The third way to quantum mechanics is the forgotten first

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ABSTRACT. Quantum mechanics can be formulated in three ways, as Heisenberg, Schrödinger and Feynman did respectively. For the last way, an unknown (i.e. forgotten) forerunner exists, that we have found in a paper by Gregor Wentzel, published before the famous works by Heisenberg and Schrödinger, and contemporary with the fundamental works of L. de Broglie. In that paper, one can find the basic formulae and their interpretation as they were adopted by Feynman twenty years later. We believe that Wentzel's work was forgotten for several reasons: (I) Schrödinger's equation was much simpler to deal with (Wentzel himself contributed to its development in the same way as L.Brillouin and H.Kramers did). (II) The first application was rejected by Heisenberg and Kramers. (III) The approximation used by Wentzel was too naïve and failed. Nevertheless, the foundation laid by Wentzel was sound, as it has been shown by Feynman's work. Therefore, Wentzel has to be considered as one of the founders of quantum mechanics.

Our exposition aims at explaining some details. It is accompanied by two appendices. They respectively provide a summary of the quoted paper by Wentzel and of the theory of canonical transformations, needed to understand the link between Wentzel's and Feynman's formulations.

RÉSUMÉ La mécanique quantique peut être introduite de trois manières, celle de Heisenberg, celle de Schrödinger, et celle de Feynman. Pour la dernière, il existe un précurseur inconnu, c'est-à-dire oublié, que nous avons trouvé dans un travail de Gregor Wentzel, publié avant les travaux célèbres de Heisenberg et Schrödinger, contemporain des travaux fondamentaux de L.de Broglie. On peut y trouver les formules fondamentales et leur interpretation employées par Feynman vingt ans après. Nous croyons que le travail de Wentzel a été oublié pour plusieurs raisons. Premièrement, l'équation *de Schrödinger était plus simple à analyser (Wentzel lui-même contribuait simultanément à ce developpement avec L.Brillouin et H.Kramers). Deuxièmement, l'application première fut rejetée par Heisenberg et Kramers. Troisièmement, l'approximation usée par Wentzel était trop simple et échoua. Néanmoins, la fondation par Wentzel était correcte, comme les travaux de Feynman l'ont montré. Par conséquence, Wentzel doit être considéré comme l'un des fondateurs de la mécanique quantique.*

Notre exposition veut expliquer quelques détails. Elle est accompagnée de deux annexes, l'un résumant le travail cité de Wentzel, l'autre la théorie des transformations canoniques nécessaire pour comprendre la connection des formules de Wentzel avec celles de Feynman.

1 Introduction

Traditionally, we teach about three ways to quantum mechanics, attributed to Heisenberg [33,5,4,20], Schrödinger [52,53,55,56], and Feynman [30].

In the Heisenberg picture, Hilbert-space operators are substituted for the classical variables, and these operators may be represented by matrices. The classical canonical equations of motion are translated by the correspondence [20] between the Poisson bracket and the commutator into

$$\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}t} = \frac{\mathrm{i}}{\hbar}(\mathbf{HQ} - \mathbf{QH}) \ , \tag{1}$$

$$\mathbf{Q}[t + \delta t] = \exp[\frac{\mathrm{i}}{\hbar}\mathbf{H}\delta t]\mathbf{Q}[t]\exp[-\frac{\mathrm{i}}{\hbar}\mathbf{H}\delta t] \ .$$

The state of the system in question is represented by a fixed Hilbert vector, which is not necessarily made explicit, if the representation is produced by the matrix elements themselves.

The central issue of the Schrödinger approach is the equation of motion for a state Hilbert vector, whose representation in the function space L^2 is the wave function $\psi[x, t]$:

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = \mathbf{H}\psi[x, t] \ , \tag{2}$$

$$\psi[x, t + \delta t] = \exp[-\frac{\mathrm{i}}{\hbar}\mathbf{H}\delta t]\psi[x, t] \ .$$

This Hilbert vector moves on the unit sphere in Hilbert space,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \psi^*[x,t]\psi[x,t]\mathrm{d}x = 0 \ ,$$

and it is used to calculate the distribution of the measured classical variables. Eigenstates to the Hamilton operator characterize stationary states. The approach is equivalent to that of Heisenberg [54].

The third method was published more than twenty years later by Feynman [30,28]. It aims at the immediate calculation of transition probabilities, which later on can be translated into the expressions for a corresponding wave function. In this picture, transitions are given by a set of mediating trajectories in configuration or phase space, each contributing to the phase of a Hilbert vector, whose squared amplitude is the desired probability. The phase of an individual path is the classical action integral, and one obtains

$$\psi[x,t+\delta t] = \int (x|x')_{\delta t}\psi[x',t]\mathrm{d}x'$$

with the transition contribution

$$(x|x')_{\delta t} = K\exp[\frac{\mathrm{i}}{\hbar}\int_{x'}^{x} L[x,\dot{x},t]\mathrm{d}t] \ . \qquad (3)$$

The total amplitude is the sum of the interfering contributions, formally

$$(q_A|q_E) = \int \mathcal{D}q[t]\exp[\frac{\mathrm{i}}{\hbar}\int L[q^i,\dot{q}^i,t]\mathrm{d}t] \ .$$

With a suitable choice of K, this relation is equivalent, in simple examples, to Schrödinger's equation ([30], eq.18). The main point of Feynman's approach is the notion of interference of paths, not the mere equivalence of a mechanical path through configuration space with transversals to the surfaces[1] of constant S. The mathematics of the integration over paths is a problem in its own right [46,31,1].

Feynman cites suggestions and remarks of Dirac [21,22,23], which are taken as hints to use a kind of Huygens' principle to evaluate the

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[1] The character S is used for both the solutions of the Hamilton-Jacobi equation $H[q,\frac{\partial S}{\partial q},t] + \frac{\partial S}{\partial t} = 0$ and for the action integral $S = \int_{x'}^{x} L[x,\dot{x},t]\mathrm{d}t$ depending on the path. We will try to avoid confusion.

evolution in time of the quantum-mechanical wave function. He – and all followers – did not cite nor recognize an early work by Gregor Wentzel [63], where exactly the formulas written later by Feynman are derived with the aim of obtaining the characteristics of wave propagation by techniques of point mechanics. Wentzel's paper reached the editor of *Zeitschrift für Physik* on February 2nd, 1924. It is the first paper which describes a method like Feynman's to construct transition probabilities. This method was not applied in full [64,65]. Nevertheless, the method was appropriate, and proven to be manageable by Feynman. Thus, Wentzel's paper should be acknowledged in the history of quantum mechanics. Heisenberg's famous article on matrix mechanics dates from July 29th, 1925 [33], and is clearly second to Wentzel's. Schrödinger published about the quantum wave equation from January 27th, 1926 on [52,53,55,56]. Only DeBroglie wrote about matter waves already in september 1923 [15,16,17], prior to G.Wentzel.

In the following, we first intend to sketch the derivation used by Wentzel, and secondly, we will conjecture about the question, why this article was never associated to the development of Feynman's path integral approach. For the embedding of our particular topic into the history of quantum mechanics, we recommend the book by F.Hund [35].

2 The value of interfering paths

Mechanics and geometrical optics are governed by integral principles which attribute a value to each path in configuration or phase space. The actual motions or propagations are identified by local extrema of this value which we call action. The simplest of these principles is Fermat's principle, where the action is the integral over the refraction index. In point mechanics, the integral in time of the Lagrange function $L[q, \dot{q}, t] = E_{\text{kinetic}} - E_{\text{potential}}$ is the general rule. For time-independent total energy E, it contains the Maupertuis-Jacobi principle, which identifies the refraction index with $\sqrt{E - E_{\text{potential}}}$, thus providing the link between mechanics and geometrical optics. The name of Fermat is associated with the method of identifying the rays with minima of the integral over the refraction index. The name of Hamilton is associated with the connection to wave propagation. The surfaces of constant phase define transversal rays which are solutions to Fermat's principle. The phase is proportional to some action. For wave phenomena, the ray is produced by the interference implicit in Huygens' principle, which excludes all other points of space in the limit of infinitesimally small wavelength.

This is the argument to understand the meaning of the extremum principle for the action. Mechanics seems to fail only in explaining dispersion and interference phenomena.

All this was known and used already in the discussion of early quantum theory. The question then was how to distinguish between pure paths of particles and true wave phenomena. In the realm of geometrical optics this is not possible, and the history of this recognition was deeply influenced by Einstein [25,26]. Einstein constructs his results with interfering paths too, but these paths are real paths from different source events. In addition, the experiment which he proposed in [25] was wrongly evaluated and ended in disappointment.

In the historical evolution, quantum mechanics turned out to be wave mechanics. After the construction of the Schrödinger equation, i.e. the appropriate wave equation, the wave function was interpreted as providing a probability phase [6,7], whose interferences produced transition probabilities. Therefore, the notion used by Wentzel for reinstating the particle concept is to interpret the result of interfering paths as transition probability, i.e. to translate the language of wave theory into the language of particle statistics.

Feynman defines the value of a path $q^i[t]$ through the configuration space by the action integral $S = \int L[q^i, \dot{q}^i, t]\mathrm{d}t$. This action integral yields the contribution of the path in question to the transition amplitude $(q_A|q_E)$ from the configuration E to the configuration A. This contribution differs from path to path by a phase, and the total amplitude is the sum of the interfering contributions.

In the first paper of 1924 [63], Gregor Wentzel anticipated this idea in a strikingly explicit fashion. As we already noted, the discussion of the quantisation postulate at that time opened a broad acknowledgement of the correspondence between geometrical optics and point mechanics based on the variational principles of Fermat, Maupertuis and Jacobi [8,9,12,24]. The main question was to implement wave characteristics, and the general proposition was to use Huygens principle [13,18,26,27,57]. This principle was usually formulated as integral theorem for the wave equation. The interference interpretation never involved the contributions of individual virtual paths.

Wentzel is the first to consider the logical argument for the contribution of such paths in phase space to a probability amplitude. The central issue is the measure of the deviation from the classical path. Wentzel chooses the integral $\int \sum_i Q^i \mathrm{d}P_i$, in the canonical coordinates where the momenta P_i are constants of motion (see appendix B). In extended phase space, which can be constructed by (1) defining time as an additional coordinate q^0, (2) its conjugate momentum $p_0 = -W$ as minus the energy, the Hamiltonian $H[q,p,q_0]+p_0$ vanishes. Here, in any system of canonical coordinates we obtain

$$L\mathrm{d}t = \sum_{i=0}^{n} p_i \mathrm{d}q^i = \sum_{i=0}^{n} p'_i \mathrm{d}q'^i + \mathrm{d}F$$

and the variations $\delta S = -\delta S_1$ of the integrals

$$\begin{aligned} S &= \int_E^A \sum_{i=0}^{n} p_i \mathrm{d}q^i \ , \\ S_1 &= \int_E^A \sum_{i=0}^{n} q^i \mathrm{d}p_i = \sum_{i=0}^{n} \big((q^i p_i)|_{\mathrm{at}A} - (q^i p_i)|_{\mathrm{at}E} \big) - S \ , \end{aligned}$$

are canonical invariants. The phase ϕ, at the moment identified with that yielding the quantum-mechanical interference, is postulated by Wentzel to be the partly invariant[2] measure

$$\phi = -\frac{1}{h} \int_E^A \left(\sum_{i=1}^{n} q_i \mathrm{d}p^i - t\mathrm{d}W \right) \tag{4}$$

in these coordinates, for any path[3]. The quantum interference is supposed to be analogous to the wave interference. Wentzel writes: *Indem wir die klassische Wellenphase durch unsere Quantenphase ersetzen, ist es nun leicht, die wellentheoretische Interferenzformel in die Sprache der Quantenstatistik zu übersetzen: Stehen dem Lichtquant verschiedene Wege s von E nach A zur Verfügung, so ist die Wahrscheinlichkeit, daß es auf einem beliebigen der Wege s nach A gelangt und dort absorbiert*

[2] Canonical transformation possibly add a term not depending on the paths between given endpoints. Hence, the interference of the different contributions is not affected. The path-independent term corresponds to a phase factor in quantum mechanics. Wentzel's expression differs form Feynman's by such a path-independent term.

[3] The sign is corrected in Wentzel's second article.

wird, nicht etwa gleich der Summe der Apriori-Wahrscheinlichkeiten der einzelnen Lichtwege s, sondern J mal so groß[4], *wo*

$$J = \frac{(\sum \mathbf{f}_s e^{2\pi i \phi_s})(\sum \mathbf{f}_s^* e^{-2\pi i \phi_s})}{|\sum \mathbf{f}_s|^2} \ . \tag{5}$$

Wentzel identifies the principle of zero deviation from "mechanics" with the Fermat-Jacobi principle of least refraction-corrected path, as was general use in the discussion of the wave-particle duality at that time. The decisive step however is to write an interference formula for all different paths with the same start and the same end, independent of their being "mechanical", i.e. solution of the equation of motion, or not. Such an interference formula did not exist before. This constitutes the difference to all other attempts to interpret wave phenomena by particle motion at that time. In the particular situation described by Wentzel, the probability is related to the amplitude of light ($\mathbf{f}_s$ is the vectorial amplitude of the classical wave), but both the variables entering the formulae and the basic philosophy are not merely characteristic for optics alone, but for mechanics in general. Wentzel writes expressis verbis: *Die formale Übereinstimmung des Zählers mit dem Amplitudenquadrat superponierter Wellen sichert dem Ansatz eine ausnahmslose Gültigkeit, was die Beschreibung irgendwelcher Interferenzphänomene anbelangt*[5]. We want to underline that the explicit use of the interference formula introduced by Wentzel is the decisive step to quantum particle mechanics. The same interference concept is the basic idea of Feynman's approach in 1948 too. The gap to this approach consists in the explicit technics for handling a path integral, i.e. for really calculating a transition amplitude.

3 The response to Wentzel's article

Before the publication, in 1925, of Heisenberg's method to calculate matrix elements, dispersion theory was one of the central topics

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[4] By replacing the classical phase of the wave with our quantum phase, it turns out to be simple to translate the interference formula of wave theory into the language of quantum statistics: If the quantum of light may propagate along different paths from E to A, the probability for going to A along any of them and being absorbed there is not given by the sum of the a priori probabilities of the individual paths s, but J times that value.

[5] The formal coincidence of the numerator with the square of the amplitude of superposed waves ensures the ansatz to be universally valid for the representation of interference phenomena of any kind.

of discussion. The question was how to get interference and dispersion with a flow of particles (the quanta of light, named "*Nadelstrahlung*") [26,24,42,47,57,48,59,49]. The classical theory of dispersion explained the relation between absorption lines and anomalous dispersion, but neither the number nor the narrowness of lines. Guth [32] characterized that time by the battle between wave and particle structure of light, the particle aspect always increasing its realm. Smekal [57] expected still a much longer way to show that wave theory was not indispensable for optics[6]. There had been several attempts to outline a particle-based explanation of wave phenomena [26,24,42,47,48,49,44,60,45,61]. None of them really reached the stage of a mathematically constructed theory, and none of them reached the stage of a basis for mechanics in general such as Wentzel's.

Wentzel's paper was always understood to be part of that discussion, even by Wentzel himself, and the by far ampler importance of his postulates, for mechanics in general, went along unnoticed. In addition, Wentzel used his scheme to argue for the adoption of an intermediate orbit between initial and final state of an atom interacting with light, and his results were not backed by experiment. Kramers and Heisenberg used only the initial orbit and could fit the data [38]. The formulas which Wentzel developed for dispersion with his interference concept in mind [65] were cited by Kramers and Heisenberg [38], but rejected in a lengthy footnote[7]. The success of the formula found by Kramers and Heisenberg and the change in direction of the following evolution of quantum mechanics made this judgement final. So Wentzel himself cites his paper only once, in the following article "*Zur Quantentheorie des Röntgenbremsspektrums*" [64]. Already there the exposition of the method is banned to an appendix[8]. In the journal *Physikalische Berichte*

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[6] "*Bis zur Verwirklichung derartiger Zukunftshoffnungen, welche in mancherlei Hinsicht geeignet waeren, das Dogma von der Unentbehrlichkeit wellentheoretischer Ueberlegungen in der Optik der Reflexion und Interferenz zu zerstoeren, ist aber vielleicht noch ein sehr weiter Weg.*" (For the realization of such hopes for the future, which would be appropriate under several respects, namely, to destroy the dogma of the indispensability of wave-theoretical arguments in the optics of reflection and interference, perhaps there is still a much longer way.)

[7] In short: "*Es gibt keine experimentellen Gründe, die Gültigkeit einer einfacheren Formel anzuzweifeln.*" (There is no experimental motive to doubt the validity of the simpler (old) formula.)

[8] "*Zur Quantentheorie unperiodischer Systeme im allgemeinen*"

we find an abstract by Wentzel himself, which gives the impression too, that he only marginally recognized the importance of his approach for mechanics[9].

Wentzel was quoted by his colleague in München K.F.Herzfeld [34]. It is not completely clear whether Herzfeld cites Wentzel in reference to a program or to a theory in his article[10]. As far as we found out, Herzfeld [68] was the last one to give a full account of Wentzel's approach, naming it "Corpuscular theory of interference". But new interest did not arise. Citing Wentzel, Herzfeld mentioned also Beck [2]. This shows that the interference notion seemed to him to be more important than the integral over the manifold of paths.

Other authors quoted Wentzel's concept occasionally. For instance, when Epstein and Ehrenfest [27] wrote that coherence and interference would resist any attempt to understand, Smekal [58] answered by citing Wentzel as an example for such an understanding. A.Landé [43] too presumably felt the importance of the concept, but he criticized it with an argument based on causality. However, this argument would invalidate also Feynman, if correct and applicable. Pauli [50] cites Wentzel and Herzfeld only together with Ornstein and Burger [47,48,49], which proves that he did not notice the far more general importance and the explicitness of Wentzel's concept[11]. Pringsheim [51] and Kulenkampff

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[9] *Es wird versucht, die Interferenzerscheinungen vom Standpunkte der Lichtquanten aus als fundamentale statistische Phänomene zu verstehen. Die Möglichkeit dazu ergibt sich daraus, daß die Lichtphase* $\int \mathrm{d}s/\lambda$ *durch die Bohrsche Frequenzgleichung* $hc/\lambda = \Delta W$ *eine einfache mechanische Bedeutung erhält...* (It is attempted to understand the interference phenomena from the viewpoint of light quanta as fundamental statistical phenomena. From here the possibility emerges that the phase of the light obtains a simple mechanical meaning through Bohr's frequency equation).

[10] Herzfeld writes: *Die zweite Aufgabe besteht ... in der quantentheoretischen Deutung des Huygensschen Prinzips. Diese Aufgabe ist aber ... nicht verschieden von der allgemeinen, welche die quantentheoretische Deutung der Interferenz stellt. Sobald diese gelöst ist* [63], *ist damit auch die Brechung usw. erklärt* (The second task is the quantum interpretation of Huygens' principle. This task is not distinct from the general one to explain the interference in quantum theory. As soon as this problem is solved, refraction etc. is explained too).

[11] "*Die Versuche von G.Wentzel, K.F.Herzfeld und L.S.Ornstein u. H.C.Burger, die Ausbreitung des Lichtes in dispergierenden Medien vom reinen Lichtquantenstandpunkt aus zu behandeln, können vorläufig wohl noch kaum als befriedigend angesehen werden.*" (The attempts by G.Wentzel, K.F.Herzfeld

[41] only quote the experimental argument against Wentzel's dispersion theory.

Presumably under the influence of the successes of quantum mechanics Wentzel changed his attitude and restricted his work to the use of the classical ("mechanical") solutions and to the correspondence of the Hamilton-Jacobi function to a single phase. In this way, Wentzel also found the approximation scheme now known as Wentzel-Brillouin-Kramers method [66,11,10,39]. Here at last, the concept of interfering paths is forgotten. All the studies concerning the correspondence between classical mechanics and quantum mechanics now concentrated on geometrical optics and canonical transformation theory [36,37,62]. The Schrödinger equation was fitted best to the task of calculating spectra and underlying energy levels, and both the problem of second quantization and of calculating transitions in more general problems were still ahead.

Even in F.Hund [35], who explicitely aims to answer the question whether quantum mechanics could have evolved differently, we find no hint to Wentzel's path integrals.

Dirac was the next to express the idea of getting probabilities by superposition of paths [21]. Dirac connected it mainly to the canonical tranformation theory in the direction of the Wentzel-Brillouin-Kramers approximation, i.e. to geometrical optics. The interference principle is formulated, but not explicitly. The identification of a phase with the action integral is the result of the construction of the unitary evolution operator. This construction is not possible without the knowledge of quantum mechanics existing at that time in the Heisenberg or Schrödinger form. Feynman [30] asserts that his work was inspired by Dirac's publications. Now Dirac was always very sparing of citations, so it is difficult to draw conclusions from his not mentioning Wentzel. In the end, nobody recognized the outline already formulated by Wentzel; even he himself apparently never came back to the driving idea of his early work on quantum optics.

The reader of older literature is often suspected of falling into the trap of reading things into a book instead of reading out of a book. However, any book is complete only with the reader and changes with

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and L.S.Ornstein and H.C.Burger to deal with the propagation of light in dispersive media from a pure light-quantum viewpoint at present cannot yet be considered satisfactory.)

its reader. At any time, literature is lost if not read anew, with the knowledge and the capabilities added by the time between, not only knowledge of more physics, but also capability of deeper reading[12].

We want to thank H.-J.Treder for discussion and help with his private library, and to acknowledge the contribution of L.Mihich (Pavia) and F.Antoci for help and hints with the citations.

Appendices

A Wentzel's paper

At the beginning of his article "Zur Quantenoptik" Wentzel observes that since Einstein's derivation of Planck's radiation law certain probabilities are attributed to the emission and absorption processes, but no more precise assertions are made. He intends to propose a general hypothesis for such probabilities, that in his opinion can help in overcoming the contradiction existing in theoretical optics: wave theory of interference and polarization on one side, quantum theory of the spectral lines on the other side. To this end, he interprets the interferences as the offspring of underlying quantum-statistical laws.

In Section 1 of his paper Wentzel remembers that the most important foundation of the quantum theory is certainly the law that an atomic system cannot radiate if it finds itself in what he calls a mechanical state, i.e. a state in which the laws of classical mechanics are obeyed.Radiative processes are instead invariably associated with "transitions" for which the laws of classical mechanics do not hold. But not only the acts of emission and of absorption are "non-mechanical", since the very presence of light propagating through a transparent medium will cause non-mechanical perturbations in the atoms involved in the process.

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[12] J.L.Borges writes in the essay *El libro* [3]: *Cada vez que leemos un libro, el libro ha cambiado, la connotación de las palabras es otra. Además, los libros están cargados de pasado …Si leemos un libro antiguo es como si leyéramos todo el tiempo que ha transcurrido desde el día en que fue escrito y nosotros.* (Every time when we read a book, the book has changed, the connotation of the words is different. In addition, the books are loaded with past …If we read an ancient book, it is as if we read all the time elapsed between the day when it was written and us.)

In order to provide an invariant measure of the deviations of the intra-atomic motions from Hamiltonian mechanics, Wentzel considers the canonical coordinates β_k and the conjugate momenta α_k associated with the atomic systems involved in the propagation of light. For simplicity, the α_k are assumed to be constant in the mechanical states. Then the desired measure is provided by the integral $\int \sum_k \beta_k \mathrm{d}\alpha_k$. This integral is extended to the particular path in phase space that corresponds to the deviations from mechanics caused by a light quantum going from an emitting atom E to an absorbing atom A in a certain way. Wentzel attributes to any path of this kind a phase

$$\varphi = \frac{1}{h} \int \sum_k \beta_k \mathrm{d}\alpha_k \ , \tag{6}$$

where h is Planck's constant. φ provides the sought-after bridge between the quantum behaviour and the wave-like phenomena.

Wentzel introduces the total energy W of the atomic systems as one of the momenta (α_1), and the time t as the coordinate β_1 conjugated to it; the phase φ is then defined as[13]

$$\varphi = \frac{1}{h} \left(\int t \mathrm{d}W + \int \sum_2 \beta_k \mathrm{d}\alpha_k \right) . \tag{7}$$

As a check of his ideas within geometrical optics Wentzel envisages the simple system constituted by the atoms E and A exchanging a light quantum of energy ΔW that travels in vacuo with the velocity c along the path joining the two atoms, and from his definition (7) of the phase, by retaining only the first addendum, he recovers Bohr's $\Delta W = h\nu$ principle. Section 1 ends with the definition of the refractive index n, and with the remark that Fermat's principle $\delta \int n \mathrm{d}s = 0$ can now be rewritten as $\delta \sum \int \beta_k \mathrm{d}\alpha_k = 0$, i.e. as the requirement that for the rays

[13] Since Wentzel does not write the upper limit to the summation index k, it is possible to interpret this new definition in two ways: either it is the outcome of a canonical transformation performed in ordinary phase space, or it corresponds to an extension of the phase space itself by the addition of energy and time as a further conjugate pair. In the latter case the phase introduced by Wentzel is just the one considered by Feynman (apart for a wrong sign, and a path-independent term). In the appendix to a subsequent paper [64] Wentzel clearly chooses the latter option, and also corrects the sign error of eq.(7).

of geometrical optics the integrated deviation from mechanics shall be a minimum.

In Section 2, Wentzel defines his interference formula.

If the light quantum has several paths at his disposal for going from an emitting atom E to an absorbing atom A, the overall probability of the process is not equal to the sum of the a priori probabilities associated with the individual paths, which is given by

$$| \mathcal{F}_0 |^2 = | \sum_s \mathbf{f}_s |^2 \ , \tag{8}$$

where $\mathbf{f}_s$ is the vector amplitude of the classical wave associated with the s-th path. The overall probability is instead supposed to be J times the a priori probability, where

$$J = \frac{(\mathcal{F}\tilde{\mathcal{F}})}{| \mathcal{F}_0 |^2} \tag{9}$$

and the complex amplitude $\mathcal{F}$ is given by

$$\mathcal{F} = \sum_s \mathbf{f}_s \exp(2\pi \mathrm{i} \varphi_s) \tag{10}$$

where φ_s is the quantum phase defined by (6) or by (7). Wentzel emphasizes the general validity of his formula for interference processes of any kind, and the advantage of ensuring a priori that the "wavelength" measured through the interferences and through the photoelectric effect are one and the same thing.

He further notices an essential feature: in his conception the emitting and the absorbing systems are intrinsically coupled. To him it is also noteworthy that no interference is conceivable without the presence of the absorbing system. Section 2 ends with a long Note dealing with the issue of the coherence length, as it can be confronted from the proposed viewpoint.

In Section 3 Wentzel outlines a theory of discrete spectra through a specialized use of his phase and interference formulae. While contemplating only the degrees of freedom of the emitting atom, he introduces the action variables I_k and the conjugated angle variables

$$w_k = t \cdot \frac{\partial W}{\partial I_k} + u_k \ , \tag{11}$$

as it is customary when dealing with "conditionally periodic" systems. The u_k appearing in (11) are undetermined phases, which are constant in the mechanical motions. Wentzel tentatively[14] postulates that they remain constant also during the transitions, and assumes that, in order to get the transition probability, one shall modify the interference formula proposed in Section 2, since one shall not only sum the amplitudes associated with the individual paths, but also take the average over the undetermined phases u_k. Under these assumptions, he finds that the probability of transition can be nonvanishing only when the action variables change by an integer multiple of Planck's action quantum:

$$\Delta I_k = n_k h \ . \tag{12}$$

Therefore, if initially "quantized", the atom will find itself after the transition in another quantized state.

Under the mentioned assumptions Wentzel calculates the expression for the amplitude $\mathcal{F}$ and finds agreement with Bohr's correspondence principle for the intensity and for the polarization. Then he compares his result with the predictions of the classical wave theory, and asserts that through his theory one can describe refraction, reflection and double refraction just as it is done classically; in fact, he adds, Huygens' principle is just based on interferences.

Wentzel ends the Section and the paper by observing that, while in the former quantum theory the action quantum h had to be introduced twice, i.e. once in the $\Delta W = h\nu$ principle, and a second time in the quantum conditions, his theory allows to introduce it just once, in the expression (6) for the quantum phase.

B Canonical transformations

Canonical coordinates (q^i, p_k) are coordinates of the phase space. They are defined by the canonical form of the equations of motion,

$$\frac{\mathrm{d}q^i}{\mathrm{d}t} = \frac{\partial H[q,p,t]}{\partial p_i} \ , \quad \frac{\mathrm{d}p_k}{\mathrm{d}t} = -\frac{\partial H[q,p,t]}{\partial q^k} \ .$$

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[14] Already in the subsequent paper [64] dealing with continuous spectra Wentzel changes his mind, and assumes that the emission and the absorption processes are characterized by the variation in time of both the action variables and of the undetermined phases.

Because of the sign, we identify configuration coordinates q^i and conjugate momenta p_k. Canonical transformations are transitions from one set of canonical coordinates (q, p) to another one (Q, P). The method of interest to generate canonical transformations is a Legendre transform. We assume the existence of a generating function $F[q, P, t]$ depending on the old configuration coordinates q^i and the new momenta P_k. The function $F[q, P, t]$ generates the transformation by

$$Q^i = \frac{\partial F[q, P, t]}{\partial P_i} \ , \quad p_k = \frac{\partial F[q, P, t]}{\partial q^k} \ .$$

The new Hamilton function is then given by

$$\bar{H}\mathrm{d}t - \sum_{i=1}^{n} P_i \mathrm{d}Q^i = H\mathrm{d}t - \sum_{i=1}^{n} p_i \mathrm{d}q^i + \mathrm{d}(F - \sum_{i=1}^{n} Q^i P_i)$$

or

$$\bar{H} = H + \frac{\partial F}{\partial t} \ .$$

Any new Hamiltonian $\bar{H}[Q, P, t]$ can be constructed as long as the partial differential equation

$$\bar{H}[\frac{\partial F}{\partial P}, P, t] = H[q, \frac{\partial F}{\partial q}, t] + \frac{\partial F[q, P, t]}{\partial t}$$

can be solved. The Hamilton-Jacobi transformation aims at a vanishing new Hamiltonian and constant configuration coordinates and momenta:

$$S = S[q, P, t] \ , \quad \bar{H} = H[q, \frac{\partial S}{\partial q}, t] + \frac{\partial S}{\partial t} = 0 \ , \quad p_i = \frac{\partial S}{\partial q^i} \ , \quad Q^i = \frac{\partial S}{\partial P_i} \ ,$$

$$\bar{H}\mathrm{d}t - \sum_{i=1}^{n} P_i \mathrm{d}Q^i = H\mathrm{d}t - \sum_{i=1}^{n} p_i \mathrm{d}q^i + \mathrm{d}(S[q, P, t] - \sum_{i=1}^{n} Q^i P_i) \ .$$

Then we get

$$\sum_{i=1}^{n} Q^i \mathrm{d}P_i = -L\mathrm{d}t + \mathrm{d}S \ .$$

By this transform, the congruence of paths in phase space can be mapped onto the initial values $Q^i, P_i (i = 1, \ldots, n)$ of coordinates and momenta at some time t_0. If we characterize the states by ordinary coordinates $q[t]$ and conjugate momenta $p[t]$, the integral

$$\int_E^A \sum_{i=1}^n Q^i[q,p,t] \mathrm{d}P_i[q,p,t]$$

vanishes for the classical solution. The definition of the generating function yields

$$\begin{aligned}\int_E^A \sum_{i=1}^n Q^i[q,p,t] \mathrm{d}P_i[q,p,t] &= -\int_E^A L[q,\dot{q},t]\mathrm{d}t \\ &\qquad + S[q_A,P_A,t_A] - S[q_E,P_E,t_E] \\ &= -\delta \int_E^A L[q,\dot{q},t]\mathrm{d}t \ .\end{aligned}$$

Because the left-hand side vanishes for the solutions of the canonical equations (the "mechanical" paths in Wentzel's language), the right-hand side is not merely the action integral corrected by the path-independent term $S_A - S_E$, but the deviation in the action integral, which is zero for the extremal.

If the Hamilton function does not depend on time explicitly, we may look for action-angle variables by separating the time from the Hamilton-Jacobi function. We obtain

$$H = H[q,p] \ , \quad \frac{\partial H}{\partial t} = 0 \ , \quad S[q,P,t] = W[q,P] - E[P]t$$

with the new transformation

$$\bar{H}\mathrm{d}t - \sum_{i=1}^n P_i \mathrm{d}Q^i = H\mathrm{d}t - \sum_{i=1}^n p_i \mathrm{d}q^i + \mathrm{d}(W[q,P] - \sum_{i=1}^n Q^i P_i) \ , \qquad (13)$$

$$p_i = \frac{\partial W}{\partial q^i} \ , \quad Q^i = \frac{\partial W}{\partial P_i} \ , \quad H[q, \frac{\partial W}{\partial q}] = E[P] = \bar{H}[P] \ .$$

The canonical equation now determine the motion to follow

$$\dot{Q}^i = \frac{\partial \bar{H}}{\partial P_i} = \mathrm{const} \ , \quad \dot{P}_i = \frac{\partial \bar{H}}{\partial Q^i} = 0 \ .$$

One may interpret Wentzel's introduction of the energy as canonical momentum as follows. We solve $E = E[P]$ for the first momentum P_1, and use E instead of the old P_1 for the first new momentum, separating it from the others $(P_2, \ldots, P_n)$. We then obtain

$$\dot{Q}_1 = \frac{\partial \bar{H}}{\partial E} = 1 \;\rightarrow\; Q_1 = t + t_0 \;, \quad \dot{Q}_2, \ldots, \dot{Q}_n = 0 \;.$$

and our special formula changes from (13) to

$$\bar{H}\mathrm{d}t - \sum_{i=2}^{n} P_i \mathrm{d}Q^i - E\mathrm{d}t = -L\mathrm{d}t + \mathrm{d}(W - Et - \sum_{i=2}^{n} Q^i P_i) \;,$$

$$\sum_{i=2}^{n} Q^i \mathrm{d}P_i + t\mathrm{d}E = -L\mathrm{d}t + \mathrm{d}W \;\; - E\mathrm{d}t = -\bar{L}\mathrm{d}t + \mathrm{d}W \;,$$

where $\bar{L} = \sum_1^n p_k \dot{q}^k$ is the reduced Lagrangian. This is Wentzel's formula in the ordinary phase space interpretation.

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