On Quantum Optics

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Since *Einstein*'s deduction of *Planck*'s radiation law, in quantum statistics certain probabilities are usually attributed to emission and absorption processes without making more precise assertions over them. We want to propose here a general ansatz for such probabilities, which seems adequate to help in overcoming the contradiction present in theoretical optics – wave theory of interference and polarisation on one side, quantum theory of spectral lines on the other one. *We understand interference as expression of fundamental quantum statistical laws.* To this end, the instrument will be *a quantum interpretation of the phase of light in wave theory.*

§ *1. The phase.* Let us consider the path of a light ray from the emitting atomic system E to the absorbing atomic system A. For the undulatory theory of interference, the phase

$$\varphi = \int_E^A \frac{\mathrm{d}s}{\lambda} = \frac{\nu}{c} \int_E^A n \mathrm{d}s \tag{1}$$

(ν = frequency, λ = wave length, n = refraction index, $\mathrm{d}s$ path element) is essential. We claim that the phase φ can be seen in quantum theory as a pure mechanical quantity.

Surely, the most important fundament of quantum theory is the statement that an atomic system cannot radiate as long as it finds itself in

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a mechanical state, that is, that absorption and emission involve always *non-mechanical "transitions"*. However, not only the acts of emission and absorption will be non-mechanical; also along its whole path the light will produce continuous non-mechanical perturbations in the atoms of the traversed medium. In order to obtain an invariant measure for the size of these perturbations, i.e. for the deviations of the internal atomic motions from *Hamilton*ian mechanics, the motion of all atomic systems touched by the process of light propagation will be described by a system of canonical momentum and position coordinates α_k, β_k, most simply a system whose momenta α_k are constant in mechanical states (α_k = integration constant of *Hamilton*'s partial differential equation of the total system). The desired measure for the deviations from mechanics results in the integral $\int \sum_k \beta_k \mathrm{d}\alpha_k$, which should be extended over all non-mechanical processes, i.e. over all variations of α_k. We claim that the phase φ, up to a universal dimensional factor h (the *Planck* quantum of action), is identical with this integral:

$$\varphi = \frac{1}{h} \sum \int \beta_k \mathrm{d}\alpha_k \ . \tag{2}$$

It is well known that, according to Jacobi, it is possible to introduce the energy W as one of the momentum coordinates (α_1); more exactly, the sum of the energy of all participating atomic systems, by this way supposing them *all to be coupled in principle*. Because the position coordinate β_1 conjugate to W is the time t, instead of eq.(2) we obtain

$$\varphi = \frac{1}{h} \left[\int t \mathrm{d}W + \sum_2 \int \beta_k \mathrm{d}\alpha_k \right] \ . \tag{3}$$

For the moment, we consider only the systems E and A, i.e. we consider the case of the light propagation in vacuo. Provisionally, it is supposed that the acts of emission and absorption happen in no time. The emission, i.e. a reduction of the energy (at E) by a certain amount $-\Delta W$ takes place at the instant t_E; by the *energy principle*, at the instant t_A the system has to return to its original energy by the reassumption of the energy amount $+\Delta W$ (at A). Hence, eq.(3) yields

$$\varphi = \frac{1}{h} \left[\Delta W (t_A - t_E) + \sum_2 \int \beta_k \mathrm{d}\alpha_k \right] \ . \tag{4}$$

But $t_A - t_E$ is equal to the length of the light path l divided by the light velocity c (in vacuo). Consequently:

$$\begin{aligned}\varphi &= \frac{\Delta W}{hc} \cdot l + \frac{1}{h} \sum_2 \int \beta_k \mathrm{d}\alpha_k \\ &= \frac{l}{\lambda_0} + \dots\end{aligned} \tag{5}$$

where

$$\lambda_0 = \frac{hc}{\Delta W} = \frac{c}{\nu} \tag{6}$$

is the vacuum wave length corresponding to *Bohr*'s $h\nu$ principle.

The assumption of instantaneity of emission and absorption is not essential; it is sufficient to suppose that each infinitesimal element of energy $\mathrm{d}W$ needs the time l/c to get from E to A; then eq.(5) follows from eq.(3). *The only property of the "light quantum" that is essential here is its propagation velocity c.*

Apart from the contributions of the degrees of freedom $k = 2, 3, \dots$, which we will not consider until §3, eq.(5) coincides with eq.(1) for $n = 1$. By identification of eq.(1) with eq.(2) for arbitrary refracting media too, we obtain by use of eq.(6):

$$n = \frac{c}{\Delta W} \cdot \frac{\mathrm{d}}{\mathrm{d}s} \sum_1 \int \beta_k \mathrm{d}\alpha_k \ . \tag{7}$$

Hence, the refraction index measures quantally the deviation from mechanics per path and energy unit. Its dependence on ΔW and on the medium is part of the subject of a quantum theory of dispersion which Mr.*K.F.Herzfeld* will publish soon in this journal[1]. We can write *Fermat*'s principle $\delta \int n \mathrm{d}s = 0$ as $\delta \sum \int \beta_k \mathrm{d}\alpha_k = 0$, and we may state that on the light paths of geometrical optics the overall deviation from mechanics is at minimum.

§ *2. The interference formula.* By substitution of the classical wave phase by our quantum phase, it is simple to translate the interference formula of wave theory into the language of quantum statistics: If the

[1] The author acknowledges many instigations to this work from the discussions with Mr.*Herzfeld* about the possibilities of a quantum theory of interference and dispersion.

light quantum may choose different paths s from E to A, the probability that it passes to A on some path and is absorbed there is not equal to the sum of the a priori probabilities of the individual light paths, but J times as much, where

$$J = \frac{(\mathbf{F}\tilde{\mathbf{F}})}{|\ \mathbf{F}_0\ |^2}, \tag{8}$$

$$\mathbf{F}_0 = \sum_s \mathbf{f}_s\ , \quad \mathbf{F} = \sum_s \mathbf{f}_s e^{2\pi \mathrm{i}\cdot\varphi_s} \tag{9}$$

Here, φ_s are the phases eq.(2), taken over the individual paths s, and $\mathbf{f}_s$ the vectorial amplitudes of the classical waves to whose quantal meaning we come back in §3. In orthogonal xyz-coordinates the factor J reads:

$$J = \frac{\begin{array}{l}(\sum \mathbf{f}_{sx}\cos\varphi_s)^2 + (\sum \mathbf{f}_{sx}\sin\varphi_s)^2 + (\sum \mathbf{f}_{sy}\cos\varphi_s)^2 \\ +(\sum \mathbf{f}_{sy}\sin\varphi_s)^2 + (\sum \mathbf{f}_{sz}\cos\varphi_s)^2 + (\sum \mathbf{f}_{sz}\sin\varphi_s)^2\end{array}}{(\sum \mathbf{f}_{sx})^2 + (\sum \mathbf{f}_{sy})^2 + (\sum \mathbf{f}_{sz})^2} \tag{10}$$

The formal coincidence of the numerator with the square of the amplitude of superposed waves ensures the ansatz, eq.(8), to be universally valid for the representation of interference phenomena of any kind. In contrast with the wave theory our ansatz has the advantage that it guarantees a priori the *identity of the "wave lengths" measured by interference and by the photo effect. Schrödinger*[2] has shown that these wave lengths show the correct *Doppler shift* too, if the systems E and A are moving.

For our ansatz, the assumption is essential that the emitting and the absorbing system are coupled in principle, according to a general thesis recently formulated by *Smekal*[3]. Firstly we had to assume in § 1 a mechanical coupling, in order to relate mutually the evolution in time of the different atomic systems in a unique way. Furthermore, formula (8) puts the quantum processes in the individual systems in mutual dependence. It is highly remarkable that according to our interpretation the presence of the absorbing system A is indispensable for any interferences to happen; in vacuo, such interferences are not only unobservable, but not existing in principle. An intensity of light measured by the number of light quanta per unit of time and area could never show interferences slanted to the light path, as one can easily recognize with the example of standing waves.

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[2] Phys.ZS.**23**, 301, 1922.

[3] Wiener Anzeiger 1922, Nr.10, S.79

In our conception, the *emissions of two different atoms E, E' are* obviously *incoherent in principle*, if light paths touching all three atoms E, E', A do not play a special rôle[4].

§ *3. On the theory of spectra.* Now we intend to consider in detail the *dependence of the interference phenomenon on the character of the emitting atom E.* We will suppose in particular that the system E is conditionally periodic and that the corresponding partial differential equation is separable. In conformity to this, the position coordinates β_k are chosen to be the so-called *angle variables* w_k which, apart from their linearity in time, are determined by the fact that the system is periodic in the w_k with period 1. It is well known that the conjugate momentum constants $\alpha_k = I_k$ are identical with the "phase integrals" ($\oint p_k \mathrm{d}q_k$) of quantum theory.

In order to decompose the phase φ according to eq.(3) in these

[4] The finite *coherence length* of the emissions of an atom can be taken into account by a supplementary postulate. The phase differences in eqs.(8), (9) for two light paths s and s' can be written:

$$\varphi_s - \varphi_{s'} = \Delta W/hc \cdot \int n \mathrm{d}s,$$

where the integral is taken over the *closed* curve $E \to s \to A \to s' \to E$. We now require this curve *to be closed not only in space, but also in time*, meaning that the energy quantum ΔW is present in the system also for the times $t_{Es} - t_{Es'}$ and $t_{As} - t_{As'}$ respectively, i.e. is stored in the systems E and A. To this end,

$$| t_{Es} - t_{Es'} | < \tau_E \ , \quad | t_{As} - t_{As'} | < \tau_A$$

is required, if τ_E and τ_A are the times spent by the quantum ΔW in the atoms E and A, respectively. In general, $\Delta W/h$ will not be an eigenfrequency of the system A, and τ_A will be practically zero. The absorption times t_A to be substituted in eq.(4) have to coincide practically. This corresponds to the circumstance that in wave theory those wave trains interfere which arrive *simultaneously* at A. On the other hand, only those rays s and s' interfere whose emission times t_E differ less than τ_E. *The mean life time of the initial state of E plays the part of a coherence time.* In fact, the equation

$$\text{coherence length} = \text{life time} \times \text{light velocity}$$

corresponds well to the known values: 10^2 cm $= 10^{-8}$ s $\cdot 10^{10}$ cm/s.

coordinates too, we use the relation

$$w_k = t \cdot \frac{\partial W}{\partial I_k} + u_k \ , \tag{11}$$

where the u_k mean undetermined phase constants. In this case, the expression eq.(2), as far as system E is concerned, reads:

$$\varphi = \frac{1}{h} \int \sum w_k \mathrm{d}I_k = \frac{1}{h} \big[\int t \mathrm{d}W + \sum \int u_k \mathrm{d}I_k \big] \ . \tag{12}$$

It is assumed now that the *phase constants* u_k *remain unchanged during the transition* (i.e. while the I_k are changed by ΔI_k)[5]. Then, eq.(12) yields

$$\varphi = \frac{1}{h} \big[\int t \mathrm{d}W + \sum u_k \Delta I_k \big] + \ldots \tag{13}$$

For given values of ΔI_k, the transition still can occur with different values of u_k. Consequently, we have to generalize our probability ansatz (§2) so that the probability of a transition with *any* value of u_k differs from the sum of the individual probabilities by a factor J again, where the average must be taken now not only over the light paths s, but also over the phase constants u_k. For the vectorial coefficients $\mathbf{f}_s$ (in eq.(9)) we will be able to write in general:

$$\mathbf{f}_s = \mathbf{E}_s(u_k) \cdot \mathrm{d}u_1 \mathrm{d}u_2 \ldots \tag{14}$$

Hence, in place of eq.(9) we find

$$\begin{aligned} \mathbf{F}_0 &= \sum_s \int \ldots \int \mathrm{d}u_1 \mathrm{d}u_2 \ldots \mathbf{E}_s(u_k) \ , \\ \mathbf{F} &= \sum_s \int \ldots \int \mathrm{d}u_1 \mathrm{d}u_2 \ldots \mathbf{E}_s(u_k) e^{2\pi \mathrm{i} [\sum u_k \Delta I_k / h + \int t \mathrm{d}W / h + \ldots]} \ . \end{aligned} \tag{15}$$

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[5] Hence, we require that – *in action-angle coordinates* – the second of the two *Hamilton* equations

$$I_k = \text{const.}, \quad u_k = \text{const.}$$

remains valid also during non-mechanical transitions. But we have to restrict this requirement explicitly to the *spontaneous* transitions; e.g., if we would extend it to processes which light arouses in the atoms of refracting media (§ 1), the refraction index eq.(7) would yield 1 always. As opposite to the *spontaneous* processes, one can take the *adiabatic* ones, where the I_k are constant on average over long time intervals, but where the u_k are varying in general (note added in proof).

Now, by postulate the system E in toto is periodic with period 1 in the w_k and by eq.(11) in the u_k too . Consequently, the function $\mathbf{E}_s(u_k)$ can be expanded into a Fourier series of the following kind

$$\mathbf{E}_s(u_k) = \sum_{n_k} \mathbf{D}^{(s)}_{n_k} e^{-2\pi \mathrm{i} \sum n_k u_k} \tag{16}$$

(n_k integer). If eq.(16) is substituted into eq.(15) and integrated over all u_k from $-\infty$ to $+\infty$, assuming

$$\sum_s \mathbf{D}^{(s)}_0 \neq 0$$

the probability expression, eq.(8), vanishes, except for the case of all ΔI_k being integer multiples of h. Hence, our formula yields that the momenta I_k jump only by integer multiples of h, in conformity with the well-known quantum conditions of separable systems:

$$\Delta I_k = n_k \cdot h \ . \tag{17}$$

Once "quantized", an atom will proceed to a quantized state again and again.

By substitution of eq.(17) in eq.(15) and eq.(8), respectively, we obtain a measure for the probability of transitions between quantized states, i.e. for the *intensity of the corresponding spectral lines.* Now, since the whole integrand of eq.(15) is periodic in the u_k by eqs.(16) and (17), it is sufficient to integrate over the unit cube $0 \leq u_k \leq 1$; then one obtains:

$$\mathbf{F} = \sum_s \mathbf{D}^{(s)}_{n_k} \cdot e^{2\pi \mathrm{i} \int t \mathrm{d}W/h + \cdots} = \sum_s \mathbf{D}^{(s)}_{n_k} \cdot e^{2\pi \mathrm{i} \int \frac{\mathrm{d}s}{\lambda}} \ . \tag{18}$$

Here, the coefficient $\mathbf{D}^{(s)}$ is the amplitude of a certain harmonic in eq.(16), namely the one of the order $n_k = \Delta I_k / h$. *The formula eq.(18) becomes identical with Bohr's correspondence principle for intensity and polarisation, if the vector* $\mathbf{E}_s$ *of eqs.(14) and (16) is identified with the wave-theoretical light vector radiated from the system E in position $w_k = u_k$ via the path s to A.* According to *Bohr*, it remains undetermined whether the classical radiation, eq.(16), has to be calculated for the initial state, or for the final state, or else for an intermediate state. If

the amplitude $\mathbf{D}$ of the harmonic in question vanishes for all intermediate states, the correspondence principle sharpens to a *selection rule*.

The expression, eq.(18) is formally identical with the classical periodic light vector; only the mechanical orbital frequency is replaced by the quantum frequency $\Delta W/h$, by averaging the former in the integration procedure over u_k and introducing the latter by the probability ansatz. In the interference formula, eq.(10), one can replace the vectors $\mathbf{f}_s$ directly by the classical light amplitudes $\mathbf{D}^{(s)}_{n_k}$. This offers the possibility to take over the classical boundary conditions for $\mathbf{D}^{(s)}$ at the boundaries of different media (discontinuities of n) by correspondence; then evidently result the laws of refraction, reflection, dichroism (polarisation) just as in undulatory theory. In fact, *Huygens*' principle rests upon interferences.

For the moment, the translation of the preceding considerations to *non-periodic systems* E is opposed by the difficulty that in this case the definition of a privileged system of position coordinates similar to that of angle variables is not so simple. It would be only necessary to uniquely define a system of normal coordinates whose phase constants u_k (see above) shall remain unchanged during the transition. The author intends to consider this question soon in more detail elsewhere, in the example of the continuous Röntgen spectrum.

While up to now in quantum theory the *Planck* quantum of action is required at *two* essentially distinct points, namely in the $h\nu$ principle and in the quantum conditions, it is introduced here only *once*, namely in the expression, eq.(2), for the phase φ. *Here, we obtained the $h\nu$ principle, the quantum conditions, and the correspondence principle from the one expression, eq.(2), together with the probability ansatz, eqs.(8, 15)*; the $h\nu$ principle without restrictive assumptions, the quantum conditions and the correspondence principle by privileging the system of angle coordinates.

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