On the position operator for massless particles

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ABSTRACT. It is commonly stated that the position operator for massless particles has non-comutting components. To account for this, we have shown that the commutation relations between coordinates and momenta differ for massive and massless particles. Here we find a position operator for massless particles which has commuting components.

RESUMÉ. On exprime communément que l'opérateur de position des particules sans masse a des composantes non commutatives. En fait, nous avoons montré que les relations de commutation des coordonnées et de l'impulsion des particules massives différent de celles des particules sans masse. Nous avons construit un opérateur de position pour les particules sans masse dont les composantes sont commutatives. Ensuite nous avons déduit certaines propriétés intéréssantes de cet opérateur.

1 Introduction and survey

The notion of position operator has its roots in the early days of the birth of quantum mechanics. Although in the Copenhagen interpretation of quantum mechanics, the concept of position, and therefore path of the particle, is meaningless, nevertheless there must exist an operator called position operator having the property that its expectation value in the classical limit would behave classically. In other words, any macroscopic object has position. In quantum mechanics, one deals with *elementary systems* which means any system whose state has a definite transformation under Poincare group (or under Gallileo group in the non-relativistic case). An *elementary particle*, then, can be defined as

an elementary system which has no constituents. In this way, electron is an elementary particle while Hydrogen atom is an elementary system only. In dealing with elementary systems one works only with generators of Poincare group as physical observables rather than the *position* of the system. Clearly it is natural to search for a *position operator* as an observable whose eigenvalues are the possible positions of an elementary particle or the center of mass position of an elementary system. Unfortunately when one restricts himself to the positive energy manifold, the operator $i\vec{\nabla}_p$ is no longer hermitian.

The problem of finding the position operator in the framework of nonrelativistic quantum mechanics, where the symmetry of space-time is the Gallileo group, is simple.[1,3] Serious work on relativistic case began after the works of Pryce and Newton-Wigner.[2] They found a position operator, which we call it Pryce-Newton-Wigner operator, having the foregoing property. Until now, a lot of theoretical works has been done on this operator.[3]

In spite of these investigations concerning the position operator, the following problem observed by Pryce and Newton-Wigner, is still unsolved. When one tries to write down the position operator for massless particles with non-zero helicity (e.g. photons), one encounters inconsistency. Technically speaking, one is not able to write a position operator having commuting components for such particles. This is a serious problem, as photon would not be localizable. If you measure some component of the photon's position, its other components could not be determined precisely, as Heisenberg's uncertainty principle dictates. It can be shown that the localizability problem is related to causality.[4]

Some people have tried to overcome the problem by rejecting or weakening the Newton-Wigner postulates for the derivation of the position operator.[5] They assume that the probability of finding the particle in a volume V consisting of volumes V_1 and V_2 with empty intersection is not equal to the sum of the probabilities of finding the particle in V_1 and V_2. This is not a physically reasonable assumption, and still has the causality problem.

In spite of the lack of localizability for massless particles, it has been shown[6] that by defining precisely the concept of localizability for massless particles, there exist localized wavefunctions with any desired accuracy, in accordance with the experimental facts.

In this work, we shall show that if one proposes that for massless

particles the canonical commutation relation is not the standard one, it can be shown that one arrives at a position operator for massless particles which has commuting components. In the appendix we present a classical argument in favour of the new commutation relation.

Before doing so it is instructive to review breifly the procedure of constructing position operator for relativistic massive particles. In quantum mechanics, observables are identified by hermitian linear operators with their eigenvalues as allowed results of any measurement of that observable. The essential problem is: what are the observables and their corresponding operators. This can be answered in a formal way. Events occure in space and time and thus it is natural to look for the symmetries of the space-time, which is according to the special theory of relativity, the Poincare group.

The Poincare group consists of space and time translations, rotations and boosts generated by hermitian operators $\vec{P}$, H, $\vec{J}$ and $\vec{K}$ respectively. To these operations space inversion with unitary operator Π and time reversal with antiunitary operator $\mathcal{T}$ must be added. All of our knowledge about these operators are their commutation relations:

$$
\begin{array}{llll}
[P_i, P_j] = 0 & [P_i, H] = 0 & [J_i, J_j] = i\epsilon_{ijk}J_k & [K_i, K_j] = -i\epsilon_{ijk}J_k \\
[J_i, P_j] = i\epsilon_{ijk}P_k & [J_i, K_j] = i\epsilon_{ijk}K_k & [J_i, H] = 0 & [K_i, P_j] = i\delta_{ij}H \\
[K_i, H] = iP_i & \Pi^2 = 1 & \mathcal{T}^2 = 1 & [\Pi, \mathcal{T}] = 0 \\
\Pi\vec{P}\Pi = -\vec{P} & \Pi H\Pi = H & \Pi\vec{J}\Pi = \vec{J} & \Pi\vec{K}\Pi = -\vec{K} \\
\mathcal{T}\vec{P}\mathcal{T} = -\vec{P} & \mathcal{T}H\mathcal{T} = H & \mathcal{T}\vec{J}\mathcal{T} = -\vec{J} & \mathcal{T}\vec{K}\mathcal{T} = \vec{K}
\end{array}
\tag{1}
$$

with clear physical meanings. Note that $\mathcal{T}$ is antiunitary, i.e. acting on any function leads to its complex conjugate:

$$
\mathcal{T}f = f^*
\tag{2}
$$

Irreducible representations of the Poincare group which are identified as particles according to Wigner, can be constructed using the Casimir operators:

$$
\mathcal{C}_1 = H^2 - P^2
\tag{3}
$$

$$
\mathcal{C}_2 = (\vec{P} \cdot \vec{J})^2 - (H\vec{J} + \vec{P} \times \vec{K})^2
\tag{4}
$$

Now following Foldy let[7]

$$
\vec{J} = \vec{Q} \times \vec{P} + \vec{S}
\tag{5}
$$

$$\vec{K} = \frac{1}{2}(H\vec{Q} + \vec{Q}H) + H^{-1}\vec{P} \times \vec{S} - t\vec{P} \tag{6}$$

where $\vec{Q}$ must be identified as the position operator of the particle, $\vec{L} = \vec{Q} \times \vec{P}$ as the orbital angular momentum and $\vec{S}$ as the spin. Using the canonical commutation relation:

$$[Q_i, P_j] = i\delta_{ij} \tag{7}$$

and after some algebra, one can show:

$$\mathcal{C}_2 = -m^2 S^2 \tag{8}$$

$$[S_i, S_j] = i\epsilon_{ijk}S_k \tag{9}$$

the last relation enables one to interpret $\vec{S}$ as spin. From these relations the position operator can be derived:

$$\vec{Q} = H^{-1}(\vec{K} + t\vec{P} - \frac{i}{2}H^{-1}\vec{P}) - m^{-1}H^{-1}(H+m)^{-1}\vec{P} \times (H\vec{J} + \vec{P} \times \vec{K}) \tag{10}$$

This is the Pryce-Newton-Wigner position operator. Note that this is meaningless in the limit $m \to 0$. Its time derivative is the velocity:

$$\frac{d\vec{Q}}{dt} = i[H, \vec{Q}] = H^{-1}\vec{P} \tag{11}$$

It is a vector:

$$[J_i, Q_j] = i\epsilon_{ijk}Q_k \tag{12}$$

$$\Pi\vec{Q}\Pi = -\vec{Q} \tag{13}$$

and under boosts:

$$[K_i, Q_j] = -iH^{-1}P_iQ_j \tag{14}$$

and time reversal:

$$\mathcal{T}\vec{Q}\mathcal{T} = \vec{Q} \tag{15}$$

It can be shown that this position operator is unique up to canonical transformations.[1]

2   Position operator for massless particles

For massless particles the Pryce-Newton-Wigner position operator does not work as it can be seen from the fact that in the $m \to 0$ limit, it does not have a good behaviour. In fact if one starts with massless representations of Poincare group the position operator obtained has not commuting components if one uses the canonical commutation relation (7). Thus this leads to non-localizability (which is equal to the lack of causality). It can be seen that this is because the correct commutation relation for position and momentum is not used. It can be argued *heuristically* that the correct one is as follows

$$[Q_i, P_j] = iH^{-2}P_iP_j \qquad (16)$$

instead of (7). In this section we shall show that using this relation one will arrive at a position operator with commuting components.

For massless representations one has:

$$H^2 = P^2 \qquad (17)$$

$$\vec{P} \cdot \vec{J} = H\Sigma \qquad (18)$$

$$H\vec{J} + \vec{P} \times \vec{K} = \vec{P}\Sigma \qquad (19)$$

where Σ is the helicity operator having eigenvalues $\pm h$ (for photon $h = 1$). It can be shown that these equations leads to a non-commuting position operator except in the case of zero helicity.[3]

As it is shown in the previous section for massless particles the commutation relation (7) must be replaced by one given in equation (16). Thus the problem is finding an operator satisfying relations (11)-(16). This operator can be only of the forms $\vec{f}(\vec{P}) \cdot \vec{K}$, $g(H)\vec{P} \times \vec{J}$ and $h(H)\vec{P} \times (\vec{P} \times \vec{K})$ because of the character of position operator under space inversion and time reversal. The above equations may be used to solve for f, g and h. The final result after symmetrization is as follows:

$$\vec{Q} = \frac{1}{2}(H^{-3}\vec{P}(\vec{P} \cdot \vec{K}) + (\vec{K} \cdot \vec{P})\vec{P}H^{-3}) + tH^{-1}\vec{P} \qquad (20)$$

This position operator has commuting components and all other commutation relations are correct. Now, we should be careful about two points: First; since we find the complete solution of commutation relations, our position operator is unique up to a canonical transformation

(which leaves equation (16) unchanged, not equation (7)). Second; according to equation (16) our position operator is not an ordinary *vector* under translations. It is not a free vector, i.e. when one translate the reference frame the position operator of massless particles does not move rigidly. (This is apparent from the fact that $\vec{L} = 0$)

This new position operator has at least two new intresting results which we shall discuss below. First, since the standard position–momentum commutation relation is changed for the massless particles, the uncertainty relation for position and momentum may differ from the well known one. It is a standard result of quantum mechanics that

$$(\Delta Q_i)(\Delta P_j) \geq \frac{1}{2}| < [Q_i, P_j] > | = \frac{\hbar}{2}| < H^{-2}P_iP_j > | \qquad (21)$$

where we have recovered the $\hbar$ factor.

In order to calculate the right hand side of this relation we introduce the momentum eigenstates as

$$P_i|\vec{k}> = k_i|\vec{k}> \qquad (22)$$

and write the general state of the system as follows

$$|\alpha> = \int d^3k S(\vec{k})|\vec{k}> \qquad (23)$$

It can be easily seen that

$$< H^{-2}P_iP_j> = \int d^3k|S(\vec{k})|^2 \frac{k_ik_j}{k^2} \qquad (24)$$

In the case in which S is only a function of the lenght of $\vec{k}$, using the orthonormality condition of the state vector we have

$$< H^{-2}P_iP_j> = \frac{1}{3}\delta_{ij} \qquad (25)$$

so

$$(\Delta Q_i)(\Delta P_j) \geq \frac{1}{6}\hbar\delta_{ij} \qquad (26)$$

In the general case where S depends on the direction of $\vec{k}$ (i.e. when there is a preffered direction $\vec{k}_0$) like $S(\vec{k}) \sim exp(-\alpha(\vec{k} - \vec{k}_0)^2)$ the uncertainty relation for position and momentum reads as

$$(\Delta Q_i)(\Delta P_j) \geq \hbar(A\delta_{ij} + Bk_{0i}k_{0j}) \qquad (27)$$

where

$$A = \frac{1}{6} + \ terms\ involving\ k_0$$

$$B = 0 + \ terms\ involving\ k_0$$

Thus our new position operator suggests a new uncertainty relation for position and momentum. The experimental consequences of this new relation can be in principle, verified for massless particles with a wavefunction which peaks at a very high momentum, for example. In such a case the role of the second term at the right hand side of the relation (27) is important.

The second important result of our new position operator for massless particles is about its eigenfunctions. To simplify the calculations, we work in the momentum representation where

$$\vec{K} = \frac{1}{2}i(\frac{\partial}{\partial \vec{P}}H + H\frac{\partial}{\partial \vec{P}}) - H^{-1}\vec{S} \times \vec{P} \tag{28}$$

and consider a zero helicity massless particle. The Shcrodinger picture eigenvalue problem for the position operator is then

$$iP^{-2}\vec{P}\vec{P} \cdot \frac{\partial \Phi_{\vec{q}}(\vec{P})}{\partial \vec{P}} + iP^{-2}\vec{P}\Phi_{\vec{q}}(\vec{P}) = \vec{q}\Phi_{\vec{q}}(\vec{P}) \tag{29}$$

where $\Phi_{\vec{q}}(\vec{P})$ is the position eigenfunction in the momentum representation with the eigenvalue $\vec{q}$. The form of this equation suggests that the wavefunction is nonzero only when $\vec{P}$ and $\vec{q}$ are parrallel. So we set

$$\Phi_{\vec{q}}(\vec{P}) = \Phi_{\vec{q}}^{(0)}(P)\delta\left(\frac{\vec{P} \cdot \vec{q}}{Pq} - 1\right) \tag{30}$$

Inserting this relation in (29) one arrives at

$$\frac{d\Phi_{\vec{q}}^{(0)}(P)}{dP} = \left(-iq - \frac{1}{P}\right)\Phi_{\vec{q}}^{(0)}(P) \tag{31}$$

which can be easily solved. The wavefunction is thus

$$\Phi_{\vec{q}}(\vec{P}) = \frac{N}{P}e^{-iPq}\delta\left(\frac{\vec{P} \cdot \vec{q}}{Pq} - 1\right) \tag{32}$$

The $\vec{x}$-representation position eigenfunctions can be obtained via Fourier transformation

$$\Psi_{\vec{q}}(\vec{x}) = \int \frac{d^3 P}{P} \Phi_{\vec{q}}(\vec{P}) e^{-i\vec{P}\cdot\vec{x}} = \frac{N}{2} \delta\left(\frac{\vec{x}\cdot\vec{q}}{q} - q\right) \qquad (33)$$

That is our position operator for massless particles has the peculiar property that is delta function in the direction of its eigenvalue and is constant in the direction prependicular to the eigenvalue.

3   Conclusion

It is shown that on the basis of classical arguments one is forced to propose a new commutation relation between position and momentum for massless particles. Using the new commutation relation (16) one arrives at a position operator for massless particles which has commuting components. The effect of the new commutation relation on the position-momentum uncertainty relation is investigated. Also the localized states, i.e. the eigenfunctions of this new position operator are derived.

4   Appendix

In this appendix we presenta classical reasoning in favour of the relation (16). Consider a classical system with coordinate $\vec{Q}$, energy H and momentum $\vec{P}$. According to the well known results of classical mechanics, translation of the reference frame by $\vec{\epsilon}$ affects the coordinates as

$$Q'_i = Q_i + \epsilon_j \{P_j, Q_i\}$$

where $\{,\}$ represents Poisson brackets. If the standard Poisson bracket between coordinates and momenta is satisfied

$$\{Q_i, P_j\} = \delta_{ij}$$

we have

$$\vec{Q}' = \vec{Q} + \vec{\epsilon}$$

which reads as: *translation of reference frame is equal to translation of the particle*. Why these two operations are equal? This is because for a massive particle one can always transform to the rest frame of the particle, in which the particle is attached to the space-time.

Now the difficulty for massless particles is apparent. Any massless particle must move with unit velocity (for such particles $H^2 = P^2$ and thus $u^2 = H^{-2}P^2 = 1$) and it is a well-known result of Poincare transformations that there is no rest frame for such particles – they move with unit velocity in any reference frame. Thus one cannot use the standard Poisson brackets for coordinates and momenta. Let us see what is the correct one for massless particles.

Let the velocity of the particle be $\vec{u}$ with $u^2 = 1$ and suppose we want to calculate P_1, Q_2. So we choose $\vec{\epsilon} = \epsilon \hat{e_1}$ and transform to a frame in which $u_1' = 0$. For simplicity we assume that the translation of reference frame is dynamic, i.e.:

$$\epsilon = u_1 t'$$

During time t', the particle moves in $\hat{e_2}$ direction by $u_2' t'$ which when transformed to the initial frame is equal to $u_2 t \gamma^{-2}(u_1)$. From this amount $u_2 t$ must be subtracted because we assume that the translation to be dynamic. The net change in Q_2 is:

$$u_2 t (1 - u_1^2) - u_2 t = -u_1^2 u_2 t = -u_1 u_2 \epsilon$$

so we conclude that $\epsilon\{P_1, Q_2\} = -\epsilon u_1 u_2$ or in general:

$$\{P_j, Q_i\} = H^{-2} P_i P_j$$

The quantum mechanical analogous of this relation can be achieved via Dirac's canonical quantization rule $\{,\} \to -i[,]$ as

$$[Q_i, P_j] = iH^{-2}P_i P_j$$

This equation is the analogous to equation (7) and must be used for massless particles. It is worthwhile to note that it is covariant (i.e. is compatible with equations (1)) and thus it is independent of the way it is constructed.

References

[1] T.F. Jordan, J. Math. Phys., **21**(8), 2028, (1980).

[2] M.H.L. Pryce, Proc. Roy. Soc. **195A**, 62 (1948); T.D. Newton and E.P. Wigner, Rev. Mod. Phys. **21**, 400 (1949).

[3] See H. Bacry, *Localizability and Space in Quantum Physics*, Springer-Verlag (1988); and G.N. Fleming, Phys. Rev. B139, 963, (1965); and J.M. Jauch and C. Piron, Helv. Phys. Acta, 40, 559, (1967); and T. Sohkawa, Prog. Theor. Phys., 60, 257, (1978); and D.R. Grigore, J. Math. Phys., 30, 2646, (1989).

[4] G.C. Hegerfeldt, Phys. Rev. D10, 3320, (1974); and G.C. Hegerfeldt, Phys. Rev. Lett., 54, 2395, (1985); and S.N.M. Ruijsenaars, Ann. Phys. (N.Y.), 137, 33, (1981); and G.C. Hegerfeldt and S.N.M. Ruijsenaars, Phys. Rev. D22, 377, (1980).

[5] M. Comi, Il Nuovo Cimento, **56A**, No.3, 299, (April 1980); and references therein.

[6] J. Mourad, Phys. Lett. A, **182**, 319, (1993).

[7] L.L. Foldy, Phys. Rev., **102**, No.2, (1956).