Fast iterative method for the recurrence of an elementary integrable system (I)

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ABSTRACT. The recurrence on a torus is studied by making use of iteration formulae providing an essential tool to calculate the rigorous (non statistical) state of the system in the long run. It is a fast method, in the sense that each one of the iterations corresponds to a position of the representative point of the system one half (at least) closer to the initial position than the preceding iteration. The corresponding inverse formulae are also calculated. Both are applied in geometric theory of numbers leading to an almost intuitive method of approximation of irrationals. An application is also given providing the calculation and ordering of Poincaré recurrent cycles of a dynamical system with two normal modes.

RÉSUMÉ. On étudie la récurrence sur un tore en faisant emploi de formules d'itération fournissant une méthode rigoureuse (c'est à dire non statistique) pour déterminer l'état du système à tout instant. Il s'agit d'une méthode rapide, en ce sens qu'à chaque itération le point représentatif du système se trouve à une distance de sa position de départ qui est égale ou inférieure à la moitié de la distance correspondante fournie par l'itération précédente. On déduit aussi les formules de l'itération inverse. Ces deux ensembles d'équations sont appliqués en théorie géométrique des nombres conduisant à une méthode presque intuitive d'approximation d'irrationnels. Les mêmes formules sont aussi appliquées pour calculer et ordonner les cycles récurrents de Poincaré d'un sytème dynamique à deux modes normaux.

I - Introduction

Since the appearance of the recurrence theorem that bears the name of Poincaré [1],[2], the logical inferences from that fascinating and yet simple result have been the safe ground from which the most various and unorthodox speculations have taken flight.

In fact, shortly after its appearance, this result was expectedly found at the epicenter of a controversy between Boltzmann and Zermelo [3],[4], which turned around what then seemed an intrinsic paradox: the actual recurrence, after some very long interval of time, of the initial state of a real physical system for which the hypothesis of Poincaré's theorem hold true (Needless to say, this last and essential requirement was the more controversial).

The way found by Boltzmann to overcome this difficulty is well known and lies in a famous example in which he took the molecules of 1 cm3 of a gas (with normal pression and temperature) and calculated the expected value for the time needed for the molecules to be "near enough" their initial state (The indiscernability was taken of the order of 10 percent of the average distance between the molecules for positions, and 1/500 for the velocities). This calculation, though performed with the rudimentary "ergodic" tools of the early times (as far as we know, it has never been improved since then), leaves no doubt as for the conclusion: such interval of time (the so-called Poincaré cycle) is an enormous one, far beyond what we nowadays assume to be the age of the solar system. The unavoidable conclusion was that from a "practical" point of view (if not in absolutely satisfactory philosophical grounds), recurrence is actually avoided and removed: the time needed for the system (any system ?) to recur over its initial state is overwhelmingly greater than the duration of its own life-time.

Yet two features of Boltzmann's reasoning deserve a careful thought: the first one is that the "small" system considered by Boltzmann has nevertheless a huge number of degrees of freedom (10^{18} molecules in the cm^3); the second one, which follows from the first, is that the calculation performed by Boltzmann is naturally a statistical one, for this seems the mathematical method more suited for dealing with such situations. Actually, even in a calculation as the one given in 1947 by Kac [5], where he integrates the recurrence times of a "simple" dynamical system over all admissible initial conditions, the exact (non probabilistic) values of such intervals of time are totally disregarded.

As far as we know, a deterministic calculation of a Poincaré cycle of a system with few degrees of freedom has never been given, in spite of all the importance attached by Quantum Mechanics (or, perhaps better, the Theory of Quanta) to systems with a small number of vibration modes. In fact, even the study of such a "simple" system as the Lissajous beats on the plane seems to have deserved no more than a statistical treatment. On the other side, and still confining ourselves to systems with few normal modes, it is hard to get rid of the feeling, underlying Kac's theorem, that such recurrence cycles may not be "very large" and that they must depend, perhaps very sharply, on the values of the normal frequencies and the initial conditions. In fact, such conjecture lies at the very basis of the present work, in which we present a deterministic (by opposition to statistical) method providing these recurrence times predicted by Poincaré for a very broad class of dynamical systems. The dynamical system we make use of in our work is the simplest one, with only two normal modes. Yet we shall see that even such an "elementary" situation brings about some considerable difficulties which express all the richness and complexity of the underlying symmetries of the problem. Its generalisation for a higher number of modes (which we shall not treat in this work) does not seem to require any substantial change in the fundamental ideas, though we may undoubtedly expect a somewhat frightening weight for the corresponding formalism.

Perhaps we could condense the present paper (part I) by simply stating that it consists of a thorough investigation of the function $(\chi) \bmod 2\pi$, that is, the rotation of a circle, of its inverse function and of some related properties. From this, it is not unbecoming to expect our methods to be extended as to include some homeomorphisms of the circle, such conjecture (which we shall not develop here) being based on Denjoy's theorem [6] and subsequent developments - see, for instance, [7],[8] - by which some very general homeomorphisms of the circle can be reduced to a rotation.

Another aspects of these results are those developed in part II and concerning some applications in dynamical systems (ordering the recurrence Poincaré cycles) and geometric theory of numbers. In fact, the determination of a Poincaré cycle of our system (up to a certain order of approximation of its initial state) provides an approximation (up to a certain order, related to the former one) of an irrational number. This amounts to saying that, under very broad terms, both problems are one and the same.

To end up this introduction, let us point out that we expect our methods to have also an useful application in two other mathematical fields related to the above mentioned ones, namely, ergodic theory and computational calculus. In ergodic theory it is our hope that by making use of our simple algorithms (namely, those presented in paragraphs

II and III of the present paper) some simple ergodic problems of iteration that are only handled under a probabilistic or statistical way (viz, Staeckle and Kac theorems) may be studied up to a rather accurate approximation in a rigorous deterministic way. As for the computational applications, our belief is based upon the fact that our algorithms should prove more performing or reliable, namely, in straightforward calculations of long time simulations or of approximations of irrational numbers where other methods seem less numerically accurate [9].

The structure of the present paper (part I) is the following:

In II we derive a deterministic method that allows us to carry out the rigorous calculation of the consecutive instants of time (all of them multiples of a certain elementary interval taken as unity) and the associated values of the reduced phase at which the representative point of a 2-normal mode system is found within an increasingly smaller neighbourhood of its initial state. The inverse problem is treated in III and IV: given a certain instant of time, we calculate the exact values of the reduced phase taken by the representative point at that instant. Both this method and the preceding one are the fundamental tools employed throughout part II in approximating irrational numbers and ordering the succession of the Poincaré recurrence cycles.

II - The iteration formulae.

Let us consider a dynamical system with two normal modes. The evolution of its representative point on the torus may be followed by its projection over two planes, that is, by two representative points denoted by RP_o and RP, each one of them describing a circle centered in the origin of coordinates with constant velocity. We assume that at initial time $t = 0$ the phases of RP_o and RP are null and that in the course of time they increase in clockwise sense. We denote by τ the constant period of the periodical motion of RP and by τ_o that of RP_o. In what follows our concern will be the positions of the representative point of the system at instants $t = k\tau_o$ (multiples of the period of RP_o). This means that we want to calculate the phases of RP at $t = k\tau_o$, since at these instants RP_o will meet again the zero value of its initial phase. For this elementary dynamical system the theorem of Liouville clearly holds true and so does the recurrence property.

In this paper we always consider "reduced" phases, that is, mod(2π), with their values in the interval $(0, 2\pi)$. More precisely, $\phi(t)$ means the reduced phase of the representative point RP at instant t. Let then be the RP starting with zero phase at $t = 0$, and let us denote by δ_o the value of its reduced phase at $t = \tau_o$:

$$\delta_o \equiv \phi(t = \tau_o) = (2\pi \frac{\tau_o}{\tau}) \bmod 2\pi$$

For the sake of simplicity, we shall always assume in what follows that $\delta_o \epsilon (0, \pi)$, since the case $\delta_o \epsilon (\pi, 2\pi)$ can be reduced to the preceding one by means of a trivial shift in the phase, and the equality $\delta_o = \pi$ means that the quotient of the periods τ and τ_o is a rational number. Furthermore, we shall take a time scale unit $\tau_o = 1$. (Nevertheless, and whenever need is felt of recalling the dynamical meaning of τ_o, we shall write it explicitly.)

Now, for any value of $\delta_o = (0, \pi)$, it is clear that it always exists an infinite sequence of positive integers $[l] \equiv [l_1, l_2, l_3, \ldots]$ such that

$$2 \leq l_1 < l_2 < l_3 < \ldots < l_\chi < \ldots$$

and

$$\phi(t = l_\chi \tau_o) - \phi(t = (1 + l_\chi)\tau_o) > \pi$$

To each of these values $\chi = 0, 1, 2, \ldots$, we associate the following couple of real numbers

$$V = V(\chi) \equiv [\phi(l_\chi \tau_o) - 2\pi < 0, \quad \phi((1 + l_\chi)\tau_o) > 0]$$

and on some oriented line we represent at left the first of these values (the negative one), and at right the positive one. If we take these horizontal lines $\chi = 0, 1, 2, \ldots$ altogether and place the origin of each one of them on the same vertical line we obtain the diagram shown in figure 1. V will be called the "phase shift" and takes values in the interval $V \epsilon (-\delta_o, +\delta_o)$, while the domain for the reduced phase is the interval $[0, 2\pi]$. The integer labeling each horizontal will be called its *order number* and is denoted by q.

In this phase shift diagram we are going to determine **two infinite sequences of points,** $P_1, P_2, \ldots, P_k, \ldots and P'_1, P'_2, \ldots P'k, \ldots$ corresponding to the positions of RP at certain instants $t = T_K$ (for points P_k) and $t = T'_k$ (for points P'_k), all located near $V = 0$, and belonging to certain horizontals with order number $q = q_k$ (for points P_k) and $q = q'_k$

(for points P'_k). We shall denote by $V = \delta_k$ (for the P_k) and by $V = \delta'_k$ (for the P'_k) their phase shifts. Then, *by definition*, we have:

a) P_k $(k = 1, 2, \ldots)$, is the first position (in the course of time) of RP such that

$$| \delta_j | < \frac{1}{2} | \delta_{j-1} | \quad (j = 1, 2, 3, \ldots) \tag{2.1}$$

b) P'_k $(k = 1, 2, \ldots)$ is the first position of RP such that

$$| \delta_j | + | \delta'_j | = | \delta_{j-1} | \quad (j = 1, 2, 3, \ldots) \tag{2.2}$$

From the above it immediately follows that δ_k and δ'_k have opposite signs and also that

$$0 < | \delta_1 | < \frac{\delta_o}{2} < | \delta'_1 | < \delta_o$$

$$0 < | \delta_k | < \frac{1}{2} | \delta_{k-1} | < | \delta'_k | < | \delta_{k-1} | < \frac{\delta_o}{2^k} \quad , \quad (k \geq 2)$$

Figure 2 shows the position of points P_k, P'_k (with $k = 1, 2, 3$). One also sees that the “fundamental” direction of the diagram that goes through the position of RP in $t = \tau_o = 1$ intersects a certain finite number of horizontals (in the figure, this number is equal to $\nu_1 + 2 = 1 + 2$, where ν_1 is the integral part of the quotient $| \frac{\delta'_1}{\delta_1} |$) in points with positive phase shift, and intersects all the other horizontals in points with negative phase shift. The last of the former ones and the first of the later ones are obviously P_2 and P'_2, and P_2 is simply that of the two which is the nearest to $V = 0$,that is, with the least value for the absolute value of the phase shift.

We may remark that, *due to the autonomy of our dynamical system*, the positions of RP over the horizontals with order number q such that $jq_2 = 3j = (\nu_1 + 2)j \leq q < 3(j + 1) = (\nu_1 + 2)(j + 1)$ (with $j = 1, 2, \ldots$) are obtained from the corresponding points over the first $\nu_1 + 2 = 3$ horizontals by displacing them at left of a length $j\delta_2$, which means, more precisely, that to those values of the phase shift is now added the algebraic value of the additional phase shift $j\delta_2$ (in the case of figure 2, $\delta_2 < 0$).

If, in order to abridge our notation, we call “block $[i]$” the ensemble of horizontals with order numbers $q = 0, 1, \ldots, q_i - 1$, then we may say that a block $[i]$ is followed by at least another ensemble of q_i horizontals

$(q = q_i, q_i+1, q_i+2, \ldots, 2q_i-1)$ which exactly reproduce the same block $[i]$ now displaced, as a whole, by a phase shift δ_i. We shall then speak of the *"displaced blocks"* $[i]_{\delta_1}, [i]_{2\delta_1}, \ldots, [i]_{k\delta_1}, \ldots$. To be more precise, $[i]_{k\delta_1}$ is formed by the horizontals $q = kq_i, kq_i + 1, kq_i + 2, \ldots, (k+1)q_i - 1 \quad (k \geq 1)$, and the phase shift of the position of RP in each one of these is equal to that of the corresponding position of RP in $[i]$, increased (algebraically) by $k\delta_1$. Of course, for $k = 0$, we have $[i]_{k\delta_i} = [i]$.

Let us point out that (with the obvious exception of $t = 0$) there is no position of RP in block $[i]$ with phase shift V such that $\mid V \mid < \mid \delta'_i \mid$, which means that no position of RP can there be found in the $\mid \delta'_i \mid$ -neighbourhood of $V = 0$, and that in the most "unfavourable" situation where $q'_i < q_i$. We only need to keep this remark in mind to conclude from equations (2.7) below that between any two consecutive iterations there is no better approximation of the RP to its initial situation.

Obviously, similar definitions are given for the points P'_i: Block $[i]'$ is thus formed by the horizontals with order numbers $q = 0, 1, \ldots, q'_i - 1$. It then follows that $[i]'$ is *either* a block $[i]$ extended with some additional horizontals (if $q'_i > q_i$), *or* a block $[i]$ in which a certain number of its last horizontals are lacking (if $q'_i < q_i$). As for the displaced block $[i]'_{k\delta_1}$, its properties are similar to those of $[i]_{k\delta_1}$, and it comprehends the horizontals with order numbers $q = kq_i, kq_i + 1, kq_i + 2, \ldots, kq_i + q'_i - 1$.

Before proceeding with the deductions, we must introduce a notation of leading importance for what follows. In fact, one expects the quotients of the form $\mid \delta'_k/\delta_k \mid$ to play a prominent role in our methods of calculus. We shall then write

$$\mid \frac{\delta'_k}{\delta_k} \mid \equiv \nu_k + \mu_k$$

where ν_k is the integral part of the quotient. We shall also denote by $\bar{\nu}_k$ and $\bar{\bar{\nu}}_k$ the two consecutive integers between which the real number $\nu_k + \mu_k$ is found, $\bar{\nu}_k$ being the nearest to $\nu_k + \mu_k$. In other words this means that we have:

$$\mu_k < \frac{1}{2} \Rightarrow \bar{\nu}_k = \nu_k, \bar{\bar{\nu}}_k = \nu_k + 1$$

and

$$\mu_k > \frac{1}{2} \Rightarrow \bar{\nu}_k = \nu_k + 1, \bar{\bar{\nu}}_k = \nu_k$$

(In the example of figure 2 we have $\bar{\nu}_o = \nu_o, \bar{\nu}_1 = \nu_1 + 1 = 2, \bar{\nu}_2 = \nu_2 = 2$). Let then be the quotient

$$\frac{2\pi}{\delta_o} \equiv \nu_o + \mu_o \tag{2.3}$$

where $\nu_o \geq 2$, and $0 < \mu_o < 1$ is the integral part of the irrational number $2\pi/\delta_o$. It is then clear that

$$\begin{aligned} \mu_o > \frac{1}{2} \Rightarrow \bar{\nu}_o \equiv \nu_o + 1 \Rightarrow \begin{cases} 0 < (1-\mu_o)\delta_o \equiv \delta_1 \\ 0 > -\mu_o\delta_o \equiv \delta_1' \end{cases} \\ \mu_o < \frac{1}{2} \Rightarrow \bar{\nu}_o \equiv \nu_o \Rightarrow \begin{cases} 0 < (1-\mu_o)\delta_o \equiv \delta_1' \\ 0 > -\mu_o\delta_o \equiv \delta_1 \end{cases} \end{aligned} \tag{2.4}$$

As for δ_2 and δ_2', their expression is similarly obtained from the quotient $\mid \delta_1'/\delta_1 \mid \equiv \nu_1 + \mu_1$, and we have

$$\begin{aligned} \mu_1 > \frac{1}{2} \Rightarrow \bar{\nu}_1 \equiv \nu_1 + 1 \Rightarrow \begin{cases} 0 < (1-\mu_1) \mid \delta_1 \mid \equiv \delta_2 \\ 0 > -\mu_1 \mid \delta_1 \mid \equiv \delta_2' \end{cases} \\ \mu_1 < \frac{1}{2} \Rightarrow \bar{\nu}_1 \equiv \nu_1 \Rightarrow \begin{cases} 0 < (1-\mu_1) \mid \delta_1 \mid \equiv \delta_2' \\ 0 > -\mu_1 \mid \delta_1 \mid \equiv \delta_2 \end{cases} \end{aligned}$$

More generally we have for $k = 1, 2, \ldots$

$$\mid \frac{\delta_k'}{\delta_k} \mid \equiv \nu_k + \mu_k \tag{2.5}$$

$$\mu_k > \frac{1}{2} \Rightarrow \bar{\nu}_k \equiv \nu_k + 1 \Rightarrow \begin{cases} 0 < (1-\mu_k) \mid \delta_k \mid \equiv \delta_{k+1} \\ 0 > -\mu_k \mid \delta_k \mid \equiv \delta_{k+1}' \end{cases}$$

$(2.6a)$

$$\mu_k < \frac{1}{2} \Rightarrow \bar{\nu}_k \equiv \nu_k \Rightarrow \begin{cases} 0 < (1-\mu_k) \mid \delta_k \mid \equiv \delta_{k+1}' \\ 0 > -\mu_k \mid \delta_k \mid \equiv \delta_{k+1} \end{cases}$$

$(2.6b)$

Upon these expressions one may easily check formulae (2.1),(2.2). By now taking in consideration what precedes we are finally able to particularize the structure of blocks $[i]$ and $[i]'$ defined above. We have thus:

a) Block $[i+1]$ (with $i = 2, 3, \ldots$) is formed by the sequence of $\bar{\nu}_i + 1$ blocks $[i], [i]_{\delta_1}, [i]_{2\delta_1}, \ldots, [i]_{(\bar{\nu}_i - 1)\delta_1}, [i]'_{\bar{\nu}_i \delta_1}$

b) Block $[i+1]'$ (with $i = \ldots 2, 3, \ldots$) is formed by the sequence of $\bar{\bar{\nu}}_i + 1$ blocks $[i], [i]_{\delta_1}, [i]_{2\delta_1}, \ldots, [i]_{(\bar{\bar{\nu}}_i - 1)\delta_1}, [i]'_{\bar{\bar{\nu}}_i \delta_1}$

We may leave aside the direct (and easy) proof of these assertions, since they easily follow by means of a reasoning similar to the one exposed in figure 2.

If we then denote by q_i and T_i the order number and the instant of time corresponding to point P_i, and by $q'_i \equiv q_i + p_i$ and $T'_i \equiv T_i + \theta_i$ (with p_i and θ_i positive or negative integers) the same entities attached to P'_i, then the same kind of argument leads to

$$P_{i+1}: \quad q_{i+1} = (\bar{\nu}_i + 1) q_i + p_i, \quad T_{i+1} = (\bar{\nu}_i + 1) T_i + \theta_i \qquad (2.7a)$$
$$P'_{i+1}: \quad q'_{i+1} = (\bar{\bar{\nu}}_i + 1) q_i + p_i, \quad T'_{i+1} = (\bar{\bar{\nu}}_i + 1) T_i + \theta_i \qquad (2.7b)$$

with $i = 2, 3, \ldots$

Let us recall that only $i - 2$ independent operations (the iterations given by formula (2.7)) are needed in order to obtain the values $\delta_i, \delta'_i, T_i, T'_i, q_i, q'_i, (i \geq 3)$ from $\delta_2, \delta'_2, T_2, T'_2, q_2, q'_2$ and that these, in turn, are directly obtained from the "fundamental datum" δ_o, that is, the quotient of the two periods: $\phi(t = \tau_o) = (2\pi\tau_o/\tau) \bmod 2\pi$

Let us also stress that T_k is the first instant of time at which the RP is found within an arbitrarily small $\mid \delta_k \mid -$ neighbourhood of its initial zero phase shift (with $\mid \delta_k \mid < \frac{\delta_o}{2^k}, k \geq 1$).

Our iteration formulae (2.7) thus provide a simple, deterministic (that is, non statistical) method for the rigorous determination of the first $\mid \delta_k \mid$-recurrence of the dynamical system formed by RP and RP_o. Not only do they give the rigorous values of the instants of recurrence and of the corresponding deviations of RP from the initial position, but they are also "systematic" in the sense explained above, that is, they assure the non existence of "better" approximations between any two consecutive iterations.

Concerning the order of approximation carried by the δ_k, we must add that formulae (2.2),(2.5) provide a rigorous expression that advantageously replaces the upper bound $\delta_o/2^k$. In fact, one gets from them

$$\mid \delta_n \mid = \delta_o . \Pi_{k=1}^{n} (1 + \nu_k + \mu_k)^{-1} \leq \frac{\delta_o}{2^n}$$

III - The inverse problem.

In the present paragraph we are going to solve the following problem: Given $\delta_o \equiv \phi(t = \tau_o) = (2\pi\tau_o/\tau) \bmod 2\pi$ and an instant of time $t = T$ (a positive integer, since $\tau_o = 1$), find $\chi(t = T)$ and $\phi(t = T)$. We shall of course assume that the method presented in the foregoing paragraph has already provided the basic tools needed in the sequel, viz, the sequences $\delta_k, \delta'_k, T_k, T'_k, q_k, q'_k$, totally determined by the sole knowledge of the value of δ_o.

We shall start by dismissing the two cases

$$0 \leq T < T_1 (or T'_1)$$

and

$$0 \leq T < T_2 (or T'_2)$$

The first one is trivial and is besides included in the second . Now this later one (to which, as we shall see, all the others can be reduced) will be, for this very reason, carefully investigated later on (see, for instance, equations $(\alpha), (\alpha'), (\beta), (\gamma)$).

For all other cases, it then exists an integer $N \geq 2$ such that

$$T \Rightarrow \exists N \geq 2 : T_N \leq T < T_{N+1} \tag{3.1}$$

The integer T then unambiguously determines two other integers, a_N and b_N in the following way:

$$T = a_N T_N + b_N \qquad (3.\ (\alpha_1)a)$$

where a_N is the greatest integer such that

$$1 \leq a_N < \bar{\nu}_N \qquad (3.\ (\alpha_1)b)$$

It then follows that

$$\text{if} \quad a_N = \bar{\nu}_N \Rightarrow b_N = 0, 1, \ldots, T'_N - 1$$

and for all other values of a_N,

$$b_N = 0, 1, \ldots, T_N - 1$$

Let us separately consider the two possible situations for δ_N:

$$\delta_N < 0$$

Then

$$\begin{aligned} b_N = 0 &\Rightarrow \phi(t=T) = 2\pi - a_N \mid \delta_N \mid = 2\pi + a_N\delta_N \\ &\Rightarrow q(t=T) = a_N q_N \Rightarrow \chi(t=T) + 1 = a_N q_N \end{aligned}$$

And for the other values of b_N, that is, $b_N = 1, 2, \ldots, T'_N - 1$ or $1, 2, \ldots, T_N - 1$, we have

$$\begin{aligned} &\Rightarrow \phi(t=T) = \phi(t=b_N) - a_N \mid \delta_N \mid = \phi(t=b_N) + a_N\delta_N \\ &\Rightarrow \chi(t=T) = a_N q_N + \chi(b_N) \end{aligned}$$

Let us now assume that

$$\delta_N > 0$$

$$\begin{aligned} \text{if} \quad b_N = 0 &\Rightarrow \phi(T) = a_N\delta_N \\ &\Rightarrow \chi(T) = q(T) = a_N q_N \end{aligned}$$

And for all other values of b_N, that is, for $b_N = 1, 2, \ldots, T'_N - 1$ or $T_N - 1$, we have

$$\begin{aligned} &\Rightarrow \phi(t=T) = \phi(t=b_N) + a_N\delta_N \\ &\Rightarrow \chi(T) = a_N q_N + \chi(b_N) \end{aligned}$$

We then see that the foregoing formulae concerning $\chi(T)$ can be summarized as follows:

$$\begin{aligned} &\text{if} \quad b_N = 0 \Rightarrow \chi(T) = \begin{cases} a_N q_N - 1, & \text{if} \quad \delta_N < 0 \\ a_N q_N, & \text{if} \quad \delta_N > 0 \end{cases} \\ &\text{if} \quad b_N \neq 0 \Rightarrow \chi(T) = a_N q_N + \chi(b_N) \end{aligned} \qquad (3.\ (\beta_1))$$

As for the formulae concerning the reduced phase, they can be abridgedly written under the form

$$\begin{aligned} &\text{if} \quad b_N = 0 \Rightarrow \phi(T) = \begin{cases} 2\pi + a_N\delta_N, & \text{if} \quad \delta_N < 0 \\ a_N\delta_N, & \text{if} \quad \delta_N > 0 \end{cases} \\ &\text{if} \quad b_N \neq 0 \Rightarrow \phi(T) = a_N\delta_N + \phi(t=b_N) \end{aligned} \qquad (3.\ (\gamma_1))$$

(Of course, $b_N \neq 0$ means $b_N = 1, 2, \ldots, T'_N - 1$ or $b_N = 1, 2, \ldots, T_N - 1$).

If $b_N = 0$, the method comes to its end. If not, we then have two possibilities, wether we previously got

$$a_N = \bar{\nu}_N \quad \text{or} \quad a_N \neq \bar{\nu}_N (\quad \text{that is} \quad , a_N < \bar{\nu}_N).$$

In fact, and starting from the value of b_N obtained above, one may determine two other integers, a_{N-1} and b_{N-1}, defined as follows:

$$b_N = a_{N-1} T_{N-1} + b_{N-1} \tag{3. $(\alpha_2)a$}$$

where a_{N-1} is the greatest integer such that

$$\begin{array}{l} 0 \leq a_{N-1} \leq \bar{\bar{\nu}}_{N-1}, \quad \text{if} \quad a_N = \bar{\nu}_N \\ 0 \leq a_{N-1} \leq \bar{\nu}_{N-1}, \quad \text{if} \quad a_N \neq \bar{\nu}_N (\quad \text{that is, if} \quad a_N < \bar{\nu}_N) \end{array} \tag{3. $(\alpha_2)b$}$$

One is then found in one of the three possible cases:

$$a_{N-1} = \bar{\nu}_{N-1} \quad , \quad a_{N-1} = \bar{\bar{\nu}}_{N-1} \quad , \quad a_{N-1} \neq \bar{\nu}_{N-1}, \bar{\bar{\nu}}_{N-1}$$

Let us assume that we have $a_{N-1} = \bar{\nu}_{N-1}$. It then follows $b_{N-1} = = 0, 1, \ldots, T'_{N-1} - 1$. Now

$$\begin{aligned} \text{if} \quad b_{N-1} = 0 \Rightarrow \phi(t = b_N) &= \begin{cases} 2\pi - a_{N-1} \mid \delta_{N-1} \mid, & \text{if} \quad \delta_{N-1} < 0 \\ a_{N-1}\delta_{N-1}, & \text{if} \quad \delta_{N-1} > 0 \end{cases} \\ \Rightarrow \chi(t = b_N) &= \begin{cases} a_{N-1} q_{N-1} - 1, & \text{if} \quad \delta_{N-1} < 0 \\ a_{N-1} q_{N-1}, & \text{if} \quad \delta_{N-1} > 0 \end{cases} \end{aligned}$$

and

$$\begin{aligned} \text{if} \quad b_{N-1} \neq 0 \Rightarrow \phi(t = b_N) &= \phi(t = b_{N-1}) + a_{N-1}\delta_{N-1}, \forall \delta_{N-1} \\ \Rightarrow \chi(t = b_N) &= \chi(t = b_{N-1}) + a_{N-1} q_{N-1}, \forall \delta_{N-1} \end{aligned}$$

If we now consider the case $a_{N-1} = \bar{\bar{\nu}}_{N-1}$ a mere checking of the preceding formulae shows that they remain unchanged.

Finally, for all other values of a_{N-1}, that is, if

$$a_{N-1} \neq \bar{\nu}_{N-1}, \bar{\bar{\nu}}_{N-1} \quad ,$$

the same formulae still hold true, with the only difference that we must replace T'_N by T_N.

Let us sum up the foregoing conclusions:

$$\text{if} \quad b_{N-1} = 0 \Rightarrow \chi(t = b_N) = \begin{cases} a_{N-1}q_{N-1} - 1, & \text{if} \quad \delta_{N-1} < 0 \\ a_{N-1}q_{N-1}, & \text{if} \quad \delta_{N-1} > 0 \end{cases} \tag{3.(β_2)a}$$

$$\text{if} \quad b_{N-1} \neq 0 \Rightarrow \chi(t = b_N) = a_{N-1}q_{N-1} + \chi(t = b_{N-1}) \tag{3.(β_2)b}$$

As for the reduced phase, it is obtained by means of the following formulae:

$$\text{if} \quad b_{N-1} = 0 \Rightarrow \phi(t = b_N) = \begin{cases} 2\pi + a_{N-1}\delta_{N-1}, & \text{if} \quad \delta_{N-1} < 0 \\ a_{N-1}\delta_{N-1}, & \text{if} \quad \delta_{N-1} > 0 \end{cases} \tag{3.(γ_2)a}$$

$$\text{if} \quad b_{N-1} \neq 0 \Rightarrow \phi(t = b_N) = \phi(t = b_{N-1}) + a_{N-1}\delta_{N-1} \tag{3.(γ_2)b}$$

In equations $(3.(\beta_2), (\gamma_2))$, the value $b_{N-1} \neq 0$ obviously means that $b_{N-1} = 1, 2, \ldots, T_{N-1} - 1$ or $b_{N-1} = 1, 2, \ldots, T'_{N-1} - 1$.

The method is thus carried on as long as b_r is not zero , and we have (with $r = N-1, N-2, \ldots, 3, 2.$) two integers a_r and b_r unambiguously determined by b_{r+1} in the following way:

$$b_{r+1} = a_r T_r + b_r \tag{3. (α_3)a}$$

where a_r is the greatest integer such that

$$0 \leq a_r \leq \begin{cases} \bar{\bar{\nu}}_r, & \text{if} \quad a_{r+1} = \bar{\nu}_{r+1}, \quad \text{or} \quad \bar{\bar{\nu}}_{r+1} \\ \bar{\nu}_r, & \text{if} \quad a_{r+1} \neq \bar{\nu}_{r+1}, \quad \text{or} \quad \bar{\bar{\nu}}_{r+1} \end{cases} \tag{3. (γ_3)b}$$

It then follows that b_r can only take certain values, which are:

$$\text{if} \quad a_r = \bar{\nu}_r \quad \text{or} \quad \bar{\bar{\nu}}_r \Rightarrow b_r = 0, 1, 2, \ldots, T'_r - 1$$
$$\text{if} \quad a_r \neq \bar{\nu}_r \quad \text{or} \quad \bar{\bar{\nu}}_r \Rightarrow b_r = 0, 1, 2, \ldots, T_r - 1$$

One may then verify that we have

$$\text{if} \quad b_r = 0 \Rightarrow \chi(t = b_{r+1}) = \begin{cases} a_r q_r - 1, & \text{if} \quad \delta_r < 0 \\ a_r q_r, & \text{if} \quad \delta_r > 0 \end{cases} \tag{3.(β_3)a}$$

$$\text{if} \quad b_r \neq 0 \Rightarrow \chi(b_{r+1}) = a_r q_r + \chi(t = b_r) \tag{3.(β_3)b}$$

And for the reduced phase:

$$\text{if} \quad b_r = 0 \Rightarrow \phi(t = b_{r+1}) = \begin{cases} 2\pi + a_r\delta_r, & \text{if} \quad \delta_r < 0 \\ a_r\delta_r, & \text{if} \quad \delta_r > 0 \end{cases} \tag{3.(γ_3)a}$$

$$\text{if} \quad b_r \neq 0 \Rightarrow \phi(t = b_{r+1}) = \phi(t = b_r) + a_r\delta_r \tag{3.(γ_3)b}$$

If we now take in consideration the whole assemblage formed by equations $(\alpha_3), (\beta_3), (\gamma_3)$, with $r = N-1, N-2, \ldots, 3, 2$ we see that the solution for the problem stated at the beginning of this paragraph is reduced to finding the values $\phi(t)$ and $\chi(t)$ for the instants of time $t = b_2$, with

$$b_2 = 0, 1, 2, \ldots, T_2 - 1$$

and

$$b_2 = 0, 1, 2, \ldots, T_2' - 1$$

Now this is a straightforward investigation that can be carried on directly by making use of the fundamental lattices as the one shown in figure 1, in which we represent the positions of RP at instants

$$t = 0, 1, 2, \ldots, T_2 - 1 \quad \text{and} \quad t = 0, 1, 2, \ldots, T_2' - 1$$

By simple inspection the reader could thus easily check the following two properties:

a) Assume that we have $\bar{\nu}_o = \nu_o + 1$; let

$$t \equiv b_2 \in [0, 1, 2, \ldots, T_2 - 1] \quad \text{or} \quad t \equiv b_2 \in [0, 1, 2, \ldots, T_2' - 1]$$

and a_1 and b_1 be two positive integers defined by means of b_2 in the following way:

$$b_2 = a_1 T_1 + b_1 \tag{3. (α)a}$$

where a_1 is the greatest integer such that

$$0 \leq a_1 \leq \begin{cases} \bar{\nu}_1, & \text{if} \quad b_2 \in [0, 1, 2, \ldots, T_2 - 1] \\ \bar{\bar{\nu}}_1, & \text{if} \quad b_2 \in [0, 1, 2, \ldots, T_2' - 1] \end{cases} \tag{3. (α)b}$$

It then follows that

$$\begin{aligned} \text{if} \quad a_1 = \bar{\nu}_1, o\tau\bar{\bar{\nu}}_1 &\Rightarrow b_1 \in [0, 1, 2, \ldots, T_1' - 1] \\ \text{if} \quad a_1 \neq \bar{\nu}_1, \bar{\bar{\nu}}_1 &\Rightarrow b_1 \in [0, 1, 2, \ldots, T_1 - 1] \end{aligned} \tag{3. (α)c}$$

Furthermore, and under the same conditions, we have

$$\chi(t = b_2) = a_1 \tag{3.(β)}$$
$$\phi(t = b_2) = a_1\delta_1 + b_1\delta_0 \tag{3.(γ)}$$

b) If we now go over the case $\bar{\nu}_0 = \nu_0$ and let again

$$t \equiv b_2 \in [0, 1, 2, \ldots, T_2 - 1]$$

or

$$t \equiv b_2 \in [0, 1, 2, \ldots, T_2' - 1]$$

this integer unambiguously defines two positive integers a_1 and b_1 as follows:

$$b_2 = a_1 T_1' + b_1 \tag{3. (α')a}$$

where a_1 is the greatest integer such that

$$0 \leq a_1 \leq \begin{cases} \bar{\nu}_1, & \text{if} \quad b_2 \in [O, 1, 2, \ldots, T_2 - 1] \\ \bar{\bar{\nu}}_1, & \text{if} \quad b_2 \in [0, 1, 2, \ldots, T_2' - 1] \end{cases} \tag{3. (α')b}$$

It then follows that

$$\begin{array}{l} \text{if} \quad a_1 = 0 \Rightarrow b_1 \in [0, 1, 2, \ldots, T_1' - 1] \\ \text{if} \quad a_1 \neq 0 \Rightarrow b_1 \in [0, 1, 2, \ldots, T_1 - 1] \end{array} \tag{3. (α')c}$$

and we can also verify that

$$\chi(t = b_2) = a_1 \tag{3.(β)}$$
$$\phi(t = b_2) = a_1\delta_1 + b_1\delta_0 \tag{3.(γ)}$$

It is important to stress that (unlike what happened for $r = 3, 4, \ldots$) the formulae defining $a_{r-1} = a_1$ and $b_{r-1} = b_1$ (with $r = 2$), are different according wether we have $\bar{\nu}_0 = \nu_0$ or $\bar{\nu}_0 = \nu_0 + 1$, though, once a_r and b_r are obtained, the expressions $(\beta), (\gamma)$ providing $\chi(t = b_2)$ and $\phi(t = b_2)$ are the same. One must always keep this fact in mind when dealing with the subsequent algorithms.

It must also be remarked, in connection with the admissible values for a_1, which are (and whatever the value for $\bar{\nu}_0 = \nu_0$ or $\bar{\nu}_0 = \nu_0 + 1$, - see $(\alpha), (\alpha')$)

$$0 \leq a_1 \leq \begin{cases} \bar{\nu}_1, & \text{if} \quad b_2 \in [0, 1, 2, \ldots, T_2 - 1] \\ \bar{\bar{\nu}}_1, & \text{if} \quad b_2 \in [0, 1, 2, \ldots, T_2' - 1] \end{cases} \tag{3.2}$$

that these inequalities are equivalent to

$$0 \leq a_1 \leq \begin{cases} \bar{\nu}_1, & \text{if} \quad a_2 \neq \bar{\nu}_2, \bar{\bar{\nu}}_2 \\ \bar{\bar{\nu}}_1, & \text{if} \quad a_2 = \bar{\nu}_2, \text{ or } \bar{\bar{\nu}}_2 \end{cases} \tag{3.3}$$

Let us then resume the reasoning leading to formulae $(\alpha_3), (\beta_3), (\gamma_3)$ *to which we now add formulae* $(\alpha) - o\tau(\alpha') - (\beta)$ *and* (γ). We see that our method, starting from an integer T, unambiguously determines two sequences of integers,

$$a_N, a_{N-1}, \ldots, a_2, a_1.$$
$$b_N, b_{N-1}, \ldots, b_2, b_1.$$

(From these couples a_k, b_k, only a_1, b_1 is differently calculated according to wether we have $\bar{\nu}_0 = \nu_0$ or $\bar{\nu}_0 = \nu_0 + 1$).

Let us assume that we have $\bar{\nu}_0 = \nu_0 + 1$: Then these two sequences of integers are obtained by means of formulae $(\alpha_1), (\alpha_2), (\alpha_3)$, and (α) - *in that order.* And if all the b_r are different from zero, we have

$$T = \sum_{k=1}^{k=N} a_k T_k + b_1 \tag{3.4}$$

$$\chi(T) = \sum_{k=2}^{k=N} a_k q_k + a_1 \tag{3.5}$$

$$\phi(T) = \sum_{k=1}^{k=N} a_k \delta_k + b_1 \delta_0 \tag{3.6}$$

If, on the contrary, a certain b_s, with $s \in [N, N-1, \ldots, 3, 2]$ is zero, then we find the following "truncated" formulae:

$$T = \sum_{k=s}^{k=N} a_k T_k \tag{3.7}$$

$$\chi(T) = \sum_{k=s}^{k=N} a_k q_k + \begin{cases} -1, & \text{if} \quad \delta_s < 0 \\ 0, & \text{if} \quad \delta_s > 0 \end{cases} \tag{3.8}$$

$$\phi(T) = \sum_{k=s}^{k=N} a_k \delta_k + \begin{cases} 2\pi, & \text{if} \quad \delta_s < 0 \\ 0, & \text{if} \quad \delta_s > 0 \end{cases} \tag{3.9}$$

Going over the case $\bar{\nu}_0 = \nu_0$, we have seen that the procedure is the same, with the only difference that formulae (α) must now be replaced by (α'). This amounts to saying that the sequence of integers a_k, b_k is now generated by equations $(\alpha_1), (\alpha_2), (\alpha_3)$, and (α') - *in that order.* One then finds, by assembling together these equations, and assuming that all the b_s are non zero,

$$T = \sum_{k=2}^{k=N} a_k T_k + a_1 T_1' + b_1 \tag{3.10}$$

while the formulae for $\chi(T)$ and $\phi(T)$ - as well as their truncated version - remain the same as those given above in the case $\bar{\nu}_0 = \nu_0 + 1$.

We have thus completely answered the question stated at the beginning of this paragraph, namely, the determination of $\chi(T)$ and $\phi(T)$ given the value of δ_0 and the instant T.

IV - The inverse problem (cont).

We shall now consider another problem, in close connection with the preceding one: Given $\delta_0 \equiv \phi(t = 1)$ and an integer χ (that is, a certain horizontal line in our fundamental lattice), to find the reduced phase and the instants of time corresponding to all the positions of the RP on the χ -horizontal. This amounts to saying that we want to determine (see figure 1)
a) the integers T such that $l_\chi + 1 \leq T \leq l_{\chi+1}$.
b) the values of $\phi(t = T)$, where T is any integer given by a)

Now, only problem a) actually matters since, once it is solved, problem b) immediately follows by making use of the method presented in the preceding paragraph.

We begin by considering again the three formulae (3.4), (3.5), (3.6). (In what follows we are less concerned by the corresponding truncated formulae). We see that the integers $a_N, a_{N-1}, \ldots, a_2, a_1$ and b_1 appear in all of them, while b_1 appears in the first two, but not in the last one. In other words: the ensemble of the positions of RP on the $\chi(T)$-horizontal (that is, the values of their reduced phases and the corresponding instants of time) is completely identified by the integers $a_N, a_{N-1}, \ldots, a_2, a_1$, and each particular position on this ensemble only differs from the others by the value of b_1, the which can take any admissible value compatible with those of the $a_N, a_{N-1}, \ldots, a_2, a_1$.

This last point deserves some further comments. We first recall the sequence of inequalities given in paragraph III and which confine the integers a_r to a certain domain of admissible values:

$$1 \leq a_N < \bar{\nu}_N \tag{4.1}$$

$$1 \leq a_{N-1} \begin{cases} \bar{\nu}_{N-1}, & \text{if} \quad a_N \neq \bar{\nu}_N \\ \bar{\bar{\nu}}_{N-1}, & \text{if} \quad a_N = \bar{\nu}_N \end{cases} \tag{4.2}$$

$$0 \leq a_r \leq \begin{cases} \bar{\nu}_r, & \text{if} \quad a_{r+1} \neq \bar{\nu}_{r+1}, \bar{\bar{\nu}}_{r+1} \\ \bar{\bar{\nu}}_r, & \text{if} \quad a_{r+1} = \bar{\nu}_{r+1} \text{ or } \bar{\bar{\nu}}_{r+1} \end{cases} \tag{4.3}$$
$$(r = N-2, N-3, \ldots, 3, 2)$$

As for $r = 1$, we have seen that $a_r = a_1$ is calculated differently according to the value of $\bar{\nu}_0 = \nu_0$ or $\bar{\nu}_0 = \nu_0 + 1$: see (3. $(\alpha), (\alpha')$). And we have (see the equivalent inequalities (3.2) and (3.3))

$$0 \leq a_1 \leq \begin{cases} \bar{\nu}_1, & \text{if} \quad a_2 \neq \bar{\nu}_2, \bar{\bar{\nu}}_2 \\ \bar{\bar{\nu}}_1, & \text{if} \quad a_2 = \bar{\nu}_2 \text{ or, } \bar{\bar{\nu}}_2 \end{cases} \tag{4.4}$$

Finally, if $\bar{\nu}_0 = \nu_0 + 1$,

$$a_1 = \begin{cases} = \bar{\nu}_1, \text{ or } \bar{\bar{\nu}}_1 \Rightarrow b_1 \in [0, 1, \ldots, T'_1 - 1] \\ \neq \bar{\nu}_1, \bar{\bar{\nu}}_1 \Rightarrow b_1 \in [0, 1, \ldots, T_1 - 1] \end{cases} \tag{4.5}$$

and if $\bar{\nu}_0 = \nu_0$,

$$a_1 = \begin{cases} = 0 \Rightarrow b_1 \in [0, 1, \ldots, T'_1 - 1] \\ \neq 0 \Rightarrow b_1 \in [0, 1, \ldots, T_1 - 1] \end{cases} \tag{4.6}$$

It is now clear that the chain of upper bounds for the a_s (whose structure is entirely determined by the sequence of integers $\bar{\nu}_k, \nu_k$, with $k = N, N-1, \ldots, 2, 1$ and 0) assigns precise upper bounds for the admissible values of b_1.

Let then be an integer χ, labeling some horizontal line in the diagram of the reduced phase, and let T be an instant of time (that is, a positive integer) corresponding to some position of RP on the χ - horizontal: $l_\chi + 1 \leq T \leq l_{\chi+1}$. Assuming that χ is known, we want to determine the possible values of T. Of course, if we knew one of these admissible values of T we could (starting with T and making use of the method of paragraph III) obtain the sequence of the integers

$$N, a_N, b_N, a_{N-1}, b_{N-1}, \ldots,$$

and introduce these values in formulae (3.4), (3.5), (3.6) or (3.7), (3.8), (3.9). Now we may reason more particularly starting from χ. In fact, it exists an integer N such that

$$\chi \Rightarrow \exists N : q_N \leq \chi < q_{N+1} \tag{4.7}$$

Let us then define two integers a_N, b'_N in the following way:

$$\chi = a_N q_N + b'_N \qquad (4.\ (A_1)a)$$

where a_N is the greatest integer such that

$$1 \leq a_N < \bar{\nu}_N \qquad (4.\ (A_1)b)$$

It then follows that

$$a_N = \begin{cases} = \bar{\nu}_N \Rightarrow b'_N \in [0, 1, \ldots, q'_N - 1] \\ \neq \bar{\nu}_N \Rightarrow b'_N \in [0, 1, \ldots, q_N - 1] \end{cases} \qquad (4.\ (A_1)b)$$

Now it is of full significance to remark that N and a_N (as well as the integers $a_{N-1}, a_{N-2}, \ldots$ that will appear in the sequel) are the same $N, a_N, a_{N-1}, a_{N-2}, \ldots$ that would be obtained by the method of the preceding paragraph (namely, equations $(\alpha_1), (\alpha_2), (\alpha_3)$) starting from the value of any T on the χ -horizontal, that is, $l_\chi + 1 \leq T \leq l_{\chi+1}$. Needless to say, there is "a priori" no reason for having $b'_N = b_N, b'_{N-1} = b_{N-1}, b'_{N-2} = b_{N-2}, \ldots$ etc.

From b'_N we define, in a similar way, a_{N-1} and b'_{N-1}:

$$b'_N = a_{N-1} q_{N-1} + b'_{N-1} \qquad (4.\ (A_2)a)$$

where a_{N-1} is the greatest integer such that

$$0 \leq a_{N-1} \leq \begin{cases} \bar{\nu}_{N-1}, & \text{if} \quad a_N \neq \bar{\nu}_N \\ \bar{\bar{\nu}}_{N-1}, & \text{if} \quad a_N = \bar{\nu}_N \end{cases} \qquad (4.\ (A_2)b)$$

We then have

$$a_{N-1} = \begin{cases} = \bar{\nu}_{N-1}, \text{ or } \bar{\bar{\nu}}_{N-1} \Rightarrow b'_{N-1} \in [0, 1, \ldots, q'_{N-1} - 1] \\ \neq \bar{\nu}_{N-1}, \bar{\bar{\nu}}_{N-1} \Rightarrow b'_{N-1} \in [0, 1, \ldots, q_{N-1} - 1] \end{cases}$$

As long as the b'_s are non zero, the method is carried on:

$$b'_{r+1} = a_r q_r + b'_r (r = N-1, N-2, \ldots, 3, 2) \qquad (4.\ (A_3)a)$$

where a_r is the greatest integer such that

$$0 \leq a_r \leq \begin{cases} \bar{\nu}_r, & \text{if} \quad a_{r+1} \neq \bar{\nu}_{r+1}, \bar{\bar{\nu}}_{r+1} \\ \bar{\bar{\nu}}_r, & \text{if} \quad a_{r+1} = \bar{\nu}_{r+1} \text{ or } \bar{\bar{\nu}}_{r+1} \end{cases} \qquad (4.\ (A_3)b)$$

It then follows that

$$a_r = \begin{cases} = \bar{\nu}_r, \text{ or } \bar{\bar{\nu}}_r \Rightarrow b'_r \in [0, 1, \ldots, q'_r - 1] \\ \neq \bar{\nu}_r, \bar{\bar{\nu}}_r \Rightarrow b'_r \in [0, 1, \ldots, q_r - 1] \end{cases}$$

From formulae $(A_1), (A_2), (A_3)$ we may now infer that

$$\chi = \sum_{k=2}^{k=N} a_k q_k + b'_2$$

if we assume that all the b'_s are different from zero. If, on the contrary, we have, for some $s, b'_s = 0$, then we find

$$\chi = \sum_{k=s}^{k=N} a_k q_k$$

We now recall the remark stated at the beginning of this paragraph, and keep in mind that if we knew any T such that $l_\chi + 1 \leq T \leq l_{\chi+1}$, its value would be fixed, according to the method exposed in the preceding paragraph, *by* the same integers a_r which we have just obtained starting from given χ, *and also by* a_1 and b_1. From this we conclude that

$$a_1 = b'_2 \qquad (4.8)$$

We now see that the integers T: $l_\chi + 1 \leq T \leq l_{\chi+1}$ are given by the same formula (3.4) of paragraph III in which:

1) integers $a_N, a_{N-1}, \ldots, a_3, a_2$ are those we have just calculated directly from χ, following formulae $(A_1), (A_2), (A_3)$;

2) integer a_1 is equal to b'_2, which has also been obtained by direct calculation from χ;

3) integer b_1 may have any value compatible (in the sense explained above, in connection with formulae (4.1) to (4.6)) with the values taken by $a_N, a_{N-1}, \ldots, a_3, a_2, a_1$.

The problem stated at the beginning of this paragraph is then completely solved.

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