Fast iterative method for the recurrence of an elementary integrable system (II)

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ABSTRACT. The iteration methods presented in part I of this work are here applied in some examples of geometric theory of numbers and recurrent dynamical systems.

RÉSUMÉ. Les méthodes itératives présentées dans la première partie de ce travail sont ici appliquées en théorie géométrique des nombres et en systèmes dynamiques récurrents.

Introduction.

In the present paper[1] we give some applications of the methods developed in part I, namely, a straightforward approximation of irrationals by rationals in an intuitive and almost immediate way. Such approximations are treated in V. In VI we solve the following problem : given the value of the reduced phase taken by the representative point in a certain fixed instant of time, to find the instant at which the reduced phase is the nearest to the above one (both instants belong to a previously fixed and obviously finite interval of time). In VII the preceding technique is applied in dynamical systems for ordering (by increasing values) the reduced phases taken at all instants (multiples of an elementary instant of time taken as unity) by the representative point of a 2-mode recurrent system within a certain finite interval of time. This leads directly, in VIII, to the method allowing the calculation of the Poincaré cycles,

[1] NB:The order of the paragraphs of the present paper follows that of part I.

that is, the ordered recurrent instants of time at which the representative point of the system is found in the interior of an arbitrarily small neighbourhood of its initial state.

V. Approximation of irrationals.

In what follows we are going to make full use of the foregoing methods in an elementary application to the theory of numbers, namely, the approximation of irrational numbers by rationals.

Let then $\delta_0 : 2\pi < 1$ be an irrational. Starting from δ_0, the algorithms of paragraph II provide the sequences $T_k, T'_k, \chi_k, \chi'_k, , \delta_k \delta'_k, ...$ (k=1,2,...) . Furthermore, the method exposed in paragraph III enables us to calculate, starting from a given value for instant T (an integer, since $\tau_0 = 1$), the values of $\chi(T)$ and $\phi(t = T)$:

$$T \Rightarrow \{aa, bb\} \Rightarrow \left\{ \begin{matrix} \phi(t=T) \\ \chi(T) \end{matrix} \right\},$$

Now it is clear that

$$T\delta_0 - 2\pi\chi(T) = \phi(T), \tag{5.1}$$

an obvious equality that follows from the very structure of the fundamental lattice. An irrational number $\delta_0 : 2\pi < 1$ and a positive integer T being then given, we shall say that the positive integer ϑ provides the best approximation for $\delta_0 : 2\pi$ with denominator T, if the expression

$$\eta(x) \equiv \left| \frac{\delta_0}{2\pi} - \frac{x}{T} \right|$$

takes its minimum for $x = \vartheta$. The following result can then be proved:

For given $\delta_0 : 2\pi$ and T, the best approximation is given either by

$$\vartheta = \chi(T) \quad \text{and} \quad \eta = \frac{\phi(T)}{2\pi T}, \quad \text{if} \quad \phi(T) < \pi$$

or by

$$\vartheta = \chi(T) + 1 \quad \text{and} \quad \eta = \frac{2\pi - \phi(T)}{2\pi T}, \quad \text{if} \quad \phi(T) > \pi$$

The demonstration is obvious. Starting from (5.1) , we have

$$\frac{\delta_0}{2\pi} - \frac{\chi(T) + n}{T} = \frac{\phi(T) - 2\pi n}{2\pi T} \quad (n\forall, \ integer).$$

Furthermore, in the expression of $\eta(x)$ we may always write the integer x under the form

$$x \equiv \chi(T) + n,$$

where n is a certain integer (n = ...,-2, -1, 0, 1, 2,..) . Hence

$$\eta(x) \equiv \left| \frac{\delta_0}{2\pi} - \frac{\chi(T) + n}{T} \right| = \left| \frac{\phi(T) - 2\pi n}{2\pi T} \right| = \eta(n)$$

We must now determine the integer n that minimises this last member. Now a mere checking on the reduced phase diagram in the fundamental lattice gives the answer: if we have $\phi(T) < \pi$, then it is n=0 that provides the minimum, and we get

$$x_{\min} = \chi(T), \quad \text{and} \quad \eta_{\min} = \frac{\phi(T)}{2\pi T}$$

If, on the contrary, $\phi(T) > \pi$, it will be n=1 and it follows that

$$x_{\min} = \chi(T) + 1, \quad \text{and} \quad \eta_{\min} = \frac{2\pi - \phi(T)}{2\pi T}.$$

The second result is, so to speak, complementary to the preceding one in that we now give the numerator instead of the denominator: Given an irrational $\delta_0 : 2\pi < 1$ and a positive integer x, to find a positive integer χ such that

$$\eta(x) \equiv \left| \frac{\delta_0}{2\pi} - \frac{\chi}{x} \right|$$

is minimal. We shall see that the answer is the following:

If $\phi(l_\chi + 1) < 2\pi - \phi(l_\chi)$, the value that minimises $\eta(x)$ is $x = l_\chi + 1$ and we have

$$\eta(x) = \frac{\phi(l_\chi + 1)}{2\pi(l_\chi + 1)}.$$

If $\phi(l_\chi + 1) > 2\pi - \phi(l_\chi)$, we must distinguish between two cases:

a) $l_\chi(\delta_0 - 2\pi + \phi(l_\chi)) < (l_\chi + 1)(2\pi - \phi(l_\chi))$. Then the minimum is attained for $x = l_\chi + 1$, and we have

$$\eta(x) = \frac{\delta_0 - 2\pi + \phi(l_\chi)}{2\pi(l_\chi + 1)}$$

b) $l_\chi(\delta_0 - 2\pi + \phi(l_\chi)) > (l_\chi + 1)(2\pi - \phi(l_\chi))$. We then find $x = l_\chi$, and we have

$$\eta(x) = \frac{2\pi - \phi(l_\chi)}{2\pi l_\chi}$$

In the proof of this theorem, the method given in paragraph IV plays a prominent role. Let us recall that such method, starting from the χ value of some horizontal in the fundamental lattice, provides the knowledge of instants $T(\chi)$ (all integer numbers, since $\tau_0 = 1$) at which RP is found over that χ- horizontal :

$$X \Rightarrow T(\chi) = l_\chi + 1, l_\chi + 2, ..., l_{\chi+1}$$

Once these $T(\chi)$ are known, the procedure presented in paragraph III gives the corresponding values of the reduced phase:

$$X \Rightarrow T(\chi) \Rightarrow \phi(t = T(\chi)).$$

We start, as above, from equation (5.1) ,

$$T(\chi)\delta_0 - 2\pi\chi = \phi(T(\chi)), \tag{5.2}$$

where $T(\chi)$ may take all the values $l_\chi + 1, l_\chi + 2, ..., l_{\chi+1}$, and we also now take in consideration the case $t = l_\chi$ (see figure 5):

$$2\pi\chi - \delta_0 l_\chi = 2\pi - \phi(l_\chi). \tag{5.3}$$

Let us now define

$$t' \equiv 1 + l_\chi, \quad \Xi \equiv \min\{2\pi - \phi(t' - 1), \phi(t')\}$$

We obviously have $0 < \Xi < \frac{\delta_0}{2}$.The proof is now given by considering separately the two possible cases for the constant Ξ. Let us first assume that $\Xi \equiv \phi(t')$. Then it follows from (5.2) that

$$\frac{\delta_0}{2\pi} - \frac{X}{t' + n} = \frac{\Xi + n\delta_0}{2\pi(t' + n)}, \quad (n\forall,\ integrer)$$

Now we may, in the expression of $\eta(x)$, write the integer x under the form $x = t' + n$,with a certain integer n . We get then

$$\eta(x) \equiv \left| \frac{\delta_0}{2\pi} - \frac{\chi}{t' + n} \right| = \frac{\delta_0}{2\pi} \left| \frac{\frac{\Xi}{\delta_O} + n}{t' + n} \right| \tag{5.4}$$

Now it is clear that without any loss of generality, we may consider only the integers such that $t'+n>0$. Furthermore, an elementary inspection of the geometrical shape of the function (with argument n) in the last member of (4) immediately shows that it takes a minimum at $n=0$, so that

$$\eta_{\min} = \frac{\phi(l_\chi+1)}{2\pi(l_\chi+1)}.$$

If we now go over the case $\Xi \equiv 2\pi - \phi(t'=1)$, we get from (5.3)

$$\frac{\chi}{t'-n-1} - \frac{\delta_0}{2\pi} = \frac{\Xi + n\delta_0}{2\pi(t'-n-1)}.$$

In the expression of $\eta(x)$ we now introduce x under the form.. $x = t'-n-1$, with some integer n . Therefore,

$$\eta(x) \equiv \left|\frac{\delta_0}{2\pi} - \frac{\chi}{t'-n-1}\right| = \frac{\delta_0}{2\pi}\left|\frac{\frac{\Xi}{\delta_0}+n}{t'-n-1}\right|, \tag{5.5}$$

so that we may confine ourselves to the values of n such that $t'-1-n>0$. Again an elementary study of the function of argument n in the last member of (5.5) shows that :

If $l_\chi(\delta_0 - \Xi) < (l_\chi+1)\Xi$ the minimum value of $\eta(n)$ is found at $n=-1$, and we have

$$\eta(n=-1) = \frac{\delta_0 - 2\pi - \phi(l_\chi)}{2\pi(l_\chi+1)}$$

If, on the contrary, $l_\chi(\delta_0 - \Xi) > (l_\chi+1)\Xi$, the minimum is taken for n=0, and we have

$$\eta(n=0) = \frac{2\pi - \phi(l_\chi)}{2\pi l_\chi}, \quad \text{qed} \quad .$$

VI. Best approximation to a state of a system.

In this paragraph we are going to present another method the which, given a finite interval of time $(0,T_s)$ and an instant of time $T^* \in (0,T_s)$ where RP takes some value ϕ^* for its reduced phase, allows us to determine a certain instant $t \in (0,t_s)$ such that

$$\Delta\phi^*(t) \equiv |\phi(t) - \phi^*| \tag{6.1}$$

is minimal.

Keeping this aim in view, we shall begin by simply stating two symmetry properties of the fundamental lattice, whose proofs are a trivial consequence of the autonomy of our dynamical system. Let then T and T^* be two instants of time at which RP takes some values ϕ and ϕ^* for its reduced phase, while χ and χ^* label the corresponding horizontals where RP is found . We then have:

I) Direct symmetry : If $\phi + \phi^* < 2\pi$, then RP will have, at instant $T + T^*$, the value $\phi + \phi^*$ for its reduced phase and will be found on the horizontal $\chi + \chi^*$. If $\phi + \phi^* > 2\pi$ then at the same instant $T + T^*$, RP will have the value $\phi + \phi^* > -2\pi$ for its reduced phase and will be found on horizontal $\chi + \chi^* .. + 1$.

II) Inverse symmetry : Assume that $T^* > T$; then, if $\phi^* - \phi > 0$, RP at instant $T^* - T$ will have the reduced phase $\phi^* - \phi$ and will be on the $\chi^* - X$ horizontal. If, on the contrary, $\phi^* - \phi < 0$, at that same instant $T^* - T$ the reduced phase of the RP will be $2\pi - |\phi^* - \phi| = 2\pi + \phi^* - \phi$ and RP will be on the horizontal $\chi^* - \chi - 1$.

Let then ϕ^* be the reduced phase of RP at instant T^* :

$$\phi^* \equiv \phi(t = T^*).$$

We shall assume, as above, that our fundamental method of paragraph II has already provided the basic tools of calculus, namely the sequences $T_k, T'_k, \delta_k, \delta'_k, (k = 1, 2, ..., s)$ up to a certain integer s. Since, for some s,

$$T^* < T_s \tag{6.2}$$

then it follows from the foregoing definitions of direct and inverse symmetry that two integers N and M do exist such that

$$T_N \leq T^* < T_{N+1} \tag{6.3}$$

$$T_M \leq T_s - T^* < T_{M+1} \tag{6.4}$$

$$N + M \leq s. \tag{6.5}$$

Assuming, for the sake of simplicity, that

$$\delta_0 < \phi^* < 2\pi - \delta_0$$

(cases $0 < \phi^* < \delta_0$ and $2\pi - \delta_0 < \phi^* < 2\pi$ can be reduced to this one, and we shall deal with them in the sequel), let us first consider inequality (6.3), that is, the interval of time $(0, T*)$. During this interval, we have the sequence of instants

$$T^* - T_N < T^* - T_{N-1} < .... < T^* - T_1 < T^*,$$

and the corresponding values of their reduced phase :

$$\phi(T^* - T_k) = \phi^* - \delta_k \qquad (k = N, N-1, ..., 2, 1).$$

Of course, we have also

$$T^* - T'_N < T^* - T'_{N-1} < .... < T^* - T'_1 < T^*$$

(we only have the first inequality if $T'_N < T^*$, that is, $T'_N < T_N$), and for the reduced phase :

$$\phi(T^* - T'_k) = \phi^* - \delta'_k$$

As for the interval of time (T^*, T_s), we find in a similar way (by now making use of inequality (6.4)) the sequence of instants

$$T^* + T_1 < T^* + T_2 < ...... < T^* + T_M,$$

with

$$\phi(T + T_j) = \phi^* + \delta_j \qquad (j = 1, 2, ..., M).$$

We obviously also have

$$T^* + T'_1 < T^* + T'_2 < ...... < T^* + T'_M < T_s$$

(we only have the last inequality if $T'_M < T_M$) , and for the reduced phase:

$$\phi(T^* + T_j) = \phi^* + \delta'_j.$$

(NB:In the case where $0 < \phi^* < \delta_0$ or $2\pi - \delta_0 < \phi^* < 2\pi$, some of the second members of the formulae for the reduced phase may be negative. The exact expressions would then be obtained by simply replacing the negative values by the corresponding complementary ones , for instance, $\phi^* - \delta_k$, by $2\pi - |\phi^* - \delta_k|$, etc).

If we now go back to case (6.3) ,

$$T_N \leq T^* < T_{N+1},$$

it follows, from the above mentioned property of inverse symmetry that, during the time interval $(0, T^*)$ no instant of time exists such that $\Delta\phi^*(t) \equiv |\phi(t) - \phi^*| < |\delta_{N+1}|$ Yet it is clear that there are certain instants $t \in (0, T^*)$ such that

$$\Delta\phi^* < \delta_{0'}\ \Delta\phi^* < |\delta_1|, ..., \Delta\phi^* < |\delta_{N+1}|$$

Furthermore, we have

$$|\phi(T^* - T_N) - \phi^*| = |\delta_N|$$

As to the question of knowing wether some instant $t \in (0, T^*)$ exists such that $\Delta\phi^* < |\delta_N|$, this clearly depends upon the sign of inequality

$$T'_{N+1} >, < T_{N+1}.$$

If we have $T'_{N+1} > T_{N+1}$ (that is, $\bar{\nu}_N = \nu_{N+1}$), then there is no $t \in (0, T^*)$ such that $\Delta\phi^* < |\delta_N|$. The same happens if we have (simultaneously) the two conditions

$$T'_{N+1} < T_{N+1} \quad (\Leftrightarrow \bar{\nu}_N = \nu_{N+1}) \quad \wedge \quad T^* < T'_{N+1}.$$

If, on the contrary,

$$T'_{N+1} \leq T^* < T_{N+1} \tag{6.6}$$

then we find, at instant $t = T^* - T'_{N+1} \in (O, T^*)$,

$$\phi(T^* - T'_{N+1}) = \phi^* - \delta'_{N+1},$$

and hence

$$\Delta\phi^* = \left|\delta'_{N+1}\right| < \delta_N,$$

To put it briefly:

The instant $t \in (0, T^*)$ such that $\Delta\phi^*(t) \equiv |\phi(t) - \phi^*|$ is minimal, is determined in the following way:

$$\text{if} \qquad T'_{N+1} \leq T^* \leq T_{N+1} \Rightarrow t = T^* - T'_{N+1} \tag{6.7}$$

$$\text{with} \qquad \phi(T^* - T'_{N+1}) = \phi^* - \delta'_{N+1}$$
$$\text{and} \qquad \Delta\phi^* = \left|\delta'_{N+1}\right| < |\delta_N|\,.$$

In all other cases, that instant comes equal to $t = T^* - T_N$, with

$$\phi(T^* - T_N) = \phi^* - \delta_N \quad \text{and} \qquad \Delta\phi^* = |\delta_N|\,, \tag{6.8}$$

Let us now turn to the study of interval (T^*, T_s). The reasoning is in all ways similar and we make use of the properties expressed through direct symmetry. Then from (6.4),

$$T_M \leq T_s - T^* < T_{M+1},$$

it is apparent that no instant of time t exists in interval (T^*, T_s) such that $\Delta\phi^*(t)|\delta_{M+1}|$. Yet we still find in it certain instants where we have

$$\Delta\phi^* < \delta_{0'}\Delta\phi^* < |\delta_>|, ..., \Delta|\phi^* < |\delta_{M+1}|\,,$$

For $t = T^* + T_M$ it comes $\phi(T^* + T_M) = \phi^* + \delta_M$.

If we have either $T'_{M+1} > T_{M+1}$ or the two simultaneous conditions

$$T'_{M+1} < T_{M+1} \quad \text{and} \qquad T_s - T^* < T'_{M+1},$$

then it does not exist an instant $t \in (T^*, T_s)$ such that $\Delta\phi^* < |\delta_M|$. Nevertheless, if it happens that $T'_{M+1} \leq T_s - T^* < T_{M+1}$, then such an instant t does exist and it comes equal to T_s :

$$\phi(T^* + T_{M+1}) = \phi^* + \delta'_{M+1}.$$

Let us summarize:

The instant $t \in (T^*, T_s)$ such that $\Delta\phi^*(t) \equiv |\phi(t) - \phi^*|$ is minimal, is now given by the following formulae:

$$\text{if} \qquad T'_{M+1} \leq T_s - T^* < T_{M+1} \Rightarrow t = T^* + T'_{M+1} \tag{6.9}$$

$$\text{with} \quad \phi(T^* + T'_{M+1}) = \phi^* + \delta'_{M+1} \quad \text{and} \quad \Delta\phi^* = |\delta'_{M+1}| < |\delta_M|.$$

In all other cases, t is equal to

$$t = T^* + T_M, \tag{6.10}$$

$$\text{with} \qquad \phi(T' + T_M) = \phi^* + \delta_M \quad \text{and} \qquad \Delta\phi^* = |\delta_M| \,.$$

By now taking into account the foregoing reasonings, an answer immediately comes to view for the question stated at the beginning of this paragraph, namely, to find the instant $t \in (0, T)$ that makes minimal the expression $\Delta\phi^*(t) \equiv |\phi(t) - \phi^*|$, where $t^* \in (0, T)$.

If we have $N > M$ the instant t under discussion is found in the interval $(0, T^*)$, and the solution is given by by formulae (6.7),(6.8).

If we have $M > N$, such instant now belongs to interval (T^*, T_s) and it is given by (6.9),(6.10). The case $N = M$ calls for a more detailed discussion, in which we separately consider the two possible cases :

$$T'_{N+1} > T_{N+1} \quad \text{or} \qquad T'_{N+1} < T_{N+1}.$$

The first case is obvious : we have two instants at which RP is found in the required conditions, that is, such instants are $T^* + T_N$ and $T^* - T_N$ and we have

$$\phi(T^* + T_N) = \phi^* + \delta_N, \quad \phi(T^* - T_N) = \phi^* - \delta_N, \tag{6.11}$$

and there is no instant $t \in (0, T_s)$ such that

$$\phi^* - \delta_N < \phi(t) < \phi^*, \quad \phi^* < \phi(t) < \phi^* + \delta_N \tag{6.12}$$

The second case is less obvious, for we are bound to consider the four different possible cases:

$$T_s - T^* <, > T'_{N-1} \qquad T^* <, > T'_{N+1}$$

We then find:

If $T_s - T^* < T'_{N+1} \quad \wedge \quad T^* < T'_{N+1}$ we go back to the foregoing formulae (6.11),(6.12) .

If $T_s - T^* > T'_{N+1} \quad \wedge \quad T^* > T'_{N+1}$ it is easily seen that we again get formulae (6.11),(6.12), with the only difference that T_N is now replaced by T'_{N+1} and δ_N by δ'_{N+1} .

If $T_s - T^* < T'_{N+1} \quad \wedge \quad T^* > T'_{N+1}$, the instant we look for is equal to $T^* + T'_{N+1}$, since we have

$$\phi(T^* - T'_{N+1}) = \phi^* - \delta'_{N+1} \Rightarrow \Delta\phi^* = \left|\delta'_{N+1}\right|$$

and there is no $t \in (0, T_s)$ such that $\Delta\phi^* < \left|\delta'_{N+1}\right|$

Finally, if $T_s - T^* > T'_{N+1} \quad \wedge \quad T^* < T'_{N+1}$ that instant is now given by $T^* + T'_{N+1}$ In fact, we have

$$\phi(T^* + T'_{N+1}) = \phi^* + \delta'_{N+1} \Rightarrow \Delta\phi^* = \left|\delta'_{N+1}\right|,$$

and there is no $t \in (0, T)$ such that $\Delta\phi^* < |\delta'_{N+1}|$.

Let us recall that we can easily overcome the restriction $\delta_0 < \phi^* < 2\pi - \delta_0$ according to what has been said above: The final formulae are straightforwardly generalised, while the reasoning underlying the method remains of course the same. Besides, in the next paragraph we shall explicitly deal with case $0 < \phi^* < \delta_0$ in some detail.

To finish this paragraph, let us add an obvious corollary of the foregoing results, namely, that given any two instants of time belonging to interval $(0, T_s)$, t_1 and t_2 (with, for instance, $t_1 < t_2$), the absolute value of the difference of the reduced phase at those instants can not be lesser than $|\delta_r|$, with $r < s$ (and hence $|\delta_r| < |\delta_s|$) and such that $T_r \leq t_2 - t_1 < T_{r+1}$.

VII. Ordering the recurrence times.

We have presented in the preceding paragraph a method which, given a finite interval of time $(0, T_s)$, allows us to calculate a certain instant $t \in (0, T_s)$ such that

$$\Delta\phi^*(t) \equiv |\phi - \phi^*| \equiv |\phi(t) - \phi(T^*)|$$

is minimal, T^* being any fixed instant of time belonging to $(0, T_s)$. Now it must be remarked that this method also provides the exact knowledge of the value of ϕ. From it we may infer wether $\phi - \phi^*$ is positive or negative, that is, if the position of RP at time t is "at right" or "at left" of its position at instant T^*. In other words, our method enables us to calculate t such that the algebraic (non absolute) value of the expression $\phi(t) - \phi^*$ is minimal and also to know if it is positive or negative. Such remark proving very useful in the reasoning below, we are going in what follows to particularize it in only one case. By so doing, the conclusions will not lack any generality, all the more since the other cases are neither very different nor very numerous.

Let then be $T^* \in (0, T_s)$ and $\phi^* \equiv \phi(T^*)$, and assume that the method of the foregoing paragraph has already provided $T \in (0, T_s)$

such that $\Delta\phi^* \equiv |\phi(T) - \phi^*|$ is minimal with, for instance, RP "at left" of ϕ^* :

$$\phi(T) - \phi^* < 0$$

According to what has been said above, the question is now to determine some instant $\widetilde{T} \in (0, T_s)$ such that $|\phi(\widetilde{T}) - \phi^*|$ is minimal and $\phi(\widetilde{T}) - \phi^* > 0$ (RP "at left" of ϕ^*). Let us assume that we have

$$N > M \quad \wedge \quad T'_{N+1} \leq T^*$$

This case (which was studied in (6.6) among all other possible cases) corresponds to $T = T^* - T'_{N+1}$ with

$$\phi(T = T^* - T'_{N+1}) = \phi^* - \delta'_{N+1}$$

and thus

$$\phi(T = T^* - T'_{N+1}) < \phi^* \Rightarrow \delta'_{N+1} > 0$$

It is then clear that the instant we are looking for will be either $T^* - T_N$ or $T^* - T'_N$. More precisely :

$$\text{if} \qquad \delta_N < 0 \Rightarrow \quad \left\{ \begin{array}{c} \widetilde{T} = T^* - T_N \\ \phi(\widetilde{T}) = \phi^* - \delta_N > \phi^* \end{array} \right\}$$

$$\text{if} \qquad \delta_N > 0 \Rightarrow \quad \left\{ \begin{array}{c} \widetilde{T} = T^* - T'_N \\ \phi(\widetilde{T}) = \phi^* - \delta'_N > \phi^* \end{array} \right\}$$

In a similar way the reader could easily obtain the corresponding results for the other cases.

Now this procedure will prove of fundamental usefulness in ordering the values of the rp taken by RP in a bounded interval $(0, T_s)$. By this we mean that we want to find an ensemble of instants of time

$$\widetilde{T}_1 \equiv T_s, \widetilde{T}_2, \widetilde{T}_3, ...,$$

all belonging to $(0, T_s)$ and such that

$$\widetilde{\phi}_1 \equiv \delta_s < \widetilde{\phi}_2 < \widetilde{\phi}_3 < ......,$$

and there is no $t \in (0, T_s)$ such that $\widetilde{\phi}_r < \phi(t) < \widetilde{\phi}_{r+1}$, where, by definition,

$$\widetilde{\phi}_r \equiv \phi(t = \widetilde{T}_r) \qquad r = 1, 2, ...$$

The "modus operandi" rests mostly upon the method described in paragraph VI and the above considerations. Let us start with instant $t = \widetilde{T}_1 \equiv T_s$ and assume that we have

$$\delta_s < 0.$$

(Case $\delta_s > 0$ is a straightforward transposition of this one). Let us look for an instant $t \in (0, T_s)$ at which RP takes the least value for its reduced phase. It is clear that such instant is to be found among the three following possible values:

$$T'_s \quad , \quad T_{s-1} \quad , \quad T'_{s-1} \quad ,$$

and a mere checking leads to the desired value:

a) if $T'_s < T_s$, that instant comes equal to $\widetilde{T}_2 \equiv T'_s$ and hence $\widetilde{\phi}_2 \equiv \delta'_s$;

b) if $T'_s > T_s$ and $\delta_{s-1} > 0$, then we will have $\widetilde{T}_2 \equiv T_{s-1}$, and $\widetilde{\phi}_2 \equiv \delta_{s-1}$;

c) if $T'_s > T_s$ and $\delta_{s-1} < 0$, then $\widetilde{T}_2 \equiv T'_{s-1}$, with $\widetilde{\phi}_2 \equiv \delta'_{s-1}$.

In any case, it is clear that we have $\widetilde{\phi}_1 \equiv \delta'_s < \widetilde{\phi}_2$ and there is no instant $t \in (0, T_s)$ such that $\widetilde{\phi}_1 < \phi(t) < \widetilde{\phi}_2$.

From now on, we have only to apply strictly the method described at the beginning of this paragraph in order to obtain $t = \widetilde{T}_3 \in (0, T_s)$ such that $\phi(t) - \widetilde{\phi}_2 \equiv \widetilde{\phi}_3 - \widetilde{\phi}_2$ is minimal and positive.

In a similar way we could find $t = \widetilde{T}_4 \in (0, T_s)$ such that $\phi(t) - \widetilde{\phi}_3 \equiv \widetilde{\phi}_4 - \widetilde{\phi}_3$ is minimal and positive - and so on.

VIII. The Poincaré cycles.

Once thus obtained the (ordered) sequence

$$\widetilde{\phi}_1 \equiv \delta_s < \widetilde{\phi}_2 < \widetilde{\phi}_3 < ....$$

and the (non ordered) sequence

$$\widetilde{T}_1 \equiv T_s, \widetilde{T}_2, \widetilde{T}_3, ...$$

(according to the method described in the preceding paragraph) we are ready to provide the complete solution for the problem of the Poincaré recurrence cycle as it was stated at the beginning of this work. By this we mean that we are going to calculate the ordered sequence of instants.

$$C_1 \equiv \widetilde{T}_1 \equiv T_s < C_2 < C_3 < .....$$

at which RP returns to the $|\delta_s|$ -neighbourhood of its initial situation, that is, $\phi(t = 0) = 0$. To be more precise, and if we denote by V(t) the phase shift of RP at instant t, we have:

$$|V(t = C_r)| < |\delta_s| = |\widetilde{\phi}_1|$$

and there is no $t \in (C_r, C_{r+1})$ such that $|V(t)| < |\delta_s| = |\widetilde{\phi}_1|$

We assume, following the case studied in the foregoing paragraph, that $\delta_s < 0$, and again leave to the reader the transposition for $\delta_s > 0$.

Let us define Π_2 as the integral part of $|\widetilde{\phi}_2/\widetilde{\phi}_1|$. It is then clear that the second recurrence instant of RP inside the $|\delta_s|$ neighbourhood of $V = 0$ comes equal to

$$C_2 = \Pi_2\widetilde{T}_1 + \widetilde{T}_2 = \Pi_2 C_1 + \widetilde{T}_2,$$

the phase shift being

$$V(t = C_2) = \widetilde{\phi}_2 - \Pi_2|\widetilde{\phi}_1|,$$

a positive value, and lesser than $|\widetilde{\phi}_1| = \delta_s$. More generally, let us also define Π_a as the integral part of $|\widetilde{\phi}_a/\widetilde{\phi}_1|, (a = 2, 3, ...)$. It then follows that the instants $\Pi_a T_s + \widetilde{T}_a$ are also recurrence instants, since RP is then found inside the $|\delta_s| = |\widetilde{\phi}_1|$-neighbourhood of $V = 0 : V(\Pi_a T_s + \widetilde{T}_a) = \widetilde{\phi}_a - \Pi_a|\widetilde{\phi}_1|$, and hence

$$0 < \widetilde{\phi}_a - \Pi_a|\widetilde{\phi}_1| < |\delta_s| = |\widetilde{\phi}_1|$$

But the same will happen with instants $(\Pi_a + 1)T_s + \widetilde{T}_a$, since

$$V(t = (\Pi_a + 1)T_s + \widetilde{T}_a) = \widetilde{\phi}_a - (\Pi_a + 1)|\delta_s|$$

from which it follows that

$$\delta_s < \widetilde{\phi}_a - (\Pi_a + 1)|\delta_s| < 0$$

The question is then to ascertain wether $\Pi_a T_s + \widetilde{T}_a$ and $(\Pi_a + 1)T_s + \widetilde{T}_a$ actually are consecutive recurrence instants or not. Let us recall that, since

$$\widetilde{\phi}_{a+1} - \widetilde{\phi}_a > |\delta_s|,$$

it follows that

$$\Pi_{a+1} \geq \Pi_a + 1$$

We may then infer that for any integer a such that

$$\Pi_{a+1} = \Pi_a + 1 \quad \wedge \quad \widetilde{T}_{a+1} < \widetilde{T}_a,$$

the recurrence instant immediately next to $\Pi_a T_s + \widetilde{T}_a$ is not $(\Pi_a + 1)T_s + \widetilde{T}_a$ but $(\Pi_a + 1)T_s + \widetilde{T}_{a+1}$ instead, and we have

$$V((\Pi_a + 1)T_s + \widetilde{T}_{a+1}) = \widetilde{\phi}_{a+1} - (\Pi_a + 1)|\delta_s|$$

a positive value and lesser than $|\delta_s|$. As for this last instant of recurrence, it will be followed by $(\Pi_a + 1)T_S + \widetilde{T}_a$, and we have for its phase shift

$$V((\Pi_a + 1)T_s + \widetilde{T}_a) = \widetilde{\phi}_a - (\Pi_a + 1)|\delta_s|$$

a negative value, and greater than δ_s.

We may now present the sequence of Poincaré recurrence cycles inside $|V| < |\delta_s|$;

$$C_1 \equiv \widetilde{T}_1 \equiv T_s, \quad \text{with phase shift} \qquad \widetilde{\phi}_1 \equiv \delta_s < 0$$

$$C_2 = \Pi_2 C_1 + \widetilde{T}_2, \quad \text{with phase shift} \qquad \widetilde{\phi}_2 - \Pi_2|\widetilde{\phi}_1| > 0$$

And for a = 3, 4, ...we have

$$C_{2a-3} = (\Pi_{a-1} + 1)C_1 + \widetilde{T}_{a+1}, \quad \text{with phase shift} \qquad \widetilde{\phi}_a - \Pi_a|\widetilde{\phi}_1| > 0$$

$$C_{2a-2} = \Pi_a C_1 + \widetilde{T}_a, \quad \text{with phase shift} \qquad \widetilde{\phi}_{a-1} - (\Pi_{a-1} + 1)|\widetilde{\phi}_1| < 0,$$

unless we have the two simultaneous conditions

$$\Pi_a = \Pi_{a-1} + 1 \quad \wedge \quad \widetilde{T}_a < \widetilde{T}_{a-1},$$

in which case C_{2a-3} and C_{2a-2} interchange their expressions given above, the same being true for the values of their phase shift .

Appendix.

In this Appendix we are going to show the full equivalence between the sequence of integers $\nu_k, \bar{\nu}$ (defined in paragraph II and which plays a fundamental role throughout this work) and the representation of a real number in continued fraction. Let us then have $\delta_0 \in (0, \pi)$ (the general case can obviously be reduced to this one). Let us further assume that

$$v_k = \bar{\nu}_k \quad , \quad (k = 0, 1, 2, ...).$$

We have then the following deduction :

$$\frac{2\pi}{\delta_0} = \nu_0 + \mu_0 \tag{A.1}$$

$$\left|\frac{\delta_1'}{\delta_1}\right| = \nu_1 + \mu_1 = \frac{1}{\mu_0} - 1 \Rightarrow \frac{1}{\mu_0} = 1 + \nu_1 + \mu_1$$

Hence, in introducing in (A.1),

$$\frac{2\pi}{\delta_0} = \nu_0 + \frac{1}{1 + \nu_1 + \mu_1} \tag{A.2}$$

Similarly, we find

$$\left|\frac{\delta_2'}{\delta_2}\right| = \nu_2 + \mu_2 = \frac{1}{\mu_1} - 1 \Rightarrow \frac{1}{\mu_1} = 1 + \nu_2 + \mu_2$$

and (A.2) takes the form

$$\frac{2\pi}{\delta_0} = \nu_0 + \frac{1}{1 + \nu_1 + \frac{1}{1+\nu_2+\mu_2}},$$

and so on : whenever we have $v_k = \bar{\nu}_k$, it follows that

$$\frac{1}{\mu_k} = 1 + \nu_{k+1} + \mu_{k+1}.$$

Hence the continued fraction

$$\frac{2\pi}{\delta_0} = \nu_0 + \frac{1}{1 + \nu_1 + \frac{1}{1+\nu_2+\frac{1}{1+\nu_3+\frac{1}{\cdots}}}} \equiv [\nu_0, 1 + \nu_1, 1 + \nu_2, ...]$$

If we now assume that

$$\bar{\nu}_k = \nu_k + 1, (k = 0, 1, 2, ...),$$

then we find

$$\frac{2\pi}{\delta_0} = \nu_0 + \mu_0 \tag{A.3}$$

$$\left|\frac{\delta_1'}{\delta_1}\right| = \nu_1 + \mu_1 = \frac{\mu_0}{1-\mu_0} \Rightarrow \frac{1}{\mu_0} = 1 + \frac{1}{\nu_1 + \mu_1}$$

That is, for (A.3),

$$\frac{2\pi}{\delta_0} = \nu_0 + \frac{1}{1 + \frac{1}{\nu_1 + \mu_1}} \tag{A.4}$$

Furthermore, we have

$$\left|\frac{\delta_2'}{\delta_2}\right| = \nu_2 + \mu_2 = \frac{\mu_1}{1-\mu_1} \Rightarrow \frac{1}{\mu_1} = 1 + \frac{1}{\nu_2 + \mu_2}$$

and (A.4) becomes

$$\frac{2\pi}{\delta_0} = \nu_0 + \frac{1}{1 + \frac{1}{\nu_1 + \frac{1}{1 + \frac{1}{\nu_2 + \mu_2}}}}.$$

We see that whenever $\bar{v}_k = v_k + 1$, it follows

$$\frac{1}{\mu_k} = 1 + \frac{1}{\nu_{k+1} + \mu_{k+1}}$$

from which we get the continued fraction

$$\frac{2\pi}{\delta_0} = [\nu_0, 1, \nu_1, 1, \nu_2, 1, ...].$$

In the general case (where $\bar{\nu}_k$ can take the values ν_k or $\nu_k + 1$) a similar calculation would lead to the form of the continued fraction, which can be given in the following way:

Let us have certain integers $w_1, w_2, ..., w_n, ... \equiv \{w\}$ such that

$$0 \leq w_1 < w_2 < .... < w_n < ...$$

$$\text{and} \quad \bar{\nu}_{w_k} = \nu_{w_k} + 1$$

$$\bar{\nu}_s = \nu_s \quad \text{for} \quad s \notin \{w\}.$$

Then we have

$$\frac{2\pi}{\delta_0} = [\nu_0, 1+\nu_1, 1+\nu_2, ..., 1+v_{w_1}, 1, \nu_{w_1+1}, 1+\nu_{w_1+2}, 1+\nu_{w_1+3}, ...$$

$$.........., 1+\nu_{w_2}, 1, \nu_{w_2+1}, 1+\nu_{w_2+2}, 1+\nu_{w_2+3}, ...].$$

The above formulae enable us , knowing the two sequences $\bar{\nu}_k + \nu_k$, to build the continued fraction associated to any number $2\pi/\delta_0$ and conversely: given any infinite or truncated continued fraction associated to some real number, we may from it obtain the two sequences of integers $\bar{\nu}_k + \nu_k$. By so doing, the coefficients of any continued fraction receive a dynamical, geometric meaning provided by the recurrence theorem of Poincaré.

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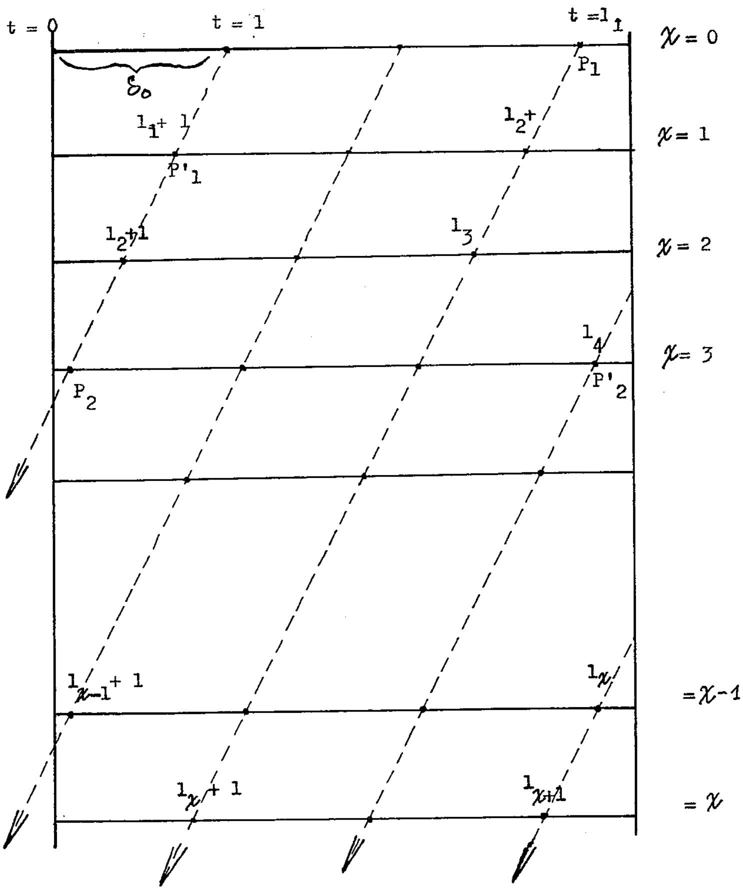

Figure 1.

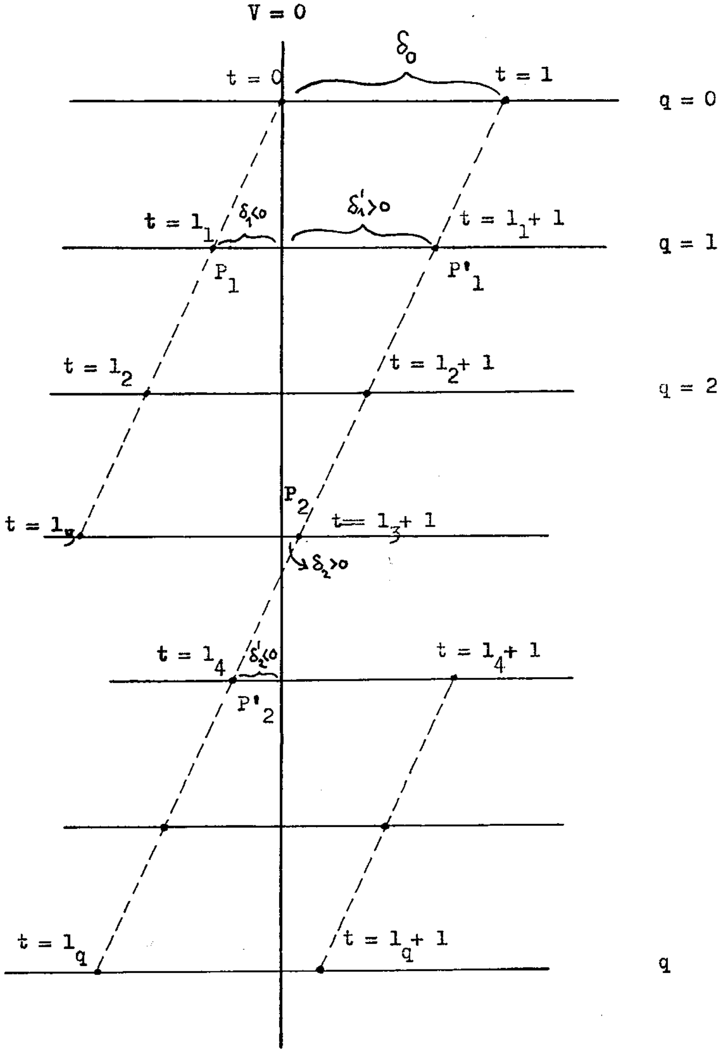
V = 0
t = 0
t = 1
q = 0
t = 1₁
t = 1₁+ 1
q = 1
P₁
P'₁
t = 1₂
t = 1₂+ 1
q = 2
P₂
t = 1₃+ 1
t = 1₄
t = 1₄+ 1
P'₂
t = 1q
t = 1q+ 1
q

Figure 2.

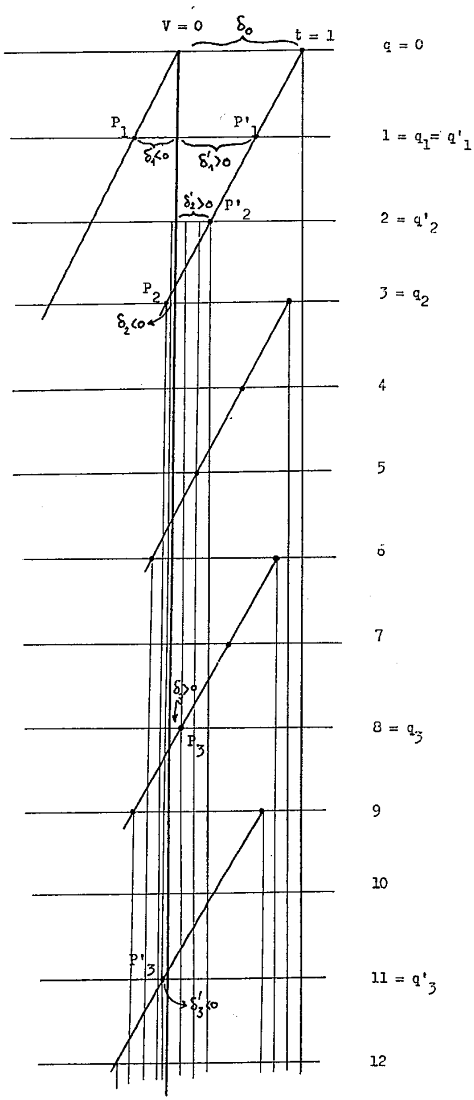
V = 0
δ_0
t = 1
q = 0
P_1
P'_1
$\delta_1 < 0$
$\delta'_1 > 0$
1 = q_1 = q'_1
$\delta'_2 > 0$
P'_2
2 = q'_2
P_2
$\delta_2 < 0$
3 = q_2
4
5
6
7
$\delta_3 > 0$
P_3
8 = q_3
9
10
P'_3
$\delta'_3 < 0$
11 = q'_3
12

Figure 3.

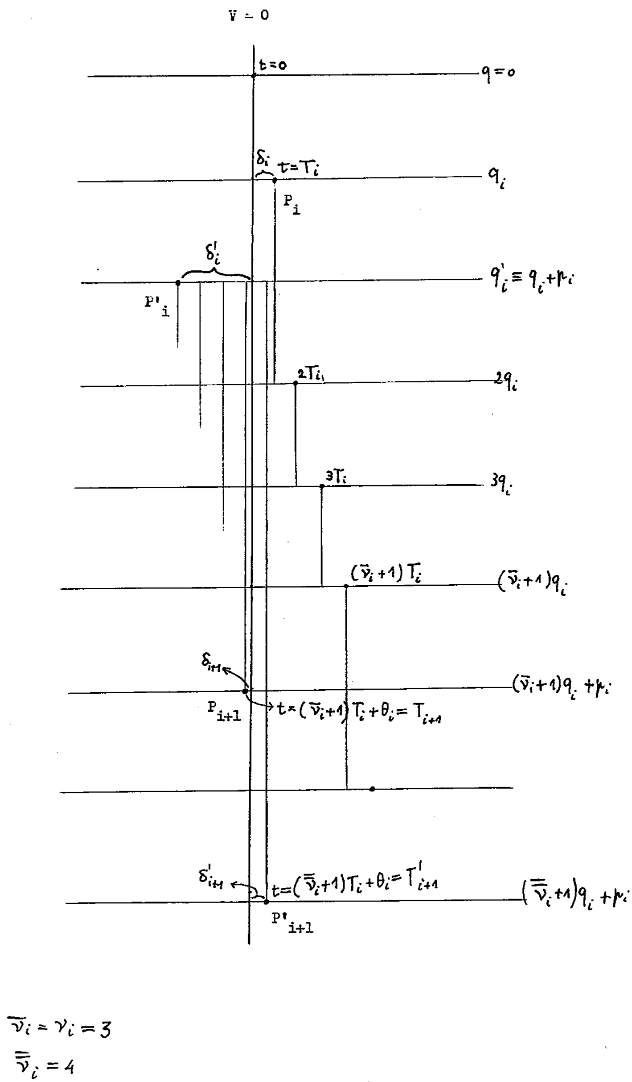
V = 0
t=0
q=0
δi
t=Ti
qi
Pi
δ'i
q'i ≡ qi+μi
P'i
2Ti
2qi
3Ti
3qi
(ν̄i+1) Ti
(ν̄i+1)qi
δi+1
(ν̄i+1)qi +μi
Pi+1
t=(ν̄i+1)Ti+θi = Ti+1
δ'i+1
t=(ν̿i+1)Ti+θi = T'i+1
(ν̿i+1)qi +μi
P'i+1
ν̄i = νi = 3
ν̿i = 4

Figure 4.

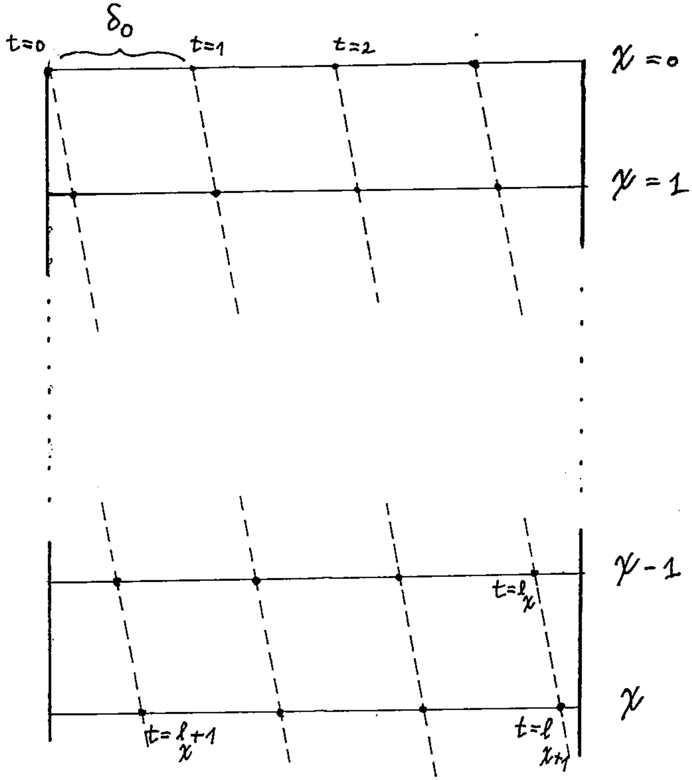

Figure 5.