by recognizing that all of electromagnetic theory follows from definitions alone, given the existence of a twice differentiable $4D$ vector field (A_j the potential) and the conservation laws of relativistic mechanics. The extrapolation is really only an extrapolation of a differentiable field A_j and the invariance properties which give the conservation laws. The simplest possible model satisfying this extrapolation is defined by the PEP Eq.(2), and has the properties described in the preceding paragraph above. The charge/current density in the system is then enclosed in the soliton vortex-like vorton subspaces which are topologically pairwise disjoint.

If one assumes that intra-system observers can measure field values only by means of probe particle/solitons, the probe soliton cannot overlap or penetrate the measured vorton subspace. It is impossible to probe and measure point by point fields inside the pairwise disjoint vorton subspaces. Fields cannot be measured in a region where charge/current density is not zero. Only a limited set of integrated or averaged soliton properties can be measured. There is uncertainty in centroid positions and other observables. Soliton futures are indeterminant, since initial field values inside vorton subspaces can not be measured and used for time integration of the PEP Eq.(2). The theory itself identifies these internal PEP fields as unmeasurable, yet the unmeasurable fields remain in the deterministic field equations which may be used in ensemble analysis.

The PEP deterministic equations but unmeasurable fields have physical meaning when used to compute the deterministic propagation of individual hypothetical solitons in an ensemble of single soliton systems. Each single soliton in the ensemble must be consistent with the initial limited set of soliton observables and their uncertainties as measured in a soliton experiment. The mass (energy), and momentum of a PEP soliton/particle which determines soliton propagation resides entirely in the solitons self interfering electromagnetic wave, generated by the sources in the vorton subspace. This suggests a compatibility with the heuristic computations of uncertainties [17-19] and probabilities using self interfering wave/particles often used in the early years of the quantum theory.

It also suggests compatibility with the more elaborate analysis of ensembles of self interfering wave/particles by Barut [3]. However, each system must be analyzed in detail to determine whether there is a quantitative correspondence principle between ensemble probabilities and the quantum theory probabilities.

The PEP model is only the simplest in a hierarchy of theoretical models which share the above properties. Each model at different levels in the hierarchy adopts a different form for the non-electromagnetic portion $[W^j_k = S^j_k - T^j_k]$ of the total system SEM tensor $S^j k$. The form of, and the fields included in, W^j_k are inaccessible to direct experimental measurement with the technology available today. The PEP model assumes $[W^j_k = 0]$. Higher levels in the hierarchy assume increasingly more complex forms for W^j_k, under the constraint that all must include a conserved flux field satisfying Eqs.(32),(33) and (34) which forms vortex-like soliton/particles with pairwise disjoint vorton subspaces. As a result, all levels of the hierarchy share the features of the PEP theory described here. Since W^j_k and the fields in it are not directly measurable, one must choose between levels in the hierarchy on the basis of differences in the symmetry groups of their field equations, and the statistical behavior of their solutions.

The point particle versus particle/wave has, over the past few decades, appeared at the heart of the question, "Does God play dice with the universe?", and its more pragmatic re-statements. One might argue that the PEP hierarchy of wave/particle theories supports the Copenhagen school because it leads to intra-system observers with a limited set of particle observables subject to uncertainty and indeterminacy. One might argue that it supports the deterministic viewpoint, and that ensemble statistics for deterministic systems will ultimately explain quantum probabilities as a correspondence principle limiting approximation. Instead, one might hope that it will eliminate some of the argument by illustrating that the Copenhagen uncertainty and indeterminacy limitations on human knowledge may be valid, but that there may exist underlying deterministic field equations which explain that uncertainty and indeterminacy.

Appendix A

The permaflux Conservation Theorem: GIVEN the $4D$ vector fields P^j and Q^j with the defined field $[G_{jk} \equiv Q_{j,k} - Q_{k,j}]$ which satisfy:

$$P^j G_{jk} = P^j (Q_{j,k} - Q_{k,j}) = 0 \qquad (\alpha 1)$$

$$P^j_{,j} = 0 \qquad (\alpha 2)$$

$$P^j \neq 0 \quad , \quad G_{jk} \neq 0 \quad , \quad G^{ik}G_{jk} \geq 0 \qquad (\alpha 3)$$

THEN in each Lorentz frame with the definitions $[P^j \Rightarrow (\sigma v, \sigma)]$, $[Q^j \Rightarrow (\mathbf{Q}, Q)]$,, $[G_{jk} \rightarrow (\mathbf{G}, \mathbf{F})]$, $\mathbf{G} \equiv \nabla \times \mathbf{Q}$ and $\mathbf{F} \equiv -\dot{\mathbf{Q}} - \nabla q$ the $3D$ pseudovector field $\mathbf{Q}$ has field lines which are: (P1) pairwise disjoint; (P2) closed or endless; (P3) conserved; and (P4) moving with a conserved entity σ at the point by point velocity $\mathbf{v}$ which satisfies $[\dot{\sigma} + \nabla \times (\sigma \mathbf{v}) = 0]$ and $[\mathbf{v} = (\mathbf{v} \cdot \mathbf{G}/G^2)\mathbf{G} + \mathbf{F} \times \mathbf{G}/G^2]$ in which the component $[(\mathbf{v} \cdot \mathbf{G})/G^2\mathbf{G}]$ contributes nothing to the displacement of the closed or endless field lines of $\mathbf{G}$.

PROOF: Properties (P1) and (P2) are proved trivially above. Properties (P3) and (P4) are proven here by $4D$ differential geometry. As proven above, $[G^{ik}G_{jk} \geq 0]$ assures that either P^j or $[\epsilon^{jkmn}P_jG_{mn}]$ is time-like and for simplicity P^j is assumed time-like.

Construct an arbitrary $2D$ surface increment Σ bounded by a closed $1D$ curve C in a region where Eqs.(α1), (α2), and (α3) are satisfied. It is intersected at each point by a field line of P^j, so that Σ may be moved point by point along these field lines in a manner which preserves its continuity.

The integral $[\int\int G_{jk}dx^jdx^k]$ over the surface Σ is invariant as it is moved along the field lines of P^j as described. To prove this, let Σ_1, C_1 represent the initial positions and Σ_2, C_2 the final positions. Map the closed $1D$ interval $0 \leq \lambda \leq 2\pi$ onto the closed curve C_1. Assign to every point on each field line of P^j the value of the parameter λ at the point where that field line intersects C_1. Map onto each and every field line of P^j intersecting C_1, the closed $1D$ interval $1 \leq \mu \leq 2$ such that $\mu = 1$ on C_1 and $\mu = 2$ on C_2 for every field line. Map intervening values of μ by moving C uniformly and continuously from C_1 to C_2 along the field lines of P^j intersecting it, assigning the same value of μ to all points on C as it moves from C_1 to C_2. This smoothly parametrizes with (λ, μ) the open ended $2D$ cylindrical surface extending from C_1 to C_2 which is foliated by the field lines of P^j intersecting C_1. The functions $x^j(\lambda, \mu)$ are precisely defined on that surface. Since μ parametrizes the field lines of P_j, it follows that: $[\partial x^j/\partial \mu = h(\mu, \lambda)P^j]$ where $h(\mu, \lambda)$ is a smooth function. The integral of $[\int\int G_{jk}dx^jdx^k]$ over that open ended

$2D$ cylinder is equal to:

$$\int\int G_{jk}[\frac{\partial x^j}{\partial \mu}\frac{\partial x^k}{\partial \lambda} - \frac{\partial x^k}{\partial \mu}\frac{\partial x^j}{\partial \lambda}]d\mu d\lambda$$
$$= \int\int G_{jk}[P^j\frac{\partial x^k}{\partial \lambda} - P^k\frac{\partial x^j}{\partial \lambda}]h(\mu,\lambda)d\mu d\lambda = 0$$

by virtue of Eq.(α1). Therefore, by applying Stoke's theorem twice, first to the opend ended $2D$ cylinder bounded by C_1 and C_2, then second to the endcaps which are Σ_1 and Σ_2, one obtains:

$$\int_{C_1} Q_j dx^j - \int_{C_2} Q_j dx^j = \int\int_{\Sigma_1} G_{jk}dx^j dx^k - \int\int_{\Sigma_2} G_{jk}dx^j dx^k = 0 \qquad (\alpha 4)$$

This proves the first sentence of this paragraph.

In any specific Lorentz frame in which one defines $[P^j \Rightarrow (\sigma\mathbf{v},\sigma)]$, $[Q^j \Rightarrow (\mathbf{Q},q)]$ and $[\mathbf{G} \equiv \nabla\times\mathbf{Q}]$, select an arbitrary $2D$ surface increment Σ bounded by a closed $1D$ curve C which lies entirely within the $3D$ space orthogonal to the time axis at time t and move Σ at velocity v for a time interval dt. This moves Σ point by point along the field lines of P^j so that Eq.(α4) applies giving:

$$\frac{d}{dt}[\int\int G_n d\Sigma] = \frac{dN}{dt} = 0$$

where the subscript n indicates the component normal to Σ at each point on Σ. This integral gives N the total number of field lines of $\mathbf{G}$ through Σ. Its time derivative may vanish for all such arbitrary Σ moving at velocity $\mathbf{v}$, if and only if, each field line of $\mathbf{G}$ is conserved, as in (P3), and moves point by point at velocity $\mathbf{v}$, as in (P4).

In each Lorentz frame, with $[\mathbf{F} \equiv -\dot{\mathbf{Q}} - \nabla q]$, Eq.($\alpha$1) takes the form $[\mathbf{v}\times\mathbf{G} + \mathbf{F} = 0]$ and $[\mathbf{v}\cdot\mathbf{F} = 0]$ so that $\mathbf{v} = \mathbf{v}\cdot\mathbf{G}/G^2 + \mathbf{F}\times\mathbf{G}/G^2$, as in (P4). *The Theorem is proved.*

The Permaflux Corollary: GIVEN the conditions of the permaflux Theorem, THEN the $2D$ surfaces everywhere foliated by, and tangent to, the field lines of P^j and $[R^j = \epsilon^{jkmn}P_j G_{mn}]$ are: (α1) endless with respect to every time-like direction; (α2) closed or endless with respect to every space-like direction; ($\alpha 3$) pairwise disjoint; and ($\alpha 4$) in each Lorentz, are at each point parallel to the two $4D$ vectors with components $[\mathbf{G}, 0]$ and $[\mathbf{F} \times \mathbf{G}/G^2, 1]$.

PROOF: By virtue of (P3) and (P4), in each Lorentz frame during the time interval dt, each moving conserved field line of $\mathbf{G}$ sweeps at a $2D$ surface increment everywhere parallel to $[\mathbf{G}, 0]$ and to $[\{\mathbf{v} \cdot \mathbf{G})\mathbf{G}/G^2 + \mathbf{F} \times \mathbf{G}/G^2\}dt, dt]$. If one divides the latter by dt, and subtracts $[\mathbf{v} \cdot \mathbf{G}/G^2]$ times the former from it, one obtains $[\mathbf{F} \times \mathbf{G}/G^2, 1]$.

If one defines $[R^j \Rightarrow (\rho\mathbf{u}, \rho)]$ in each Lorentz frame then, $[R^j G_{jk} = 0]$, follows from the definition of R^j, and takes the form $\mathbf{u} \times \mathbf{G} + \mathbf{F} = 0$, $\mathbf{u} \cdot \mathbf{F} = 0$. It follows that $[\mathbf{u} = (\mathbf{u} \cdot \mathbf{G}/G^2)\mathbf{G} + \mathbf{F} \times \mathbf{G}/G^2]$ so that $\zeta^{-1} R^j \Rightarrow [(\mathbf{u} \cdot \mathbf{G}/G^2)G + \mathbf{F} \times \mathbf{G}/G^2, 1]$. Similarly, $\sigma^{-1} P^j \Rightarrow [(\mathbf{v} \cdot \mathbf{G}/G^2)\mathbf{G} + \mathbf{F} \times \mathbf{G}/G^2, 1]$. Therefore,

$$(\zeta^{-1} R^j - \sigma^{-1} P^j) \Rightarrow (\mathbf{u} \cdot \mathbf{G}/G^2 - \mathbf{v} \cdot \mathbf{G}/G^2)[\mathbf{G}, 0]$$

and

$$[(\mathbf{v} \cdot \mathbf{G}/G^2)\zeta^{-1} R^j - (\mathbf{u} \cdot \mathbf{G}/G^2)\sigma^{-1} P^j] \Rightarrow (\mathbf{v} \cdot \mathbf{G}/G^2 - \mathbf{u} \cdot \mathbf{G}/G^2)[\mathbf{F} \times \mathbf{G}/G^2, 1]$$

It follows that, at each point in each Lorentz frame the $4D$ vectors P^j and R^j define the same $2D$ plane as the $4D$ vectors $[\mathbf{G}, 0]$ and $[\mathbf{F} \times \mathbf{G}/G^2, 1]$. This proves property ($\alpha 4$). Then properties ($\alpha 1$), ($\alpha 2$), and ($\alpha 3$) follow from (P1),(P2) and (P3) of the theorem. *The Corollary is proved.*

Acknowledgment and dedication

The author acknowledges and appreciates suggestions from A.O. Barut, J. Zeni, A. Laufer, M. Cruz, S.R. Coleman, J. Goldstone, R. Holmes, J. Neu, G. Lochak, L. Galgani and W. Honig.

This work is dedicated to Professor A.O. Barut for his helpful suggestions and in recognition of his many contributions to the understanding of the relationships between classical and quantum fields.

References

[1] G. Lochak, Found of Phys. **17**, 1189 (1987).
[2] E Schroedinger, Quantization as a Problem in Proper Values IV, Ann. d. Physik, **4-81** (1926) as translated in “Collected Papers on Wave Mechanics”, Chelsea Publishing, New York, NY (1982) p. 123.
[3] A.O. Barut, Found of Phys. Lett, **1**, 47 (1988).

[4] A.O. Barut, Ann. d. Physik, **7**, 31, (1988); Can. J. Phys. **67**, 100 (1989); Phys Lett **A143**, 349, (1990); Phys Lett **3**, 303 (1990), Found of Phys. **22**, 1267 (1992).
[5] U. Enz, in Courants, Amers, Ecueils En Microphysique, p. 131, Fond. Louis de Broglie, Paris, (1933).
[6] W.H. Rodrigues, J. Vaz, E. Recami in Courants, Amers, Ecueils En Microphysique, p. 379, Fond. Louis de Broglie (1993).
[7] W.H. Bostick, Physics Essays **4**, 45 (1991).
[8] W.H. Honig, Physics Essays **1**, 247 (1988); Physics Essays **4**, 583 (1991); Physics Essays **5**, 254 (1992); Physics Essays **5**, 514 (1992).
[9] R.F. O'Connell, Found of Phys., **23**, 463 (1993).
[10] A. Einstein, Sitzungsberichte der Preussischen Akad. d. Wissenschaften (1919), as translated in "The Principle of Relativity", H.A. Lorentz, A. Einstein, H. Minkowski and H. Weyl, Dover, NY (1952) p. 192.
[11] W. Pauli, "Relativitätstheorie", Encyklopädie der mathematischem Wissenschaften **19** B.G. Teubner, Lupzig (1921), as translated in "Theory of Relativity", Dover, NY, (1958) p. 184.
[12] For a recent review and bibliography, see: H. Zaghloul and O. Barajas, Am. J. Phys., **58**, 783 (1990).
[13] S. Chandrasekhar, Proc. Nat. Acad. Sc., **42**, 1 (1956).
[14] S. Chandrasekhar and P.C. Kendall, Astrophys., J., **126**, 2544 (1969).
[15] A. Sommerfeld, "The Mechanics of Deformable Bodies", Academic Press, NY (1950) p. 132.
[16] G.D. Jackson, "Classical Electrodynamics", John Wiley, NY (1975) p. 739.
[17] E. Schroedinger, ibid, entire book.
[18] W. Heisenberg, "The Physical Principles of Quantum Theory", English translation by University of Chicago Press (1930) reprinted by Dover Publishing Co., NY, NY, (1949).
[19] L. Schiff, "Quantum Mechanics", Chapters 1 and 2, McGraw Hill, Book Co., NY, NY, (1949). The heuristic descriptions of diffraction of wave pockets and the heuristic basis for the wave equation are reduced in later editions.

(Manuscrit reçu le 19 novembre 1994)