On the quantum measure of information *†

LEV B. LEVITIN ‡

Institute for Information and Transmission Problems
Moscow USSR

The goal of this article is to generalize Shannon's information [1,2] to quantum mechanical ensembles which are described not by probability distributions but by density matrices ; to give a quantum mechanical formulation of the "entropy defect principle" [3] ; and finally to find the influence of the irreversibility of quantum measurements on information transmission.

Consider a physical system in one of the macrostates A_i, which occurs with probability p_i, described by density matrix $\hat{\rho}, i = 1, 2, ...$ (Preparation of the definite macrostate A_i can be interpreted as the transmission of the i^{th} signal). If it is unknown which macrostate A_i the system is in then the system is described by the a priori density matrix $\hat{\rho}$

$$\hat{\rho} = \sum_i p_i \hat{\rho}^{(i)} \tag{1}$$

Based on von Neumann's entropy [4] we define the entropy defect for a quantum system via

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* Translation by Andrey Bezinger and Samuel L. Braunstein of the author's contribution to the *Proceedings of the Fourth All-Union Conference on Information and Coding Theory*, Section II (Tashkent, 1969) pp. 111-115.

† More recent perspectives of the author's work may be found in : L. B. Levitin, "Information theory for quantum systems" in *Information Complexity and Control in Quantum Physics*, editors A. Blaquière, S. Diner and G. Lochak (Springer-Verlag, New York, 1987) pp. 15-47.

‡ Current address : College of Engineering, Boston University, Boston, MA 02215, USA.

Definition 1 *The entropy defect is*

$$I_o = -\operatorname{tr}\hat{\rho}\ln\hat{\rho} + \sum_i p_i \operatorname{tr}\hat{\rho}^{(i)}\ln\hat{\rho}^{(i)} \tag{2}$$

The entropy defect shows how much, on average, the system's entropy decreases if it becomes known in which macrostates A_i the system is ; and can be interpreted as a quantity of information about the microstates of a system decribed by the ensemble $\hat{\rho}$ made up of the subensembles $\hat{\rho}^{(i)}$ (i.e. macrostates A_i).

Theorem 1

$$0 \leq I_0 \leq -\sum_i p_i \ln p_i, \tag{3}$$

with equality on the left if and only if all the matrices $\hat{\rho}^{(i)}$ are equal, and equality on the right if and only if all the matrices are orthogonal (i.e., $\operatorname{tr}\hat{\rho}^{(j)}\hat{\rho}^{(i)} = 0$ for $i \neq j$).

Thus, the entropy defect is non-negative and never exceeds the entropy of the a priori distribution of probabilities of the different macrostates.

In the quasi-classical case the entropy defect equals the amount of information about the macrostates of the system (from the set $\{A_i\}$) obtained by measuring its microstate (this is the entropy defect principle) [3] ; in quantum theory this equality does not generally hold. The amount of information about the macrostate depends on the complete set of observables measured [5].

Definition 2 *The information about the macrostate of the system obtained by measuring the complete set of observables L is given by*

$$I_L = -\sum_l \rho_{ll}\ln\rho_{ll} + \sum_{i,l} p_i \rho_{ll}^{(i)} \ln\rho_{ll}^{(i)} \tag{4}$$

where ρ_{ll} and $\rho_{ll}^{(i)}$ are the diagonal elements of the appropriate matrices in the basis determined by L.

This definition is in agreement whith Shannon's information because the elements $\rho_{ll}^{(i)}$ and ρ_{ll} have the meaning of conditioned and unconditioned probabilities, respectively, of different sets of values taken on by the complete set of variables L, i.e., of different microstates described by the eigenstates of the set of operators L.

Because of the dependence of I_L on the choice of L (or, in coding theory, on how the signal is treated by the receiver), it does not define an absolute maximum of the amount of transmitted data in contrast with Shannon's information (but only a conditional maximum for a fixed choice of L).

Definition 3 *The information about the macrostate of a physical system which has a priori probability p_i of being in the macrostage A_i is given by the quantity*

$$I = \sup_L I_L, \tag{5}$$

where the maximum is over all complete sets of observables L (i.e., over all possible bases of the Hilbert space of the system).

This I plays the same role as Shannon's information in the classical theory ; and the following theorem holds :

Theorem 2 Let $\{A_i(\tau)\}$ represent the set of signals transmitted over a channel in time τ, with each signal $A_i(\tau)$ having probability $p_i(\tau)$ and corresponding to a state of a physical system described by $\hat{\rho}^{(i)}(\tau)$. Then the channel capacity is given by

$$C = \lim_{\tau\to\infty} I(\tau)/\tau, \tag{6}$$

where $I(\tau)$ is given by Eq. (5) with $p_i(\tau)$ replacing p_i and $\hat{\rho}^{(i)}(\tau)$ replacing $\hat{\rho}^{(i)}$.

Thus, C satisfies the fundamental coding theorem, and we can consider I as information for a quantum system. The relationship between the information and the entropy defect is described by the following theorem :

Theorem 3

$$I \leq I_0, \tag{7}$$

with equality if and only if all the matrices $\hat{\rho}^{(i)}$ commute ; in this case

$$I_{L_0} = I = I_0, \tag{8}$$

for the basis L_0 in which all the $\hat{\rho}^{(i)}$ are diagonal.

This theorem gives the quantum formulation of the entropy defect principle. In general, the entropy defect I_0 is not equal to the maximum informations about the macrostate obtainable from measurement of the microstate (which yields information I). As a rule, the information is less than the entropy defect due to the irreversibility of the quantum measurement (i.e., the collapse of the wavepacket – the off-diagonal elements of the density matrix in a measurement disappear).

Theorem 4 *For a system with a known Hamiltonian*

$$\frac{dI}{dt} = 0. \tag{9}$$

That is, the information is an integral of motion: the system does not forget information ; but the optimal measurement L will be a function of time. The evolution of the system is accompanied by the "entangling" of the phase-space trajectories and L becomes more complicated to recover (these operators, as well as the system's density matrix, evolve according to the unitary transformation $\exp(-iHt/\hbar)$, where H is the system's Hamiltonian).

References

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