Quantum chaos and fundamental multivaluedness of dynamical functions

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ABSTRACT. The optical potential method is generalised and adapted to obtain a non-perturbative self-consistent description of quantum chaos in Hamiltonian systems. The method provides a reformulation of the Schrödinger equation revealing the multivaluedness of the effective Hamiltonian, i. e. its splitting into many branches, called "realisations". This splitting does not involve any decohering influence of noise, "coarse-graining", etc. Each realisation incorporates the normal complete set of eigenfunctions and eigenvalues for the entire problem. It means that one can never observe more than one realisation at a time. Then chaos appears naturally as noise-assisted "spontaneous transitions" between the different realisations or unpredictable appearance of one of them. This introduces the postulate of the fundamental dynamic uncertainty involving a modified version of the Schrödinger equation. It provides the "true" chaos with the intrinsic causal randomness appearing as a fundamental property of dynamical systems not reduced to some kind of sophisticated but predictable behaviour. The modified formulation of quantum mechanics does not contradict the ordinary one but rather extends it to the case of chaotic dynamics. Moreover, it "re-establishes" the correspondence principle which is shown to be applicable in its conventional form also to chaotic systems. The method is presented in detail for the Halmiltonian system with periodic (not small) perturbation, both in its time-independent and time-dependent versions. The generalisation to other cases of chaos is outlined as well as the ensuing universal definitions of complexity (causal randomness, probability), (non)integrability, and general solution. The method and the results obtained are directly applicable to real physical systems

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RÉSUMÉ. On généralise la méthode du potentiel optique pour obtenir description nonperturbative et autocohérente du chaos quantique dans les systèmes hamiltoniens. La méthode contient une forme modifiée de l'équation de Schrödinger qui met à l'évidence la dissociation du hamiltonien effectif en plusieurs branches appelées les "réalisations". Cette dissociation n'implique aucune "effet de décohérence" dû au bruit, "coarse-graining", etc. Chaque réalisation comporte l'ensemble complet des fonctions et valeurs propres du problème. Par conséquent le chaos apparaît naturellement comme les "transitions spontanées" entre les réalisations différentes, assistées par bruit, ou encore comme l'apparition imprévisible d'une des réalisations. Ce postulat de l'incertitude dynamique fondamentale donne le "vrai" chaos avec le hasard "causal" intrinsèque, la propriété qui ne se réduit pas à un comportement sophistiqué, mais prévisible. La formulation modifiée de la mécanique quantique, introduite par la méthode, n'est pas en contradiction avec la forme habituelle, mais sert plutôt à élargir celle-ci sur le cas de la dynamique chaotique. Elle nous permet aussi de confirmer le principe de correspondance, dans sa forme usuelle, pour les systèmes chaotiques. La méthode est présentée en détail pour le cas des systèmes hamiltoniens avec la perburbation périodique dans ses deux versions, avec et sans dépendance du temps. La généralisation sur autres types des systèmes chaotiques est indiquée, ainsi que les résultant définitions de complexité (imprévisibilité causale, probabilité), (non)intégrabilité, et la solution générale. La méthode et ses résultats sont directement applicables aux systèmes physiques réels.

1. Introduction

The recent emergence and development of the dynamical chaos concept have engendered profound changes in our understanding of both fundamental and practical aspects of the dynamical system behaviour in many different fields of physic (e. g. [1]). In particular, the behaviour of simple non-dissipative mechanical systems with few degrees of freedom presents an elementary case well suited for the study of the fundamental origins of dynamical randomness, with possible further extension to more complex situations. The description of chaos in such elementary dynamical systems within the formalism of classical mechanics has seemed to be rather successful and self-consistent [1-4]. At the same time its proposed quantum-mechanical versions, despite a large amount of the efforts made, have failed in creating a similar prosperous situation, although a number of important particular results has been obtained [4-7]. The problem of the very existence of the truly unpredictable behaviour of deterministic quantum systems remains unsettled

[8]. The fundamental difficulty stems from the unavoidable linear wave involvement in quantum postulates : such waves do not easily show global instabilities necessary for the development of chaotic regimes or, in other terms, waves lead to discreteness (one may physically realise minimum a half-wave and not, say, one fifth of it), and the discreteness is incompatible with the existing notion of instability appealing to infinitesimal values and apparently indispensable for the known definition of chaos. The methods to resolve this basic contradiction, proposed or implied (see e. g. [9-15]), use typically some "roundabout" logical issue or a reduction to another basic puzzle, leaving the problem of quantum chaos as such without definitive unambiguous answer.

This situation predetermines the importance of the search for a general self-consistent quantum (and eventually any other) chaos description providing the fundamental origin of randomness in deterministic systems. In this paper we present such an approach starting from the application of the unreduced version of the well-known optical potential method (see e. g. [16]) to the analysis of quantum chaos in Hamiltonian systems with periodic perturbation. This approach appeared originally as a part of quantum-mechanical description of charged particle scattering in crystals [17] revealing its chaotic behaviour (see, in particular, section 2.5 of the cited article). The results obtained are considerably developed and generalised in the present paper (see [18, 19] for the extended version).

Using a generic example of arbitrary periodically perturbed Hamiltonian system, we show that our method naturally leads to the new concept of the fundamental origin of chaos in dynamical systems (section 2). In particular it permits one to overcome the "pathological regularity" of quantum mechanics and to perform the ordinary semiclassical transition also for chaotic systems, in agreement with the correspondence principle (section 3). It is worthwhile to note that our approach is presented in the form ready for its application to practical study of Hamiltonian quantum chaos in various real physical systems including both basic aspects and the particular analysis of the measured quantities.

It is important that in order to obtain quantum dynamics with nonzero complexity and the conventional semiclassical transition, one does not need to reconsider the foundations of quantum mechanics as such but rather to use another, more general, form of the same formalism. This can help to moderate the painful choice described above. Moreover, in the next paper (see also [18,20] we show that it is the main unsolved

problem of the foundations of quantum mechanics, known as quantum indeterminacy and wave reduction, that can be given transparent causal solutions by application of the same method to the process of quantum measurement.

Finally, the purpose of this work is to demonstrate the universal character of the results obtained permitting one to extend the same concept of complex behaviour to other Hamiltonian and non-Hamiltonian, classical, and eventually distributed nonlinear systems. The ensuing universal notions of the fundamental dynamic uncertainty, complexity (causal randomness, probability), (non)integrability, and general solution are introduced in section 4.

2. Formulation of the method

2.1. Effective dynamical functions

Consider a conservative dynamical system with the Hamiltonian

$$H = h + V,$$

where h is the free-motion Hamiltonian, and V describes the elastic interaction. Within the total Hamiltonian H we separate the integrable part, H_0, corresponding to the regular dynamics, and the perturbation, H_p, (generally, not small) inducing chaotic behaviour of the whole system which we want to describe :

$$H = H_0 + H_p, H_0 \equiv h_0 + V_0, H_p \equiv h_p + V_p \quad .$$

We introduce then a particular representation, for definitness chosen in the form of coordinate representation, $H \equiv H(\mathbf{r})$, and divide the vector of independent variables into two parts, $\mathbf{r} = \{\mathbf{r}_\sigma, \mathbf{r}_\pi\}$, so that, in accord with the integrability of $H_0, H_0 = H_0(\mathbf{r}_\sigma)$ and $H_p = H_p(\mathbf{r}_\sigma, \mathbf{r}_\pi)$. If chaos is induced by the addition of extra degrees of freedom (dimensions), the variables $\mathbf{r}_\pi$ may correspond to these degrees, while the motion limited to the degrees $\mathbf{r}_\sigma$ is considered to be regular. In cases where such subdivision is not naturally given by the conditions of a problem, it can always be made using the well-known regularity of one-dimensional problems : in the simplest version a one-dimensional component of $\mathbf{r}$ is chosen as $\mathbf{r}_\sigma, \mathbf{r}_\sigma = x, \mathbf{r} = \{x, \mathbf{r}_\pi\}$. The subdivision of the Hamiltonian is then performed with the help of the Fourier analysis or other suitable expansion.

As we shall see later, the case of time-dependent perturbation can also be considered within the same formalism, and then $\mathbf{r}_\pi$ corresponds to the time variable, t. If the degrees of freedom remain unchanged and chaos is due to a symmetry-breaking perturbation, one can still use the above formal method of division. To obtain physically more meaningful result, one may imply, where possible, the quantum analogues of the action-angle variables ($\mathbf{r}_\sigma \to, \mathbf{I}, \mathbf{r}_\pi \to \boldsymbol{\theta}$), or those determined by the symmetry (e.g. $\mathbf{r}_\sigma = \mathbf{r}$), or other suitable choice.[1]

While studying purely dynamical origins of stochasticity, it is natural to assume that V_p is a periodic function of $\mathbf{r}_\pi$. It is also one of the most interesting practical and model cases (e. g. particle scattering in regular structures [17,21], atom excitation by electromagnetic radiation [22], the kicked rotor model [23], etc.), and it is well suited for the demonstration of our method. The dependences of V_0 and V_p or $\mathbf{r}_\sigma$ may be periodic or not.

Consider now the Schrödinger equation for our system,

$$[h_0(\mathbf{r}_\sigma) + h_p(\mathbf{r}_\pi) + V_0(\mathbf{r}_\sigma) + V_p(\mathbf{r}_\sigma, \mathbf{r}_\pi)]\Psi(\mathbf{r}_\sigma, \mathbf{r}_\pi) = E\Psi(\mathbf{r}_\sigma, \mathbf{r}_\pi), \quad (1)$$

where E is the total energy. The involvement of periodicity inspires the idea about the Fourier transformation[2] over $\mathbf{r}_\pi$ which is done in the usual way and leads to the system of equations for the component functions $\Psi_0(\mathbf{r}_\sigma), \Psi_{\mathbf{g}_\pi}(\mathbf{r}_\sigma)$:

$$[h_0(\mathbf{r}_\sigma) + V_0(\mathbf{r}_\sigma)]\Psi_0(\mathbf{r}_\sigma) = \varepsilon_\sigma \Psi_0(\mathbf{r}_\sigma) - \sum_{\mathbf{g}_\pi} V_{-\mathbf{g}_\pi}(\mathbf{r}_\sigma)\Psi_{\mathbf{g}_\pi}(\mathbf{r}_\sigma), \quad (2a)$$

$$h_0(\mathbf{r}_\sigma)\Psi_{\mathbf{g}_\pi}(\mathbf{r}_\sigma) + \sum_{\mathbf{g}_{\pi'}} V_{\mathbf{g}_\pi - \mathbf{g}_{\pi'}}(\mathbf{r}_\sigma)\Psi_{\mathbf{g}_{\pi'}}(\mathbf{r}_\sigma) = \varepsilon_{\sigma\mathbf{g}_\pi}\Psi_{\mathbf{g}_\pi}(\mathbf{r}_\sigma) - V_{\mathbf{g}_\pi}(\mathbf{r}_\sigma)\Psi_0(\mathbf{r}_\sigma), \quad (2b)$$

where $\mathbf{g}_\pi, \mathbf{g}_\pi{}' \neq 0$ are the dual "reciprocal lattice" vectors with respect to the "direct lattice" of vectors $\mathbf{r}_\pi$,

$$\varepsilon_{\sigma\mathbf{g}_\pi} \equiv E - \hbar^2(\mathbf{K}_\pi + \mathbf{g}_\pi)^2/2m, \varepsilon_\sigma \equiv E - \hbar^2\mathbf{K}_\pi{}^2/2m, \quad (3)$$

[1] We consider it to be always possible. Although our description encompasses, in principle, this latter type of chaos, the results below are specified rather for the former one ; the detailed investigation of the second type of chaos is left for next publications.

[2] In the general case it will be an expansion in terms of other complete system of functions appropriate to a problem (see [18-20]).

$$\Psi(\mathbf{r}) = \exp(i\mathbf{K}_\pi \mathbf{r}_\pi)[\Psi_0(\mathbf{r}_\sigma) + \sum_{\mathbf{g}_\pi} \Psi_{\mathbf{g}_\pi}(\mathbf{r}_\sigma) \exp(i\mathbf{g}_\pi \mathbf{r}_\pi)],$$

$$V(\mathbf{r}) = V_0(\mathbf{r}_\sigma) + \sum_{\mathbf{g}_\pi} V_{\mathbf{g}_\pi}(\mathbf{r}_\sigma) exp(i\mathbf{g}_\pi \mathbf{r}_\pi),$$

and the wave vector $\mathbf{K}_\pi$ corresponds to the standard Bloch wave representation for periodic potentials (see e. g. [16]).

Now one can use this decomposition to study the influence of "chaos bringing" perturbation $V_p(\mathbf{r}_\sigma, \mathbf{r}_\pi)$ that is responsible for the terms with $\mathbf{g}_\pi \neq 0$. We start by applying the simple method of substitution and first express $\Psi_{\mathbf{g}_\pi}(\mathbf{r}_\sigma)$ through $\Psi_0(\mathbf{r}_\sigma)$ from eq. (2b) with the help of the Green function for its homogeneous part with respect to $\Psi_{\mathbf{g}_\pi}(\mathbf{r}_\sigma)$,

$$h_0(\mathbf{r}_\sigma)\Psi_{\mathbf{g}_\pi}(\mathbf{r}_\sigma) + \sum_{\mathbf{g}_\pi{}'} V_{\mathbf{g}_\pi - \mathbf{g}_\pi{}'}(\mathbf{r}_\sigma)\Psi_{\mathbf{g}_\pi{}'}(\mathbf{r}_\sigma) = \varepsilon_{\sigma\mathbf{g}_\pi}\Psi_{\mathbf{g}_\pi}(\mathbf{r}_\sigma). \qquad (4a)$$

The Green function is given by the well-known expression :

$$G_{\mathbf{g}_\pi}(\mathbf{r}_\sigma, \mathbf{r}_\sigma{}') = \sum_n \frac{\Psi^0_{\mathbf{g}_\pi n}(\mathbf{r}_\sigma)\Psi^{0*}_{\mathbf{g}_\pi n}(r'_\sigma)}{\varepsilon^0_{\mathbf{g}_\pi n} - \varepsilon_{\sigma\mathbf{g}_\pi}},$$

where $\{\Psi^0_{\mathbf{g}_\pi n}\mathbf{r}_\sigma\}$ and $\{\varepsilon^0_{\mathbf{g}_\pi n}\}$ are the sets of engenfunctions and eigenvalues, respectively, for the auxiliary system of equations (4). The solution of the system (2b) can be presented in the form

$$\Psi_{\mathbf{g}_\pi}(\mathbf{r}_\sigma) = -\int_{s_\sigma} d\mathbf{r}'_\sigma G_{\mathbf{g}_\pi}(\mathbf{r}_\sigma, \mathbf{r}'_\sigma) V_{\mathbf{g}_\pi}(\mathbf{r}'_\sigma)\Psi_0(\mathbf{r}'_\sigma),$$

where the domain of integration s_σ coincides with the "unit cell" for $V(\mathbf{r})$ periodic in $\mathbf{r}_\sigma$ or with the whole domain of definition on $\mathbf{r}_\sigma$ for a non-periodic potential.

Now we substitute the obtained expression for $\Psi_{\mathbf{g}_\pi}(\mathbf{r}_\sigma)$ in the right-hand side of eq. (2a) and come to the conclusion that the problem is reduced to solution of the *modified Schrödinger equation* for $\Psi_0(\mathbf{r}_\sigma)$:

$$[h_0(\mathbf{r}_\sigma) + V_{\text{eff}}(\mathbf{r}_\sigma)]\Psi_0(\mathbf{r}_\sigma) = \varepsilon_\sigma \Psi_0(\mathbf{r}_\sigma), \qquad (5)$$

where the ordinary potential $V_0(\mathbf{r}_\sigma)$ is replaced by the *effective potential* (EP) $V_{\text{eff}}(\mathbf{r}_\sigma)$, also known as coherent, or optical, potential [16]. It is obtained as a sum,

$$V_{\text{eff}}(\mathbf{r}_\sigma) = V_0(\mathbf{r}_\sigma) + \vartheta(\mathbf{r}_\sigma), \tag{6a}$$

where $\vartheta(\mathbf{r}_\sigma)$ is the nonlocal part of EP expressed by an integral operator :

$$\vartheta(\mathbf{r}_\sigma) f(\mathbf{r}_\sigma) \equiv \int_{s_\sigma} d\mathbf{r}'_\sigma V(\mathbf{r}_\sigma, r'_\sigma) f(\mathbf{r}'_\sigma) \tag{6b}$$

The integral kernel $V(\mathbf{r}_\sigma, \mathbf{r}'_\sigma)$ can be presented in the form

$$V(\mathbf{r}_\sigma, \mathbf{r}'_\sigma) = \sum_{\mathbf{g}_\pi, n} \frac{V_{-\mathbf{g}_\pi}(\mathbf{r}_\sigma) V_{\mathbf{g}_\pi}(\mathbf{r}'_\sigma) \Psi^0_{\mathbf{g}_\pi n}(\mathbf{r}_\sigma) \Psi^{0*}_{\mathbf{g}_\pi n}(\mathbf{r}'_\sigma)}{\varepsilon_\sigma - \varepsilon^0_{\mathbf{g}_\pi n} - \varepsilon_{\pi \mathbf{g}_\pi} - 2\cos\alpha_{\mathbf{g}_\pi} \sqrt{(E - \varepsilon_\sigma)\varepsilon_{\pi \mathbf{g}_\pi}}}, \tag{6c}$$

where

$$\varepsilon_{\pi \mathbf{g}_\pi} \equiv \hbar^2 g_\pi^2 / 2m,$$

and $\alpha_{\mathbf{g}_\pi}$ is the angle between the vectors $\mathbf{g}_\pi$ and $\mathbf{K}_\pi$ (for one-dimensional $\mathbf{r}_\pi, \alpha_{\mathbf{g}_\pi}$ takes only two values, $\alpha_{\mathbf{g}_\pi} = 0, \pi$).

To obtain the complete solution to a problem, one should find the solutions of the modified Schrödinger equation, eq. (5), and then substitute them into the expression for $\Psi_{\mathbf{g}_\pi}(\mathbf{r}_\sigma)$, after which the general solution can be written as

$$\begin{aligned} \Psi(\mathbf{r}) &= \sum_n c_n [\Psi_{0n}(\mathbf{r}_\sigma) + \sum_{\mathbf{g}_\pi} \Psi_{\mathbf{g}_\pi n}(\mathbf{r}_\sigma) \exp(i \mathbf{g}_\pi \mathbf{r}_\pi)] \exp(i \mathbf{K}_{\pi n} \mathbf{r}_\pi) = \\ &= \sum_n c_n \exp(i \mathbf{K}_{\pi n} \mathbf{r}_\pi) [1 + \sum_{\mathbf{g}_\pi} \exp(i \mathbf{g}_\pi \mathbf{r}_\pi) \xi_{\mathbf{g}_\pi n}(\mathbf{r}_\sigma)] \Psi_{0n}(\mathbf{r}_\sigma), \end{aligned} \tag{7}$$

where

$$\Psi_{\mathbf{g}_\pi n \mathbf{r}_\sigma} = \xi_{\mathbf{g}_\pi n \mathbf{r}_\sigma} \Psi_{0n}(\mathbf{r}_\sigma) \equiv \int_{s\sigma} d\mathbf{r}_{\sigma'} x_{\mathbf{g}_\pi n}(\mathbf{r}_\sigma, \mathbf{r}'_\sigma) \Psi_0(\mathbf{r}'_\sigma), \tag{8}$$

$$x_{\mathbf{g}_\pi n}(\mathbf{r}_\sigma, \mathbf{r}'_\sigma) = \mathbf{V}_{g\pi}(\mathbf{r}_{\sigma'}) \sum_{n'} \frac{\Psi^0_{\mathbf{g}_\pi n'}(\mathbf{r}_\sigma) \Psi^{0*}_{\mathbf{g}_\pi n'}(\mathbf{r}'_\sigma)}{\varepsilon_{\sigma n} - \varepsilon^0_{\mathbf{g}_\pi n'} - \varepsilon_{\pi g_\pi} - 2\cos\alpha_{\mathbf{g}_\pi} \sqrt{(E - \varepsilon_{\sigma n})\varepsilon_{\pi g_\pi}}}$$

the coefficients c_n are to be determined from the boudary conditions, as well as certain components of $\mathbf{K}$; $\varepsilon_{\mathbf{g}_\pi n}$ and $\mathbf{K}_{\pi n}$ are specified by eqs. (3) with $\varepsilon_\sigma = \varepsilon_{\sigma n}$; and $\{\Psi_{0n}\}, \{\varepsilon_{\sigma n}\}$ are the complete sets of eigenfunctions and eigenvalues for the modified Schrödinger equation, eq. (5). And finally, a measured quantity, the probability density distribution (PDD), $\rho(\mathbf{r}) \equiv \mid \Psi(\mathbf{r}) \mid^2$, can be expressed directly from eqs. (7), (8). The resulting general formula for it can be found in ref. [17] (see eq. (18)).

Before analysing the results obtained, it would be not out of place to note that another practically important case, that of time-dependent periodical perturbation, $H_p = V_p(\mathbf{r}_\sigma, t)$, is effectively described by the same systeme of equations (2), where one should make the substitutions

$$\mathbf{r}_\pi \to t, \mathbf{g}_\pi \to k \quad (k \neq 0\text{is an integer}), \varepsilon_{\sigma\mathbf{g}_\pi} \to \varepsilon_{\sigma k} \equiv \varepsilon_\sigma - \hbar\omega_\pi k,$$

the wave function being presented in the form

$$\Psi(\mathbf{r}_\sigma, t) = \exp(-i\varepsilon_\sigma t/\hbar)[\Psi_0(\mathbf{r}_\sigma) + \sum_k \Psi_k \mathbf{r}_\sigma \exp(i\omega_\pi k t)],$$

and ω_π being the base frequency of the perturbation :

$$V_p(\mathbf{r}_\sigma, t) = V_0(\mathbf{r}_\sigma) + \sum_k V_k(\mathbf{r}_\sigma) \exp(i\omega_\pi k t).$$

This conclusion can be verified starting from the substitution of the total wave function above into the time-dependent Schrödinger equation. Then it is easily seen that this problem can also be reduced to solution of the modified Schrödinger equation (5), where the kernel of the effective potential can be expressed in a slightly different form,

$$V(\mathbf{r}_\sigma, \mathbf{r}'_\sigma) = \sum_{k,n} \frac{V_{-k}\mathbf{r}_\sigma V_k \mathbf{r}'_\sigma \Psi^0_{kn}\mathbf{r}_\sigma \Psi^{0*}_{kn}\mathbf{r}'_\sigma}{\varepsilon_\sigma - \varepsilon^0_{kn} - \hbar\omega_\pi k}, \tag{6d}$$

and $\{\Psi^0_{kn}(\mathbf{r}_\sigma)\}, \{\varepsilon^0_{kn}\}$ are determined from the auxiliary system,

$$h_0(\mathbf{r}_\sigma)\Psi_k(\mathbf{r}_\sigma) + \sum_{k'} V_{k-k'}(\mathbf{r}_\sigma)\Psi_{k'}(\mathbf{r}_\sigma) = \varepsilon_{\sigma k}\Psi_k(\mathbf{r}_\sigma), k, k' \neq 0. \tag{4b}$$

Starting from these formulas one obtains the general solution and any desired measurable quantity, similar to the time-independent formalism (see [18, 19]).

It is clear that the time-dependent case is a generalisation of the well-known group of the model "kicked" systems with similar behaviour described by the basic standard map [23] (called also the standard model, see e. g. [2, 3]). This model corresponds to the δ-like periodic kicks (i. e. the components $V_k(\mathbf{r}_\sigma) = V_0(\mathbf{r}_\sigma)$ do not depend on k) and also to some particular choices of the potential dependence on $\mathbf{r}_\sigma \equiv x$ (typically $V_0(x), V_k(x) \propto -\cos(x)$). The proposed generalisation can serve thus as a more realistic representation for many particular physical systems studied with the standard model. By analogy one may designate our time-dependent case as the generalised kicked oscillator. Because of its similarity to the time-independent case we now continue the analysis of the latter mentioning the differences between the two where necessary.

2.2. Fundamental dynamic multivaluedness

If is not surprising to see, from the above expressions, that the EP method thus formulated cannot provide directly the exact solutions and is nothing but another formulation of a problem. However, we can show now that for the chaotic systems it is this representation that is much more relevant than the ordinary one. It permits one to obtain, in a self-consistent manner, a basic source of randomness and complexity in such systems and then to study, in relation to these fundamental concepts, their particular chaotic properties in terms of observable quantities.

We see from the above formulas that the experimentally measured PDD is determined by the dynamics of motion in EP. The distinctive property of the latter, and the most important one as far as the dynamical chaos is concerned, is its self-consistent dependence on the energy eigenvalues to be determined. This dependence appears in the explicit form when one tries to find the energy eigenvalues $\{\varepsilon_{\sigma n}\}$ from the modified Schrödinger equation, eq. (5), while using the expressions (6c,d) for the integral kernel of $V_{\text{eff}}(\mathbf{r}_\sigma)$ which depends itself on ε_σ. Note that this property is restricted solely to the full non-perturbative EP formalism studied here as opposed to its various perturbative versions eventually used in many application [16] including the problem of quantum chaos [24, 25]. In fact, we deal here with the intrinsic effective nonlinearity of a chaotic system which is not taken into account either by the ordinary, non-modified formalism, or by the perturbative approaches (further discussion of the effective nonlinearity can be found in refs. [18, 20]). Now we are going to show that this peculiar property leads to the conclusion

that, instead of N_σ eigenvalues and eigenfunctions for the Schrödinger equation with the "non-chaotic" potential $V_0\mathbf{r}_\sigma$, one obtains up to

$$N_{\max} = (N_\pi N'_\pi + 1)N_\sigma \tag{9}$$

solutions for the equation with EP, where N_π and N'_π are the numbers of terms in the sums over $\mathbf{g}_\pi$ and n, respectively, in eq. [6c]. Among these $n_{\max}$ solutions, $N_0 = (N'_\pi + 1)N_\sigma$ solutions correspond to the normal set of eigenfunctions for the full-dimensional Schrödinger equation with the potential $V(\mathbf{r})$. The *additional* growth of the number of solutions by

$$N_\Delta \equiv N_{\max} - N_0 = N'_\pi(N_\pi - 1)N_\sigma \geq N_\sigma$$

cannot be explained in terms of ordinary splitting effects. These extra solutions could be, in principle, spurious, unphysical ones (e. g. unstable). This can indeed happen for *some* ranges of parameters, and this is precisely one of the mechanisms of regularity in chaotic quantum systems, studied below. However, it is difficult to imagine, and we confirm these doubts below, that this may be the case for all parameter values. We shall see that most often at least some of these additional solutions are quite real and observable.

Note that it will be more convenient to count solutions in terms of N_σ dividing all the quantities like $N_{\max}, N_0$ and N_Δ above by N_σ. We shall use the same notation for these reduced numbers of solutions. The statements above, concerning the number of solutions, can be verified by at least three different ways giving all the same result, eq. (9) : one can count solutions directly by analysing eq. (5) in the momentum or other suitable representation ; one can study a problem by a graphical method ; at last, reasonable and easily treated approximations of eqs. (5)-(6) can be proposed. Here we restrict ourselves to the most convincing and transparent graphical analysis confirmed by the results of the two other approaches (their description can be found in [17-19].

We first rewrite eq. (5) for certain n-th eigenvalue :

$$[h_0\mathbf{r}_\sigma + V_0\mathbf{r}_\sigma]\Psi_{0n}\mathbf{r}_\sigma + \int_{s_\sigma} d\mathbf{r}'_\sigma V_n(\mathbf{r}_\sigma, \mathbf{r}'_\sigma)\Psi_{0n}\mathbf{r}'_\sigma = \varepsilon_{\sigma n}\Psi_{0n}\mathbf{r}_\sigma.$$

Multiplying it by $\Psi^*_{0n}\mathbf{r}_\sigma$ and integrating over $\mathbf{r}_\sigma$ we arrive at the following formulation of a problem

$$V_{nn}(\varepsilon_{\sigma n}) = \varepsilon_{\sigma n} - \varepsilon^0_{\sigma n}, \tag{10}$$

where

$$V_{nn}(\varepsilon_{\sigma n}) \equiv \sum_{\mathbf{g}_\pi, n'} \frac{\mid V^{nn'}_{\mathbf{g}_\pi} \mid^2}{\varepsilon_{\sigma n} - \varepsilon^0_{\mathbf{g}_\pi n'} - \varepsilon_{\pi g_\pi} - 2\cos\alpha_{\mathbf{g}_\pi}\sqrt{(E-\varepsilon_{\sigma n})\varepsilon_{\pi g_\pi}}}, \tag{11a}$$

$$V^{nn'}_{\mathbf{g}_\pi} \equiv \int_{s_\sigma} d\mathbf{r}_\sigma \Psi^{0*}_{\mathbf{g}_\pi n'}(\mathbf{r}_\sigma) V_{\mathbf{g}_\pi}(\mathbf{r}_\sigma) \Psi_{0n}(\mathbf{r}_\sigma),$$

and

$$\varepsilon^0_{\sigma n} \equiv \int_{s_\sigma} d\mathbf{r}_\sigma \Psi^*_{0n}(\mathbf{r}_\sigma)[h_0(\mathbf{r}_\sigma) + V_0(\mathbf{r}_\sigma)]\Psi_{0n}(\mathbf{r}_\sigma)$$

Function $V_{nn}(\varepsilon_{\sigma n})$ possesses a number of singularities determined by the zeros of the denominators of each term of the double sum in eq. (11a). The zero, and thus the corresponding singularity position, for a term with certain n' and $\mathbf{g}_\pi$ is easily found to be at $\varepsilon_{\sigma n} = \varepsilon^{\pm}_{\sigma n'}(\mathbf{g}_\pi)$, where

$$\varepsilon^{\pm}_{\sigma n'}(\mathbf{g}_\pi) = \varepsilon_{\mathbf{g}_\pi n'} + \varepsilon_{\pi \mathbf{g}_\pi}(1 - 2\cos^2\alpha_{\mathbf{g}_\pi}) +$$
$$+2\cos\alpha_{\mathbf{g}_\pi}\sqrt{\varepsilon_{\pi\mathbf{g}_\pi}(E - \varepsilon_{\pi\mathbf{g}_\pi}\sin^2\alpha_{\mathbf{g}_\pi} - \varepsilon_{\mathbf{g}_\pi n'})} \tag{12a}$$

The superscript "±" in the notations above serves to remind us about a feature important for the following analysis : for each $\mathbf{g}_\pi$ there are two terms in the sum (11a) with the opposite directions of the vector $\mathbf{g}_\pi$ corresponding to change of sign of $\cos\alpha_{\mathbf{g}_\pi}$ in eq. (12a). This can be expressed in a straightforward fashion for the simple and rather common case of one-dimensional perturbation ($\mathbf{r}_\pi \equiv z$) :

$$\varepsilon^{\pm}_{\sigma n'}(\pm \mid \mathbf{g}_\pi \mid) = \varepsilon_{\mathbf{g}_\pi n'} - \varepsilon_{\pi g_\pi} \pm 2\sqrt{\varepsilon_{\pi g_\pi}(E - \varepsilon_{\mathbf{g}_\pi n'})}.$$

For the case of time-dependent perturbation (the generalised kicked oscillator) one obtains, using the substitution procedure described above, the same eq. (10) with

$$V_{nn}(\varepsilon_{\sigma n}) \equiv \sum_{k,n'} \frac{\mid V^{nn'}_k \mid^2}{\varepsilon_{\sigma n} - \varepsilon^0_{kn'} - \hbar\omega_\pi k}, \tag{11b}$$

where $V^{nn'}_k \equiv V^{nn'}_{\mathbf{g}_\pi} \mid_{\mathbf{g}_\pi = k}$ and the singularities are situated at

$$\varepsilon^{\pm}_{\sigma n'}(k) = \varepsilon_{kn'} + \hbar\omega_\pi k = \varepsilon_{kn'} \pm \hbar\omega_\pi \mid k \mid . \tag{12b}$$

Now to apply our graphical analysis we plot, in Fig 1, the left- and right-hand sides of eq. (10) vs $\varepsilon_{\sigma n}$ taken as continuous independent variable. Solutions are found then as abscissae of the points of intersection of the curves corresponding to the two functions plotted.

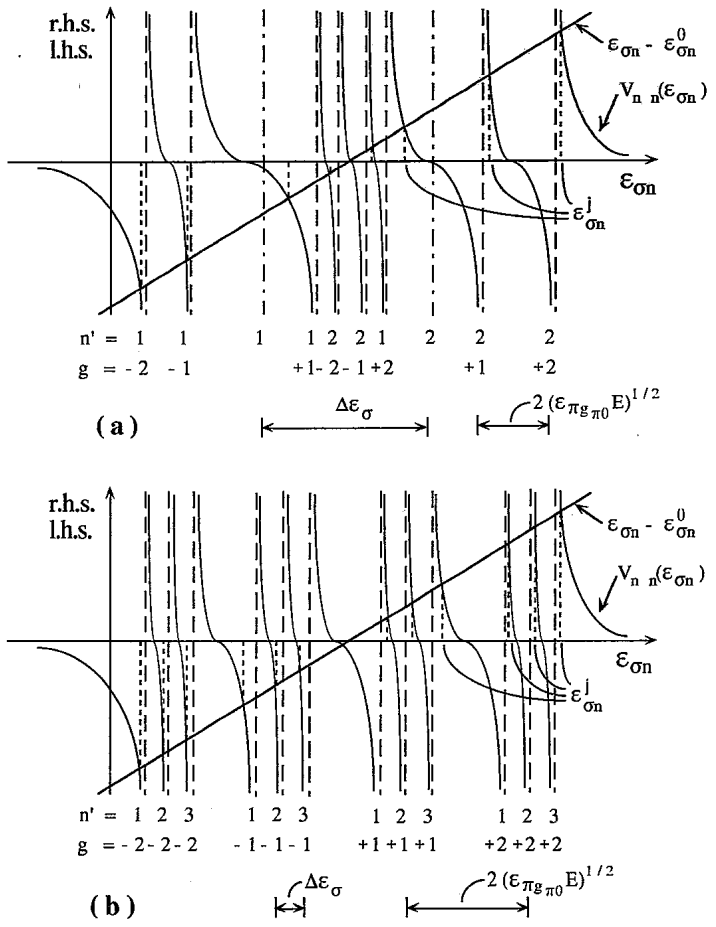

Figure 1. Graphical solution of the modified Schrödinger equation (in the reduced form, eq. (10)) for the Hamiltonian system with periodic perturbation in the parameter domains of global chaos **(a)**, and global regularity **(b)**. We plot the left- and the right-hand sides of eq. (10) vs the eigenvalue to be determined, $\varepsilon_{\sigma n}$, taken as independent variable. The illustration presented

corresponds to the case $E \gg \varepsilon_{\sigma n}$ (see text) and the following numbers of terms in the sums over $\mathbf{g}_\pi$ and n' in the expression for $V_{nn}(\varepsilon_{\sigma n})$, eq. (11a) : $N_\pi = 4, N'_\pi = 2$ (a), and $N_\pi = 4, N'_\pi = 3$ (b). The asymptotes of the function $V_{nn}(\varepsilon_{\sigma n})$ are shown by the dashed lines with the respective values of g and n' marked at the lower end of each of them, where a non-zero integer g enumerates g_π: $g_\pi = 2\pi g/d_\pi (g_{\pi 0} = 2\pi/d_\pi)$. Two vertical dash-dotted lines in (a) correspond to the values $\varepsilon_{\sigma n} = \varepsilon^0_{\mathbf{g}_\pi n'}$.

As is seen from the figure, representing eq. (10) for two characteristic cases of parameter values (see section 3), function $V_{nn}(\varepsilon_{\sigma n})$ consists of many branches due to the sums over n' and $\mathbf{g}_\pi$ of divergent terms. It is this plurality of branches which gives multiple solutions for $\varepsilon_{\sigma n}$ (we enumerate them by another index, $j : \varepsilon^j_{\sigma n}, j > 0$) instead of the single one, $\varepsilon_{\sigma n} = \varepsilon^0_{\sigma n}$, for the Schrödinger equation with the "regular" potential $V_0 \mathbf{r}_\sigma$. One part of this splitting can be explained by the ordinary multiplication of the number of eigenvalues and eigenfunctions due to the addition of the degrees of freedom corresponding to $\mathbf{r}_\pi$ to those of $\mathbf{r}_\sigma$. However, there is also another part of the splitting which just represents the basic dynamical effect introduced above. Thus for the case of Fig. 1(a), where we have restricted ourselves only to two terms of each of the sums over n' and $\pm \mid \mathbf{g}_\pi \mid$ (i. e. $N_\pi = 4, N'_\pi = 2$), in the absence of this additional splitting one could not obtain more than $N_0 = N'_\pi + 1 = 3$ solutions, whereas actually one can count $N = N_{\max} = 9$ solutions for $\varepsilon_{\sigma n}$ in full agreement with the general expression, eq. (9). To deduce eq. (9) from our graphical analysis, we just note that, as follows from eqs. (11), (12), the number of asymptotes is equal to $N_\pi N'_\pi$, and then, as is clear from the figure, the number of points of intersection of the two curves, and thus of solutions, is $N_{\max} = N_\pi N'_\pi + 1$. We shall see in the next section why and how this *maximum* number of solutions can diminish down to N_0 giving rise to qualitative changes in the chaotic behaviour of a system. Figure 1(b) reproduces these results for another characteristic parameter values (we continue this analysis in section 3). Note that the constant $\varepsilon^0_{\sigma n}$ adjusts itself, in a self-consistent manner, to its particular value for each realisation.[3]

In what follows we call this *additional* splitting of solutions and respectively of the effective Hamiltonian, potential or other relevant dy-

[3] As the value of $\varepsilon^0_{\sigma n}$ may vary for different solutions $\varepsilon^j_{\sigma n}$, the segments of the line $\varepsilon_{\sigma n} - \varepsilon^0_{\sigma n'}$ intersecting the respective branches of the function $V_{nn}(\varepsilon_{\sigma n})$ in Fig. 1, can be slightly displaced vertically one relative to another. However, this will not produce any significant changes on the scale of our schematical representation, neither in the conclusions obtained, and we do not show these details to avoid unnecessary complication.

namical function, the fundamental multivaluedness of dynamical functions (FMDF). It includes the ordinary dimensional splitting into the *complete* set of N_0 solutions as *one* component, but also, possibly, other similar complete components. We shall call the i-th component the i-th *realisation* of a dynamical system (or of a problem) and designate it by $\mathcal{R}_i$; it effectively includes, besides the corresponding complete set of solutions, also the corresponding components of EP and PDD :

$$\mathcal{R}_i \equiv \{\{\varepsilon_{\sigma n}\}^i, \{\Psi_{0n}(\mathbf{r}_\sigma)\}^i, V^i_{\rm eff}(\mathbf{r}_\sigma), \rho_i(\mathbf{r})\} \quad (i = 1, 2, ..., N_{\mathcal{R}}), \qquad (13)$$

$N_{\mathcal{R}} \geq 1$ being the number of realisations in their *ensemble* thus obtained.

It would be useful to introduce the notion of distance between two realisations. For our purposes it is quite sufficient to define it approximately as the average, or typical, magnitude of the difference between the values of the corresponding branches of EP. Then it follows from the above analysis that the separation between realisations (i. e. the distance between the neighbouring ones) is finite and varies between the energy-level separation for the unperturbed potential, $\Delta\varepsilon_\sigma$, at maximum (in the situation of global regularity, Fig. 1(b), see section 3), and the minimum of $2(\varepsilon_{\pi g_{\pi 0}} E)^{1/2} = 2\pi\hbar\sqrt{2E/m}/d_\pi$ (in the situation of global chaos, Fig 1(a)). We see that the finite separation between the realisations, proportional to $\hbar$, (i. e. their *discreteness*) is a quantum effect. As will be shown in the next section, it has the important physical consequences. It means also that in the semiclassical situation one has, at the scale of the characteristic potential values, many closely separated realisations forming a quasi-continuous distribution. Of course, these generic rules do not exclude the existence of particular realisation separations of greater (or smaller) magnitudes.

2.3. Quantum chaos as dynamic multivaluedness

It was shown above that, being reformulated in the form of the modified Schrödinger equation, eq. (5), a problem is naturally splitted into multiple realisations, at least for some parameter ranges. Now we are going to demonstrate that this splitting results in what can be interpreted as dynamical chaos :

it has the same qualitative manifestations, including especially the intrinsic dynamical randomness, for the systems with chaotic classical counterparts ;

it reproduces, within the quantum-mechanical description, the classically well-established transition chaos-regularity and its point in the

parameter space, as well as the phenomenon of asymptotically weak chaos in the domain of global regularity ;

it provides the normal semiclassical transition for chaotic systems in full agreement with the conventional correspondence principle ;

it naturally reveals the fractal structure of quantum chaos,[4] also in agreement with the existing general and particular results ;

and finaly, it gives also several forms of the "quantum suppression of chaos" which is generally not absolute, however, depends on the parameters and is compatible with the existence of the "true" dynamical chaos also in essentially quantum systems far from the semiclassical transition.

So we suppose that we are in a range of parameters giving multiple realisations $\mathcal{R}_i$ $(i = 1, 2, ..., N_{\mathcal{R}} > 1)$ defined by eq. (13). Since all the realisations have a priori "equal rights", and, from the other hand, one can always observe only one of them (because of their completeness), then each of them will actually appear in a *causally random* fashion, i. e. their unpredictability has clearly causal, dynamic origin in the discovered fundamental multivaluedness. Correspondingly, the full experimentally measured PDD, $\rho_{ex}(\mathbf{r})$, should be a result of the *probabilistic* superposition of the PDD's, $\rho_i(\mathbf{r})$, for the individual realisations :

$$\rho_{ex}(r) = \sum_{i=1}^{N_{\mathcal{R}}}{}^{\oplus} \rho_i(r), \tag{14a}$$

where $\sum^{\oplus}$ is the probabilistic sum of independent functions (random processes).[5] Then the expectation value of $\rho_{ex}(r)$, averaged over a large enough ensemble of identical repetitions of the same experiment, is obtained as (we use the same symbol for it)

$$\rho_{ex}(r) = \sum_{i=1}^{N_{\mathcal{R}}} \alpha_i \rho_i(r) \quad , \quad \sum_{i=1}^{N_{\mathcal{R}}} \alpha_i = 1, \tag{14b}$$

[4] Here we just mention this important feature referring to [18, 19] for the corresponding detailed analysis. It is important that fractal structure of quantum chaos follows from the same modified Schrödinger formalism and is obtained, in principle, by analytical means.

[5] It is easily seen that this natural "appearance" of probability and randomness in our approach provides a fundamental extension of the corresponding notions themselves (see [18, 19] for a more detailed discussion).

where the i-th realisation probability, α_i, does not, a priori, depend on i:$\alpha_i = 1/N_{\mathcal{R}}$. However, in many real situations (mostly within the semi-classical case, as is shown above) the realisations are closely separated forming a kind of quasi-continuous distribution with varying density so that individual realisations cannot be resolved experimentally. It is thus reasonable, in a general case, to designate by index i the number of a discernible realisation group with a suitable physical size, and then α_i can show a pronounced dependence on i:

$$\alpha_i = N_i/N_{\mathcal{R}},$$

N_i being the number of realisations within the i-th group. It leads, eventually, to reduction of the sum in eq. (14a) to an integral, using the *density of realisations* analogous to the well-known density of states :

$$\rho_{ex}(\mathbf{r}) = \int_{\Omega_\iota} \delta(\iota)\rho_\iota(\mathbf{r})d\iota \quad , \quad \int_{\Omega_\iota} \delta(\iota)d\iota = 1, \tag{14c}$$

where $\delta(\iota) \equiv d\alpha/d\iota$ is the density of realisations, ι can be any proper parameter characterising α, and the integrals above are taken over Ω_ι, the domain of ι variation of interest.

Note that in this way we introduce the new *concept, and the postulate, of the fundamental dynamic uncertainty* providing an explanation of origins of dynamical (in particular, quantum) chaos with the possibility of subsequent detailed description of the real chaotic system behaviour. The existence of such basic indeterminacy has been anticipated [11] as a necessary condition for the non-contradictory understanding of randomness in deterministic systems. It is important to emphasize, however, that in our approach this postulate is *naturally imposed* and specified by the discovered fundamental multivaluedness : eqs. (14) is the only reasonable issue permitting one to reconcile the existence of *many* equivalent realisations with the condition that one can observe *only one* of them. In fact, the axiom itself consists in the assumption that it is this modified form of the equation of motion (Schrödinger equation in our case), giving the multivaluedness, that is the right and more general one as compared to the ordinary form which is relevant only to the special case of regular motion. Moreover, we suppose that the set of realisations obtained from the modified Schrödinger equation is complete. This is expressed by the second equality in eq. (14b) and means that the modified formalism provides the complete description of whatever complex dynamic

behaviour (this statement can be generalised to dynamical systems of arbitrary origin [18-20]). To justify the dynamic uncertainty postulate, one should verify the correspondence between the consequences of the assumptions made and the existing experimental knowledge on chaos as well as their consistence with other fundamental principles of quantum mechanics.

To begin with, one may try to understand in more detail the physical meaning of eqs. (14). In the absence of perturbations other than V_p the system would "populate", with the propability from the set $\{\alpha_i\}$, one of the realisations (or groups of realisations) at the boundary or at the initial moment of time, and then it would remain within this realisation, theoretically, forever. In reality, however, as such additional perturbations of arbitrary origin (or noise) always exist, they will lead to "spontaneous transitions" between different realisations (the zero-noise case is discussed below).

It is clear that there way exist two limiting manifestations of the phenomenon of dynamical chaos thus defined. In the first case, a noisy perturbation is sufficiently strong or, in other words, the separation between realisations is small enough. In this case we obtain typical highly irregular behaviour with pronounced explicit manifestations of randomness in the form of fast irregularities of motion, diffusion in the set of effective occupied states, etc. This regime is realised, in particular, in the semiclassical limit, when the typical separation between realisations is going to zero (see the end of section 2.2). This situation tends qualitatively to the classical picture of dynamical chaos. The characteristic features of the latter, such as instability, divergence of trajectories, and phase-space diffusion, are reduced, in quantum-mechanical terms, to frequent random transitions between closely separated realisations. The classical picture is therefore well reproduced.

The second limiting situation corresponds to relatively large separation of realisations or, equivalently, to a relatively weak noisy perturbation, when the "spontaneous" transitions between realisations are rare, so that during the characteristic observation time they are hardly to occur. In this case we have a quasi-regular dynamics which experimentally may show practically the same features as the regular one including time reversibility, apparently total "quantum suppression of chaos", etc. We emphasize, however, that in reality, it is a manifestation of the same quantum chaos mechanism, which should show up, for example, in the occurrence of more than one realisation for an *ensemble* of identical systems or experimental repetitions for the same system. In practice it may

be difficult to separate the results of the repetitive measurements and the corresponding realisations from each other. Then the plurality of realisations may manifest itself as effective additional "noise" of dynamical origin leading to the respective peak broadenings, etc. It is important that the limiting case considered can be realised in essentially quantum conditions, far from the semiclassical limit. This shows that dynamical chaos can well be compatible with quantum mechanics, although in many essentially quantum situations it may manifest itself "less irregularly", in the particular sense specified above. It is this specific mechanism of the *relative* diminishing of *visible* manifestations of the true dynamical randomness which gives *partial* quantum suppression of chaos in our approach.

3. Quantum chaos in Hamiltonian systems with periodic perturbation

In order to further specify the features of quantum chaos within the proposed formalism of FMDF, it is convenient to analyse different types of system behaviour depending on parameters. As follows from the main postulate, the global character of the system dynamics is determined by the number of distinct realisations for that system. To demonstrate the existing possibilities, we use the example of periodically perturbed system considered above and return to its graphical analysis, presented in Fig. 1 and based on eqs. (10)-(12) (we start with the case of time-independent perturbation).

The realisations are determined by the branches of the function $V_{nn}(\varepsilon_{\sigma n})$ confined by a series of vertical asymptotes. As it follows from eq. (12a), the positions of the latter on the horizontal axis are determined by two characteristic energy intervals, $\varepsilon_1 = \Delta\varepsilon_\sigma$ and $\varepsilon_2 = 2\sqrt{\varepsilon_{\pi g_{\pi 0}}(E - \varepsilon^*)}$, where $\varepsilon^* = \varepsilon_{\pi \mathbf{g}_\pi} \sin^2 \alpha_{\mathbf{g}_\pi} + \varepsilon_{\mathbf{g}_\pi n'}$. The generic chaotic behaviour, exemplified by Fig. 1(a), correponds to $\varepsilon_2 \leq \varepsilon_1$. It is characterised by the maximum number of realisations $N_{\mathcal{R}} = N_{\mathcal{R}}^{\max}$. Moreover, the splitting of eigenvalues has a complex, irregular character : the "normal" splitting due to the addition of degrees of freedom (the sum over n' in eqs. (11)) is superimposed, in an entangled fashion, on the excessive, "chaotic" one representing the fundamental multivaluedness (the sum over $\mathbf{g}_\pi$ in eqs. (11)).

If now the parameters change so that ε_2 increases, then at $\varepsilon_2 > \varepsilon_1$ one obtains another characteristic situation, illustrated by Fig. 1(b) (for convenience and without any essential change, now we take into

account three terms in the sum over n' in eqs. (11), $N'_\pi = 3$). One can see that, whereas the total number of solutions is determined, formally, by the same rules as in the previous case (see eq. (9)) and should remain unchanged for the same N_π, N'_π, they are now subdivided into the dense groups of "normal" solutions corresponding to ordinary dimensional splitting, these groups being separated by larger intervals as far as the fundamental dynamic multivaluedness is concerned. These groups are similar to each other, and the higher is the ratio $\varepsilon_2/\varepsilon_1$, the more they are indistinguishable. Thus the two splittings, "normal" and "chaotical", are now well separated and not intermixed. And what is most important, now it is difficult to distinguish formally different, but practically very similar, realisations one from another. All these arguments permit us to identifiy the point $\varepsilon_2 = \varepsilon_1$ as the border between global chaos and regularity in the parameter space, the global chaos appearing at $\varepsilon_2 \leq \varepsilon_1$. This corresponds qualitatively to the predictions of classical description of chaos concerning the existence of the transition chaos-regularity in periodically perturbed systems [2, 3, 23]. In order to make a more detailed comparaison, we express this condition of global chaos onset through the parameters of a problem :

$$\Delta\varepsilon_\sigma \geq 2\sqrt{\varepsilon_{\pi g_{\pi 0}}(E - \varepsilon^*)} \tag{15a}$$

or

$$E \leq (\Delta\varepsilon_\sigma)^2/(4\varepsilon_{\pi g_{\pi 0}}) + \varepsilon^* \equiv E_c \approx (\Delta\varepsilon_\sigma)^2 d_\pi^2 m/(8\pi^2\hbar^2), \tag{15b}$$

where d_π is the perturbation period ($g_{\pi 0} = 2\pi/d_\pi$), and the last equality is valid if $\Delta\varepsilon_\sigma \gg \varepsilon_{\pi g_{\pi 0}}$. In the semiclassical situation when $\Delta\varepsilon_\sigma \gg \varepsilon_{\pi g_{\pi 0}}$ and $\Delta\varepsilon_\sigma = \hbar\omega_\sigma$ (ω_σ is the classical oscillation frequency for the unperturbed system), one obtains

$$E_c = \omega_\sigma^2 d_\pi^2 m/(8\pi^2). \tag{16}$$

We see that this expression for the border 'global chaos' - 'global regularity', obtained by the ordinary semiclassical limit from our purely quantum-mechanical analysis, contains only classical parameters. We shall call condition $E = E_c$ the *classical border of chaos*, although in its general form, eqs. (15), it can determine the onset of global chaos in an essentially quantum situation. Now to compare it to the equivalent expression obtained within the classical analysis of the standard model

(kicked oscillator), one should first pass to the conventional parameter **K** [2,3,23] (for the details of the relation between the periodically perturbed system of the type considered here and the standard model see e. g. [21], section 4.3.). For the case of small harmonic oscillations in the unperturbed potential one easily obstains

$$K = m\omega_\sigma^2 d_\pi^2/(2E) \tag{17}$$

(see [17], section 2.5, for details). Then the semiclassical limit of our condition for the global chaos onset, eqs. (15), (16), takes the form $K > K_c = 4\pi^2$. The classical estimations and computer calculations for the standard model and the related system give $K_c \approx 1$ [2, 3, 23]. Thus our quantum-mechanical estimations, based on the formalism of FMDF, give an apparent discrepancy with the corresponding classical results only in numerical coefficient.

This difference can be easily understood and effectively eliminated if one takes into account the nonlinear character of the unperturbed oscillations. The former means that the linear oscillation frequency in the above expression for K should be, in fact, replaced by certain effective frequency, $\omega_\sigma \to \omega_\sigma/\ell$, where the constant ℓ depends on the form of the particular potential well and is of the order of several units (it characterises the frequency spectrum width of the nonlinear oscillator, i. e. its anharmonicity). Then eq. (17) turns into

$$K > K_c = (2\pi/\ell)^2, \tag{18}$$

which coincides with the classical result at a reasonable ℓ value, $\ell = 2\pi$.

The classical border of chaos can be related to another interpretation revealing more physics. It is easily seen (cf. eqs. (3)) that the quantity $2\sqrt{\varepsilon_{\pi g_{\pi 0}}(E - \varepsilon^*)}$ represents the main portion of the discrete energy transfer between the "perturbation" degrees of freedom, $\mathbf{r}_\pi$, and those of the unperturbed problem, $\mathbf{r}_\sigma$. Then if the effective (in fact, lower) energy-level separation, $\Delta\varepsilon_\sigma$, is at resonance with this energy transfer (at $E = E_c$), it means that the chaos criterion, eqs. (15), is fulfilled for the majority of bound states and therefore it acquires the global character. This interpretation provides also a general explanation of the correspondence with the classical results. The mentioned resonance conditions turn to zero the EP denominators (see eqs. (6), (11)) giving additional problem realisations. In the semiclassical situation these resonant denominators coincide (up to the factor $\hbar$) with the

analogous classical resonance conditions known to be directly involved in classical chaos [2-4]. This automatically ensures the equivalence of the ensuing conclusions including, for example, the resonance overlapping criterion [2-4]. Yet in the general case our expressions retain their purely quantum origin.

It is important to mention another point of agreement between the results of our approach and those obtained within classical mechanics which is due, eventually, to the same intrinsic quantum-classical similarity within the EP formalism. It concerns the relation between chaos and regularity outside the classical border. It is clear from our description that within the domain of regularity chaos disappears asymptotically with the distance to the classical border. It means that some chaos always exists (the realisations are never exactly identical and the distance between them is always finite) and where it exists it is strong, but the relative number of such situations diminishes as one goes farther from the border. This is in good agreement with the well-known classical results [2,3,26]. The more detailed analysis of the expressions for EP, which will be described in the next paper (see also [18, 19]), shows that this agreement can be extended to the conclusion that "chaotically rich" features within the global regularity pass to the "stochastic layer" and "stochastic web" in the semiclassical limit, with the agreement between classically and quantum-mechanically obtained expressions for the width of the layer.

Consider now another generic possibility concerning the transition chaos-regularity. To discover it note that while analysing above eq. (12a) for the asymptote positions, we have neglected, in fact, the possibility that $E < \varepsilon^*$ under the root. Now if it is realised, the number of asymptotes is no more constant. When the parameter E diminishes from high values and becomes less than one of the discrete values of $\varepsilon^* = \varepsilon_{\pi g_\pi} \sin^2 \alpha_{\mathbf{g}_\pi} + \varepsilon_{\mathbf{g}_\pi n'}$, the corresponding pairs of asymptotes "close" leading to disappearance of the respective branches of the function $V_{nn}(\varepsilon_{\sigma n})$ and thus of a realisation of a problem. Finally, if E becomes so small that

$$E < E_q \quad , \quad E_q = \varepsilon^0_{\mathbf{g}_\pi 1} \cong \varepsilon_{\sigma 0} \equiv \min(\varepsilon_{\sigma n}), \tag{19}$$

where $\varepsilon^0_{\mathbf{g}_\pi 1}$ is the first excited energy level in the set $\{\varepsilon^0_{\mathbf{g}_\pi n}\}$, then the number of points of intersection of the two curves attains its minimum value corresponding to the ordinary dimensional splitting. In this case

$N'_\pi = 1$, and, as follows from eq. (9), $N_{\max} = N_\pi + 1 = N'_\pi + 1 = N_0$.[6] It means that we have only one realisation of a problem, and there is no any chaos at all. We shall call E_q the *quantum border of chaos*, as opposed to the classical border of chaos, E_c, introduced above and coinciding, in the semiclassical limit, with the classically obtained quantity. Contrary to this, the existence of quantum border is a purely quantum-mechanical effect : formally $E_q \to 0$ when $\hbar \to 0$; as follows from eq. (19), it is of the order of the lower energy level for the unperturbed potential, $\varepsilon_{\sigma 0}$. It represents another, more particular, but still rather general, case of the quantum suppression of chaos. The first case was described above and corresponds to partial chaos suppression due to the finite realisation separation. In contrast to this, the suppression of chaos under condition (19) is complete, and it is also a purely quantum-mechanical effect due to the finite value of the lower energy state.

It is interesting to note, by the way, that it can explain the regularity of the elementary complex constituents of matter like nuclei and atoms. The existence of this property is not evident because each of the particles in such elementary agglomerate, e. g. an electron in atom, moves in a very complex effective potential of other particles which should typically give the pronounced chaotic behaviour (we have seen that it can well exist in quantum dynamics!). Moreover, even the asymptotic disappearance of chaos, like it happens beyond the classical border, would not help ; unexcited matter seems to be absolutely regular. The same concerns, of course, the partial quantum suppression of chaos described above. In contrast to this, condition (19) of the *complete* suppression of chaos agrees very well with this demand of absolute regularity, and the energy, for example, of atomic, or nuclear, ground stage can self-consistently be just below the quantum border.[7] Once being excited, an atomic electron leaves the domain of this total chaos suppression (it is true even apart from the chaotising effect ot the exciting perturbation itself) ; this highlights a role of chaos in the processes of excitation of nuclei, atoms and solids which seems not to be recognised. From the

[6] At these low energies event this minimal splitting can easily be suppressed, $N < N_0$, which corresponds to the existence of impenetrable barriers, forbidden energy zones, etc.

[7] Recall that our approach is applicable also for this case of chaos induced by symmetry breaking ; physically, it is the asymmetrical part of the potential that plays the role of the effective "periodic perturbation" with the period determined by the symmetric part.

other hand, this means that *any* state of a bound many-body system above the ground state is, in principle, chaotic.

Now we briefly summarise the corresponding results for the parameter dependence of chaos in the case of time-dependent perturbation, obtained in the same manner from eqs. (10), (11b), (12b) and the accompanying graphical analysis. In this case we have only classical border of chaos, i. e. global quantum chaos exists at

$$0 < \omega_\pi < \omega_{0c}, \tag{20}$$

where

$$\omega_{0c} = \Delta\varepsilon_\sigma/\hbar = \omega_\sigma,$$

ω_σ stands for the "effective" classical oscillation frequency for the unperturbed potential, and the last equality applies to the semiclassical limit. Condition (20) can also be expressed through the "parameter of chaoticity" K, in which form it coincides with eq. (18), where now

$$K = (2\pi/\ell)^2(\omega_\sigma/\omega_\pi)^2. \tag{21}$$

Finally, it would be important to note that the particular method described in this paper permits one to reveal many different quantum chaotic regimes in the behaviour of the periodically perturbed Hamiltonian systems. We refer, for their detailed description, to accounts [18, 19] and to the subsequent publications. Among the most significant results we mention quantum chaos in the quasi-free motion of a system above the periodic potential barriers and "chaotic tunneling" due to the dynamic variations of the effective potential barrier height.The direct practical applicability of the obtained results is well illustrated by consideration of the respective effects in the high-energy charged particle scattering in crystals [17, 18].

4. Quantum chaos and dynamic complexity

The essence of the above results can be reduced to the introduction of a modified form of the Schrödinger formalism, eqs. (4)-(8), deduced from the ordinary Schrödinger equation by simple algebraic transformations, but providing many equivalent sets of complete solutions. Their existence has been interpreted as the fundamental causal origin of probability and randomness in deterministic quantum systems, eqs. (14), compatible with the correspondence principle and most important known

qualitative and quantitative manifestations of chaos (section 3). This solution for the puzzling problem of quantum chaos provides a unique compromise between unavoidable random elements of truly chaotic behaviour and the well-established deterministic formalism of the conventional wave mechanics.

But what is especially important, the implication of the probabilistic set of realisations permits one to introduce the precise and basically simple measure of complexity, C, of a system as a function, $C(N_{\mathcal{R}})$, of the number of realisations, $N_{\mathcal{R}}$, such that

$$C = 0 \quad \text{if} \quad N_{\mathcal{R}} = 1 \quad , \quad C > 0 \quad \text{if} \quad N_{\mathcal{R}} > 1 \quad , \quad \text{and} \quad dC/dN_{\mathcal{R}} > 0; \tag{22a}$$

for example, one may put

$$C(N_{\mathcal{R}}) = f(N_{\mathcal{R}}) \ln(N_{\mathcal{R}}), \tag{22b}$$

with $f(N_{\mathcal{R}})$ specified by some other, more particular, considerations and playing a minor role (typically, it can be just a numerical coefficient). In this definition, complexity is deduced directly from the dynamics of the system by the well-defined procedure and possesses automatically all the necessary properties.[8] In particular, complexity defined by eqs. (22) is zero both for the absolutely regular and completely randomised dynamics (in the two cases evidently $N_{\mathcal{R}} = 1$), and thus one can naturally satisfy this elementary condition presenting typical difficulty for the definition of physical complexity [27].

The proposed definition of complexity is as much consistent and useful in practical applications for real complex system analysis. Thus, in a straightforward fashion can one obtain from eqs. (22) the dependence of C on any dynamical parameter for a system, like e. g. the notorious chaoticity parameter K for the periodically perturbed quantum Hamiltonian system : $C(K) = f(K) \ln[N_{\mathcal{R}}(K)]$, where some most essential features of the dependence $N_{\mathcal{R}}(K)$ have been studied above (section 3), and they can be further specified if necessary.

This natural appearance of the dynamic multivaluedness and complexity continues in the ensuing equally transparent definitions of (non)integrability and general solution for complex dynamical systems.

[8] The presence of the logarithmic function in eq. (22b) corresponds well to the hierarchical multiplicative breeding of realisations in more involved systems with several levels of dynamic complexity.

Indeed, it is easily seen that nonintegrability can be regarded as the same plurality of solutions of a problem that basically does not allow for existence of a single solution. In reverse, for a relatively rare case of a single realisation for a problem (zero complexity) we can be sure that the problem has a unique solution which can be found by the method of effective dynamical functions itself or some other, "ordinary" method. The notion of the general solution is related to the postulated completeness of the set of realisations found within the FMDF formalism (see section 2.3) and is nothing else but the presentation for the general solution of a problem as the probabilistic sum of realisations, eqs. (14).

Finally, it is important to emphasize that all the basic notions introduced and the underlying method of effective dynamical functions show no serious limitations in their extension to arbitrary dynamical systems. Thus the FMDF formalism inherits its generality from the prototype optical potential method [16]. The ensuing concepts of dynamic multivaluedness, complexity (chaoticity), probability (causal randomness), (non)integrability and general solution (completeness) are also generalised without serious changes. It is especially important that many basic features of the qualitative physical picture obtained are easily recognisable in the known patterns of complex system behaviour : irregular change of regular dynamic regimes, transitions chaos-regularity, intermittence and "weak chaos", fractals, etc. More detailed description of this universal concept of complexity and its applications to dynamical systems of various types is the subject of further research and publications.

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